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A Common Fixed Point Theorem in G-Metric Spaces

S.K.Mohanta and Srikanta Mohanta
Department of Mathematics,
West Bengal State University,
Barasat, 24 Pargans (North), West Bengal, Kolkata 700126,
email: `smwbes@yahoo.in`

ABSTRACT

We prove a common fixed point theorem for a pair of self mappings in complete G-metric spaces. Our result will improve and supplement some recent results in the setting of G-metric spaces.

RESUMEN

Probamos un teorema de punto fijo genérico para un par de auto-aplicaciones en espacios G-métricos completos. Nuestro resultado mejorará y complementará algunos de los resultados recientes en el marco de los espacios G-métricos.

Keywords and Phrases: G-metric space, G-Cauchy sequence, G-continuity, common fixed point.

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1 Introduction

The study of metric fixed point theory has been at the centre of vigorous activity and it has a wide range of applications in applied mathematics and sciences. Over the past two decades a considerable amount of research work for the development of fixed point theory have executed by several authors. Different generalizations of the usual notion of a metric space have been proposed by Gähler [4, 5] and by Dhage [2, 3]. Unfortunately, it was found that most of the results claimed by Dhage are invalid. These errors were pointed out by Mustafa and Sims in [13]. They also introduced a more appropriate concept of generalized metric space called G-metric space [9] and developed a new fixed point theory for various mappings in this new structure. Our aim in this study is to prove a common fixed point theorem in a complete G-metric space. This theorem generalizes the fixed point results of [1], [10] and [11].

2 Preliminaries

In this section, we present some basic definitions and results for G-metric spaces which will be needed in the sequel. Throughout this paper we denote by $\mathbb{N}$ the set of positive integers.

Definition 2.1. (see[9]) Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following axioms:

(G_1) $G(x, y, z) = 0$ if $x = y = z$,

(G_2) $0 < G(x, x, y)$, for all $x, y \in X$, with $x \neq y$,

(G_3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function G is called a generalized metric , or, more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

Proposition 2.1. (see[9]) *Let* (X, G) *be a* G*-metric space. Then for any* $x, y, z,$ *and* $a \in X$*, it follows that*

(1) if $G(x, y, z) = 0$ then $x = y = z$,

(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(3) $G(x, y, y) \leq 2G(y, x, x)$,

(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(5) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,

(6) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Definition 2.2. (see[9]) Let (X, G) be a G-metric space, let (x_n) be a sequence of points of X, we say that (x_n) is G-convergent to x if $\lim\limits_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon, \text{ for all } n, m \geq n_0$. We call x as the limit of the sequence (x_n) and write $x_n \longrightarrow x$.

Proposition 2.2. (see[9]) *Let (X, G) be a G-metric space, then the following are equivalent.*

(1) $(x_n) \text{ is } G-\text{convergent to } x.$

(2) $G(x_n, x_n, x) \to 0, \text{ as } n \to \infty.$

(3) $G(x_n, x, x) \to 0, \text{ as } n \to \infty.$

(4) $G(x_n, x_m, x) \to 0, \text{ as } n, m \to \infty.$

Definition 2.3. (see[9]) Let (X, G) be a G-metric space, a sequence (x_n) is called G-Cauchy if given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon, \text{ for all } n, m, l \geq n_0$; that is, if $G(x_n, x_m, x_l) \to 0 \text{ as } n, m, l \to \infty$.

Definition 2.4. (see[9]) Let (X, G) and (X', G') be G-metric spaces and let $f : (X, G) \to (X', G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X;\ G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Proposition 2.3. (see[9]) *Let (X, G) and (X', G') be G-metric spaces, then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x, $(f(x_n))$ is G-convergent to $f(x)$.*

Proposition 2.4. (see[9]) *Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 2.5. (see[9]) *Every G-metric space (X, G) will define a metric space (X, d_G) by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \text{for all } x, y \in X.$$

Definition 2.5. (see[9]) A G-metric space (X, G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 2.6. (see[9]) *A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

3 Main Results

Theorem 3.1. Let (X, G) be a complete G-metric space, and let T_1, T_2 be mappings from X into itself satisfying

$$\max\left\{\begin{array}{l} G(T_1(x), T_2(T_1(x)), T_2(T_1(x))), \\ \\ G(T_2(x), T_1(T_2(x)), T_1(T_2(x))) \end{array}\right\} \leq r \min\left\{\begin{array}{l} G(x, T_1(x), T_1(x)), \\ \\ G(x, T_2(x), T_2(x)) \end{array}\right\} \tag{3.1}$$

for every $x \in X$, where $0 \leq r < 1$ and that

$$\inf\,[G(x, y, y) + \min\{G(x, T_1(x), T_1(x)), G(x, T_2(x), T_2(x))\} : x \in X] > 0$$

for every $y \in X$ with y is not a common fixed point of T_1 and T_2.

Then T_1 and T_2 have a common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and define a sequence (x_n) by

$$\begin{aligned} x_n &= T_1(x_{n-1}), \text{ if } n \text{ is odd} \\ &= T_2(x_{n-1}), \text{ if } n \text{ is even.} \end{aligned}$$

Then for any odd positive integer $n \in \mathbb{N}$, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(T_1(x_{n-1}), T_2(x_n), T_2(x_n)) \\ &= G(T_1(x_{n-1}), T_2(T_1(x_{n-1})), T_2(T_1(x_{n-1}))) \\ &\leq \max\left\{\begin{array}{l} G(T_1(x_{n-1}), T_2(T_1(x_{n-1})), T_2(T_1(x_{n-1}))), \\ \\ G(T_2(x_{n-1}), T_1(T_2(x_{n-1})), T_1(T_2(x_{n-1}))) \end{array}\right\} \\ &\leq r \min\left\{\begin{array}{l} G(x_{n-1}, T_1(x_{n-1}), T_1(x_{n-1})), \\ \\ G(x_{n-1}, T_2(x_{n-1}), T_2(x_{n-1})) \end{array}\right\}, \text{ by } (3.1) \\ &\leq r\, G(x_{n-1}, T_1(x_{n-1}), T_1(x_{n-1})) \\ &= r\, G(x_{n-1}, x_n, x_n). \end{aligned}$$

If n is even, then by (3.1), we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(T_2(x_{n-1}), T_1(x_n), T_1(x_n)) \\ &= G(T_2(x_{n-1}), T_1(T_2(x_{n-1})), T_1(T_2(x_{n-1}))) \\ &\leq max \left\{ \begin{array}{l} G(T_1(x_{n-1}), T_2(T_1(x_{n-1})), T_2(T_1(x_{n-1}))), \\ \\ G(T_2(x_{n-1}), T_1(T_2(x_{n-1})), T_1(T_2(x_{n-1}))) \end{array} \right\} \\ &\leq r\, min \left\{ \begin{array}{l} G(x_{n-1}, T_1(x_{n-1}), T_1(x_{n-1})), \\ \\ G(x_{n-1}, T_2(x_{n-1}), T_2(x_{n-1})) \end{array} \right\} \\ &\leq r\, G(x_{n-1}, T_2(x_{n-1}), T_2(x_{n-1})) \\ &= r\, G(x_{n-1}, x_n, x_n). \end{aligned}$$

Thus for any positive integer n, it must be the case that

$$G(x_n, x_{n+1}, x_{n+1}) \leq r\, G(x_{n-1}, x_n, x_n). \tag{3.2}$$

By repeated application of (3.2), we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \leq r^n\, G(x_0, x_1, x_1). \tag{3.3}$$

Then, for all $n, m \in \mathbb{N},\ n < m$, we have by repeated use of the rectangle inequality and (3.3) that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq \left(r^n + r^{n+1} + \cdots + r^{m-1}\right) G(x_0, x_1, x_1) \\ &\leq \frac{r^n}{1-r} G(x_0, x_1, x_1). \end{aligned}$$

Then, $lim\, G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$, since $lim \frac{r^n}{1-r} G(x_0, x_1, x_1) = 0$, as $n, m \to \infty$. For $n, m, l \in \mathbb{N}$, (G_5) implies that

$$G(x_n, x_m, x_l) \leq G(x_n, x_m, x_m) + G(x_l, x_m, x_m),$$

taking limit as $n, m, l \to \infty$, we get $G(x_n, x_m, x_l) \to 0$. So (x_n) is a G-Cauchy sequence. By completeness of (X, G), there exists $u \in X$ such that (x_n) is G-convergent to u.

Let $n \in \mathbb{N}$ be fixed. Then, since (x_m) G-converges to u and G is continuous on its variables, we have

$$G(x_n, u, u) = \lim_{m \to \infty} G(x_n, x_m, x_m) \leq \frac{r^n}{1-r} G(x_0, x_1, x_1).$$

Assume that u is not a common fixed point of T_1 and T_2. Then, by hypothesis, we have

$$\begin{aligned}
0 &< \inf\left[\,G(x,u,u)+\min\{G(x,T_1(x),T_1(x)),G(x,T_2(x),T_2(x))\}:x\in X\right]\\
&\le \inf\left[\,G(x_n,u,u)+\min\{\,G(x_n,T_1(x_n),T_1(x_n)),G(x_n,T_2(x_n),T_2(x_n))\}:n\in\mathbb{N}\right]\\
&\le \inf\left[\frac{r^n}{1-r}\,G(x_0,x_1,x_1)+G(x_n,x_{n+1},x_{n+1}):n\in\mathbb{N}\right]\\
&\le \inf\left[\frac{r^n}{1-r}\,G(x_0,x_1,x_1)+r^n\,G(x_0,x_1,x_1):n\in\mathbb{N}\right]\\
&= 0,
\end{aligned}$$

which is a contradiction. Therefore, u is a common fixed point of T_1 and T_2.

□

Theorem 3.2. Let (X,G) be a complete G-metric space, and let T_1,T_2 be mappings from X into itself satisfying

$$\max\left\{\begin{array}{c} G(T_1(x),T_1(x),T_2(T_1(x))),\\ \\ G(T_2(x),T_2(x),T_1(T_2(x)))\end{array}\right\}\le r\min\left\{\begin{array}{c} G(x,x,T_1(x)),\\ \\ G(x,x,T_2(x))\end{array}\right\} \tag{3.4}$$

for every $x\in X$, where $0\le r<1$ and that

$$\inf\left[G(x,x,y)+\min\{G(x,x,T_1(x)),G(x,x,T_2(x))\}:x\in X\right]>0$$

for every $y\in X$ with y is not a common fixed point of T_1 and T_2.

Then T_1 and T_2 have a common fixed point in X.

Proof. Let $x_0\in X$ be arbitrary and define a sequence (x_n) by

$$\begin{aligned}
x_n &= T_1(x_{n-1}),\ \text{if } n \text{ is odd}\\
&= T_2(x_{n-1}),\ \text{if } n \text{ is } even.
\end{aligned}$$

Then by the argument similar to that used in Theorem 3.1, we have for any positive integer n,

$$G(x_n,x_n,x_{n+1})\le r^n\,G(x_0,x_0,x_1). \tag{3.5}$$

Then, for all $n,m\in\mathbb{N},\ n<m$, we have by repeated use of the rectangle inequality and (3.5) that

$$\begin{aligned}
G(x_m,x_n,x_n) &\le G(x_m,x_{m-1},x_{m-1})+G(x_{m-1},x_{m-2},x_{m-2})\\
&\quad +G(x_{m-2},x_{m-3},x_{m-3})+\cdots+G(x_{n+1},x_n,x_n)\\
&\le \left(r^n+r^{n+1}+\cdots+r^{m-1}\right)G(x_0,x_0,x_1)\\
&\le \frac{r^n}{1-r}\,G(x_0,x_0,x_1).
\end{aligned}$$

Thus (x_n) becomes a G-Cauchy sequence. By completeness of (X, G), there exists $u \in X$ such that (x_n) is G-convergent to u.

Let $n \in \mathbb{N}$ be fixed. Then since (x_m) G-converges to u and G is continuous on its variables, we have

$$G(x_n, x_n, u) = \lim_{m\to\infty} G(x_n, x_n, x_m) \le \frac{r^n}{1-r} G(x_0, x_0, x_1).$$

The argument similar to that used in the proof of Theorem 3.1 establishes that u is a common fixed point of T_1 and T_2.

□

Combining Theorem 3.1 and Theorem 3.2, we state the following Theorem:

Theorem 3.3. Let (X, G) be a complete G-metric space, and let T_1, T_2 be mappings from X into itself satisfying one of the following conditions:

$$max \left\{ \begin{array}{c} G(T_1(x), T_2(T_1(x)), T_2(T_1(x))), \\ \\ G(T_2(x), T_1(T_2(x)), T_1(T_2(x))) \end{array} \right\} \le r\, min \left\{ \begin{array}{c} G(x, T_1(x), T_1(x)), \\ \\ G(x, T_2(x), T_2(x)) \end{array} \right\}$$

for every $x \in X$, where $0 \le r < 1$ and that

$$inf\, [G(x, y, y) + min\{G(x, T_1(x), T_1(x)), G(x, T_2(x), T_2(x))\} : x \in X] > 0$$

for every $y \in X$ with y is not a common fixed point of T_1 and T_2.

$$or$$

$$max \left\{ \begin{array}{c} G(T_1(x), T_1(x), T_2(T_1(x))), \\ \\ G(T_2(x), T_2(x), T_1(T_2(x))) \end{array} \right\} \le r\, min \left\{ \begin{array}{c} G(x, x, T_1(x)), \\ \\ G(x, x, T_2(x)) \end{array} \right\}$$

for every $x \in X$, where $0 \le r < 1$ and that

$$inf\, [G(x, x, y) + min\{G(x, x, T_1(x)), G(x, x, T_2(x))\} : x \in X] > 0$$

for every $y \in X$ with y is not a common fixed point of T_1 and T_2.

Then T_1 and T_2 have a common fixed point in X.

As an application of Theorem 3.3, we have the following Corollary.

Corollary 1. *Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a mapping satisfying one of the following conditions:*

$$G(T(x), T^2(x), T^2(x)) \leq r\, G(x, T(x), T(x)) \tag{3.6}$$

for every $x \in X$, where $0 \leq r < 1$ and that

$$\inf\,[G(x, y, y) + G(x, T(x), T(x)) : x \in X] > 0 \tag{3.7}$$

for every $y \in X$ with $y \neq T(y)$.

or

$$G(T(x), T(x), T^2(x)) \leq r\, G(x, x, T(x)) \tag{3.8}$$

for every $x \in X$, where $0 \leq r < 1$ and that

$$\inf\,[G(x, x, y) + G(x, x, T(x)) : x \in X] > 0 \tag{3.9}$$

for every $y \in X$ with $y \neq T(y)$.

Then T has a fixed point in X.

Proof. Take $T_1 = T_2 = T$ in Theorem 3.3. □

We now supplement Corollary 1 by examination of condition (3.6)(or, (3.8)) and condition (3.7)(or, (3.9)) in respect of their independence. In fact, we furnish Example 3.1 and Example 3.2 below to show that these two conditions are independent in the sense that Corollary 1 shall fall through by dropping one in favour of the other.

Example 3.1. Let $X = \{0\} \cup \left\{\frac{1}{2^n} : n \geq 1\right\}$. Define $G : X \times X \times X \to \mathbb{R}^+$ by

$$G(x, y, z) = \frac{1}{4} \mid x - y \mid + \frac{1}{4} \mid y - z \mid + \frac{1}{4} \mid z - x \mid, \text{ for all } x, y, z \in X.$$

Then (X, G) is a complete G-metric space.
Define $T : X \to X$ by $T(0) = \frac{1}{2}$ and $T\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}$ for $n \geq 1$.Clearly, T has got no fixed point in X. Also, it is easy to check that

$$G(T(x), T^2(x), T^2(x)) = \frac{1}{2}\, G(x, T(x), T(x))$$

for every $x \in X$.
Thus, condition (3.6) in Corollary 1 is satisfied. However, $T(y) \neq y$ for all $y \in X$ and so

$$\inf\{G(x, y, y) + G(x, T(x), T(x)) : x, y \in X \text{ with } y \neq T(y)\}$$

$$\begin{aligned}
&= \inf\{G(x,y,y) + G(x,T(x),T(x)) : x,y \in X\} \\
&= \inf\left\{\frac{1}{2} \mid x-y \mid + \frac{1}{2} \mid x - T(x) \mid : x,y \in X\right\} \\
&= 0.
\end{aligned}$$

Thus condition (3.7) in Corollary 1 does not hold.
Similarly, we can verify that condition (3.8) in Corollary 1 is also satisfied but condition (3.9) fails.
Clearly, the conclusion of Corollary 1 is not valid.

Example 3.2. Take $X = \{0,1\} \cup [2,\infty)$. Define $G : X \times X \times X \to \mathbb{R}^+$ by

$$G(x,y,z) = \frac{1}{4} \mid x-y \mid + \frac{1}{4} \mid y-z \mid + \frac{1}{4} \mid z-x \mid, \text{ for all } x,y,z \in X.$$

Then (X,G) is a complete G-metric space.
Define $T : X \to X$ by

$$\begin{aligned}
T(x) &= 0, \text{ for } x \neq 0 \\
&= 1, \text{ for } x = 0.
\end{aligned}$$

Clearly, T possesses no fixed point in X.
Since $T(y) \neq y$ for all $y \in X$, we have

$$\inf\{G(x,y,y) + G(x,T(x),T(x)) : x,y \in X \, with \, y \neq T(y)\}$$

$$\begin{aligned}
&= \inf\{G(x,y,y) + G(x,T(x),T(x)) : x,y \in X\} \\
&= \inf\left\{\frac{1}{2} \mid x-y \mid + \frac{1}{2} \mid x - T(x) \mid : x,y \in X\right\} \\
&> 0.
\end{aligned}$$

Thus condition (3.7) in Corollary 1 is satisfied.
However, for $x = 0$, we have

$$G(T(x),T^2(x),T^2(x)) = \frac{1}{2} \mid T(x) - T^2(x) \mid = \frac{1}{2} > r\, G(x,T(x),T(x))$$

for any $r \in [0,1)$.
This shows that condition (3.6) in Corollary 1 does not hold.
Similarly, we can check that condition (3.9) in Corollary 1 is also satisfied but condition (3.8) fails.
Obviously, Corollary 1 is invalid in this case.

As an application of Corollary 1, we have the following results.

Corollary 2. *Let* (X, G) *be a complete* G*-metric space, and let* $T : X \to X$ *be a* G*-continuous mapping satisfying one of the following conditions:*

$$G(T(x), T^2(x), T^2(x)) \leq r\, G(x, T(x), T(x)) \tag{3.10}$$

or

$$G(T(x), T(x), T^2(x)) \leq r\, G(x, x, T(x)) \tag{3.11}$$

for every $x \in X$*, where* $0 \leq r < 1$*. Then* T *has a fixed point in* X*.*

Proof. Suppose that T satisfies condition (3.10) for every $x \in X$. Assume that there exists $y \in X$ with $y \neq T(y)$ and

$$inf\,[G(x, y, y) + G(x, T(x), T(x)) : x \in X] = 0.$$

Then there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} \{G(x_n, y, y) + G(x_n, T(x_n), T(x_n))\} = 0,$$

which implies that,

$$G(x_n, y, y) \to 0 \text{ and } G(x_n, T(x_n), T(x_n)) \to 0 \text{ as } n \to \infty.$$

So, by Proposition 2.2, the sequence (x_n) is G-convergent to y.
But by (G_5), we have

$$\begin{aligned} G(T(x_n), y, y) &\leq G(T(x_n), x_n, x_n) + G(x_n, y, y) \\ &\leq 2\, G(x_n, T(x_n), T(x_n)) + G(x_n, y, y) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Again, by Proposition 2.2, the sequence $(T(x_n))$ is G-convergent to y. So G-continuity of T implies that, $(T^2(x_n))$ G-converges to $T(y)$.
Then by use of the rectangle inequality and (3.10) that

$$\begin{aligned} G(x_n, T^2(x_n), T^2(x_n)) &\leq G(x_n, T(x_n), T(x_n)) + G(T(x_n), T^2(x_n), T^2(x_n)) \\ &\leq G(x_n, T(x_n), T(x_n)) + r\, G(x_n, T(x_n), T(x_n)). \end{aligned}$$

Taking the limit as $n \to \infty$, and using the fact that the function G is continuous on its variables, we have

$$\begin{aligned} G(y, T(y), T(y)) &\leq G(y, y, y) + r\, G(y, y, y) \\ &= 0, \end{aligned}$$

which implies that, $y = T(y)$. This is a contradiction.
Hence, if $y \neq T(y)$, then

$$\inf\left[G(x,y,y) + G(x,T(x),T(x)) : x \in X\right] > 0.$$

If T satisfies condition (3.11), then using the same methods as above one can prove that

$$\inf\left[G(x,x,y) + G(x,x,T(x)) : x \in X\right] > 0.$$

So, using Corollary 1, we have the desired result. $\square$

The following Corollary is a generalization of the result [[1], Theorem 2.1].

Corollary 3. *Let* (X,G) *be a complete* G*-metric space, and let* $T : X \to X$ *be a mapping satisfying one of the following conditions:*

$$G(T(x),T(y),T(z)) \leq k \max \left\{ \begin{array}{l} G(x,y,z), G(x,T(x),T(x)), \\ \\ G(y,T(y),T(y)), G(z,T(z),T(z)), \\ \\ \frac{G(x,T(y),T(y))+G(z,T(x),T(x))}{2}, \\ \\ \frac{G(x,T(y),T(y))+G(y,T(x),T(x))}{2}, \\ \\ \frac{G(y,T(z),T(z))+G(z,T(y),T(y))}{2}, \\ \\ \frac{G(x,T(z),T(z))+G(z,T(x),T(x))}{2} \end{array} \right\} \tag{3.12}$$

or

$$G(T(x),T(y),T(z)) \leq k \max \left\{ \begin{array}{l} G(x,y,z), G(x,x,T(x)), \\ \\ G(y,y,T(y)), G(z,z,T(z)), \\ \\ \frac{G(x,x,T(y))+G(z,z,T(x))}{2}, \\ \\ \frac{G(x,x,T(y))+G(y,y,T(x))}{2}, \\ \\ \frac{G(y,y,T(z))+G(z,z,T(y))}{2}, \\ \\ \frac{G(x,x,T(z))+G(z,z,T(x))}{2} \end{array} \right\} \tag{3.13}$$

for all $x, y, z \in X$*, where* $0 \leq k < 1$*. Then* T *has a unique fixed point* $(\mathrm{say}\, u)$ *in* X *and* T *is* G*-continuous at* u*.*

Proof. Suppose that T satisfies condition (3.12) for all $x, y, z \in X$. Then replacing y and z by $T(x)$, we obtain from (3.12) and using (G_5) that

$$\begin{aligned} G(T(x), T^2(x), T^2(x)) &\leq k \max \left\{ \begin{array}{l} G(x, T(x), T(x)), G(x, T(x), T(x)), \\ G(T(x), T^2(x), T^2(x)), G(T(x), T^2(x), T^2(x)), \\ \frac{G(x, T^2(x), T^2(x)) + G(T(x), T(x), T(x))}{2}, \\ \frac{G(x, T^2(x), T^2(x)) + G(T(x), T(x), T(x))}{2}, \\ \frac{G(T(x), T^2(x), T^2(x)) + G(T(x), T^2(x), T^2(x))}{2}, \\ \frac{G(x, T^2(x), T^2(x)) + G(T(x), T(x), T(x))}{2} \end{array} \right\} \\ &\leq k \max \left\{ \begin{array}{l} G(x, T(x), T(x)), G(T(x), T^2(x), T^2(x)), \\ \frac{G(x, T(x), T(x)) + G(T(x), T^2(x), T^2(x))}{2} \end{array} \right\} \qquad (3.14) \end{aligned}$$

Without loss of generality we may assume that $T(x) \neq T^2(x)$. For, otherwise, T has a fixed point. So, (3.14) leads to the following cases,

(1) $G(T(x), T^2(x), T^2(x)) \leq k \, \frac{G(x, T(x), T(x)) + G(T(x), T^2(x), T^2(x))}{2}$,

(2) $G(T(x), T^2(x), T^2(x)) \leq k \, G(x, T(x), T(x))$.

In the first case, we have

$$G(T(x), T^2(x), T^2(x)) \leq \frac{k}{2-k} \, G(x, T(x), T(x)).$$

Put $r = \frac{k}{2-k}$. Then $0 \leq r < 1$.
Thus, in each case we must have

$$G(T(x), T^2(x), T^2(x)) \leq r \, G(x, T(x), T(x))$$

for every $x \in X$, where $0 \leq r < 1$.
Assume that there exists $y \in X$ with $y \neq T(y)$ and

$$\inf \left[G(x, y, y) + G(x, T(x), T(x)) : x \in X \right] = 0.$$

Proceeding exactly the same way as in the proof of Corollary 2, there exists a sequence (x_n) in X such that (x_n) is G-convergent to y and $(T(x_n))$ is G-convergent to y.
Now applying (3.12), we have

$$G(T(x_n), T(y), T(y)) \leq k \max \left\{ \begin{array}{l} G(x_n, y, y), G(x_n, T(x_n), T(x_n)), \\ \\ G(y, T(y), T(y)), G(y, T(y), T(y)), \\ \\ \frac{G(x_n, T(y), T(y)) + G(y, T(x_n), T(x_n))}{2}, \\ \\ \frac{G(x_n, T(y), T(y)) + G(y, T(x_n), T(x_n))}{2}, \\ \\ \frac{G(y, T(y), T(y)) + G(y, T(y), T(y))}{2}, \\ \\ \frac{G(x_n, T(y), T(y)) + G(y, T(x_n), T(x_n))}{2} \end{array} \right\}.$$

Taking the limit as $n \to \infty$, and using the fact that the function G is continuous on its variables, we obtain

$$G(y, T(y), T(y)) \leq k\, G(y, T(y), T(y)),$$

which is a contradiction.
Hence, if $y \neq T(y)$, then

$$\inf\,[G(x, y, y) + G(x, T(x), T(x)) : x \in X] > 0.$$

Now Corollary 1 applies to obtain a fixed point $(say\, u)$ of T.
The proof using (3.13) is similar. Uniqueness of u and G-continuity of T at u may be verified in the usual way by using any one of condition (3.12) and condition (3.13) that T satisfies. $\square$

Remark 1. We see that special cases of Corollary 3 are Theorem 2.1 of [1], Theorem 2.1 of [11] and Theorems 2.1, and 2.4 of [10].

The following Corollary is the result [[1], Theorem 2.2].

Corollary 4. *Let* (X, G) *be a complete* G*-metric space, and let* $T : X \to X$ *be a mapping satisfying one of the following conditions:*

$$G(T(x), T(y), T(z)) \leq k \max \left\{ \begin{array}{l} G(x, y, z), G(x, T(x), T(x)), \\ \\ G(y, T(y), T(y)), G(x, T(y), T(y)), \\ \\ G(y, T(x), T(x)), G(z, T(z), T(z)) \end{array} \right\} \tag{3.15}$$

or

$$G(T(x),T(y),T(z)) \leq k\,\max\left\{\begin{array}{l} G(x,y,z), G(x,x,T(x)), \\ \\ G(y,y,T(y)), G(x,x,T(y)), \\ \\ G(y,y,T(x)), G(z,z,T(z)) \end{array}\right\} \tag{3.16}$$

for all $x,y,z \in X$*, where* $0 \leq k < 1$*. Then* T *has a unique fixed point* $(\mathrm{say}\,u)$ *in* X *and* T *is* G*-continuous at* u*.*

Proof. Suppose that T satisfies condition (3.15) for all $x,y,z \in X$. Then replacing z by x ; y and x by $T(x)$ in (3.15), we have

$$\begin{aligned} G(T^2(x),T^2(x),T(x)) &\leq k\,\max\left\{\begin{array}{l} G(T(x),T(x),x), G(T(x),T^2(x),T^2(x)), \\ \\ G(T(x),T^2(x),T^2(x)), G(T(x),T^2(x),T^2(x)), \\ \\ G(T(x),T^2(x),T^2(x)), G(x,T(x),T(x)) \end{array}\right\} \\ &\leq k\,\max\left\{ G(x,T(x),T(x)), G(T(x),T^2(x),T^2(x)) \right\}. \end{aligned}$$

Without loss of generality we may assume that $T(x) \neq T^2(x)$. For, otherwise, T has a fixed point.
So, it must be the case that,

$$G(T(x),T^2(x),T^2(x)) \leq k\,G(x,T(x),T(x))$$

for every $x \in X$, where $0 \leq k < 1$.
By the same argument used in the proof of Corollary 3, we see that if $y \neq T(y)$, then

$$\inf\,[G(x,y,y) + G(x,T(x),T(x)) : x \in X] > 0.$$

Now Corollary 1 applies to obtain a fixed point $(\mathrm{say}\,u)$ of T.
The proof using (3.16) is similar. Uniqueness of u and G-continuity of T at u are obtained by the same argument used in Corollary 3. □

The following Corollary is the result [[11], Theorem 2.9].

Corollary 5. *Let* (X,G) *be a complete* G*-metric space, and let* $T : X \to X$ *be a mapping satisfying one of the following conditions:*

$$G(T(x),T(y),T(y)) \leq a\,\{G(x,T(y),T(y)) + G(y,T(x),T(x))\} \tag{3.17}$$

or

$$G(T(x), T(y), T(y)) \leq a \{G(x, x, T(y)) + G(y, y, T(x))\} \tag{3.18}$$

for all $x, y \in X$*, where* $0 \leq a < \frac{1}{2}$*. Then* T *has a unique fixed point* $(\mathrm{say}\, u)$ *in* X *and* T *is* G*-continuous at* u*.*

Proof. Suppose that T satisfies condition (3.17) for all $x, y \in X$. Then replacing y by $T(x)$ in (3.17), we have

$$\begin{aligned} G(T(x), T^2(x), T^2(x)) &\leq a \left\{G(x, T^2(x), T^2(x)) + G(T(x), T(x), T(x))\right\} \\ &\leq a \left\{G(x, T(x), T(x)) + G(T(x), T^2(x), T^2(x))\right\}, \text{ by } (G_5). \end{aligned}$$

So, it must be the case that,

$$G(T(x), T^2(x), T^2(x)) \leq \frac{a}{1-a}\, G(x, T(x), T(x)).$$

Put $r = \frac{a}{1-a}$. Then $0 \leq r < 1$ since $0 \leq a < \frac{1}{2}$.
Thus,

$$G(T(x), T^2(x), T^2(x)) \leq r\, G(x, T(x), T(x)).$$

for every $x \in X$, where $0 \leq r < 1$.
Assume that there exists $y \in X$ with $y \neq T(y)$ and

$$\inf\,[G(x, y, y) + G(x, T(x), T(x)) : x \in X] = 0.$$

As in the proof of Corollary 2, there exists a sequence (x_n) in X such that (x_n) is G-convergent to y and $(T(x_n))$ is G-convergent to y.
Now using (3.17), we have

$$G(T(x_n), T(y), T(y)) \leq a \{G(x_n, T(y), T(y)) + G(y, T(x_n), T(x_n))\}.$$

Taking the limit as $n \to \infty$, and using the fact that the function G is continuous on its variables, we have

$$\begin{aligned} G(y, T(y), T(y)) &\leq a \{G(y, T(y), T(y)) + G(y, y, y)\} \\ &= a\, G(y, T(y), T(y)), \end{aligned}$$

which is a contradiction.
Hence, if $y \neq T(y)$, then

$$\inf\,[G(x, y, y) + G(x, T(x), T(x)) : x \in X] > 0.$$

Now applying Corollary 1, we obtain a fixed point $(\mathrm{say}\, u)$ of T.
The proof using (3.18) is similar. Uniqueness of u and G-continuity of T at u are obtained by the same argument used above. □

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