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Generalized Spectrograms and τ-Wigner Transforms

BOGGIATTO PAOLO, DE DONNO GIUSEPPE, OLIARO ALESSANDRO
Department of Mathematics, University of Turin,
Via Carlo Alberto, 10, 10123 Torino, Italy
email: paolo.boggiatto@unito.it
email: giuseppe.dedonno@unito.it
email: alessandro.oliaro@unito.it

AND

BUI KIEN CUONG
Higher Education Department, Hanoi Pedagogical University 2,
Building G7-144 Xuan Thuy Rd – Hanoi, Vietnam
email: buikiencuong@yahoo.com

ABSTRACT

We consider in this paper Wigner type representations Wig_τ depending on a parameter $\tau \in [0,1]$ as defined in [2]. We prove that the Cohen class can be characterized in terms of the convolution of such Wig_τ with a tempered distribution. We introduce furthermore a class of "quadratic representations" Sp^τ based on the τ-Wigner, as an extension of the two window Spectrogram (see [2]). We give basic properties of Sp^τ as subclasses of the general Cohen class.

RESUMEN

Nosotros consideramos en este artículo representaciones de tipo Wigner Wig_τ dependiendo de um parámetro $\tau \in [0,1]$ como definido en [2]. Probamos que la clase Cohen puede ser caracterizada en terminos de la convolución de tales Wig_τ con una distribución temperada. Introducimos también la clase de "representaciones cuadraticas" Sp^τ basado en el τ-Wigner, como una extensión de dos ventanas espectrograma (ver [2]). Nosotros damos propiedades básicas de Sp^τ como subclases de la clase Cohen.

Key words and phrases: *Time-Frequency representation, τ-Wigner distribution, generalized Spectrogram.*

Math. Subj. Class.: *42B10, 47A07, 33C05.*

1 Introduction

One of the basic problems in time-frequency analysis is the representation of the energy of a signal simultaneously with respect to time and frequency. Considering for generality signals as square-integrable functions on $\mathbb{R}^d$, the classical mathematical tool used for this aim are sesquilinear maps $Q : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$. For a given signal f, the function $Q(f,f)(x,\omega)$, or for short $Q(f)(x,\omega)$, plays a role corresponding to that of density of mass in classical mechanics or that of probability distribution in statistics. In contrast however to these situations, in the case of the energy of a signal the time-frequency distribution to be used is not unique. Many proposals have been presented in the literature, each having advantages and drawbacks, see [5], [6], [7], [8], [9] for detailed presentations of these topics.

This is due essentially to the presence of the Heisenberg uncertainty principle which makes some of the natural requirements of a joint time-frequency distribution incompatible (see [11]).

Two of the most used time-frequency representations are the *Wigner distribution*:

$$Wig(f,f)(x,\omega) = Wig(f)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i t\omega} f(x+t/2)\overline{f(x-t/2)}\,dt \tag{1.1}$$

and the *Spectrogram*

$$Sp_g(f)(x,\omega) = |V_g(f)(x,\omega)|^2 \tag{1.2}$$

where $V_g(f)$ is the *Gabor transform* (also known as *short-time Fourier transform*) and is defined by

$$V_g(f)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i t\omega} f(t)\overline{g(x-t)}\,dt \tag{1.3}$$

in dependence on the "window" $g(x)$, which in the most generality can be supposed to be a tempered distribution.

This paper is based on these two representations of which we present modifications depending on parameters. We shall analyze the properties of these new representations with respect to classical requirements such as reality of values, marginal distribution conditions, and their relations with the *Cohen class*. This is a very general class of time-frequency representations, introduced by L. Cohen, see [6], and widely studied since the 1970's. It can be defined as the set of representations of the form

$$C(f) = \sigma * Wig(f) \tag{1.4}$$

where, in our context, σ will be supposed to be a tempered distribution in $\mathscr{S}'(\mathbb{R}^{2d})$ and will be called *Cohen kernel*. The wide possibility of choice of the Cohen kernel permits to cover most time-frequency representations.

We recall next that some considerations concerning shifts of the ghost frequencies led in [2] to the introduction of the representations

$$Wig_\tau(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i t\omega} f(x+\tau t)\overline{g(x-(1-\tau)t)}\,dt \tag{1.5}$$

which are a parameterized version of the Wigner representation in dependence on $\tau \in [0,1]$. It was also showed in [2] that these representations constitute the natural "quadratic form" counterparts to the τ-pseudo-differential operators which are extensions of the Weyl calculus on $\mathbb{R}^d$; classical references on this subject are Shubin [14] and Wong [15], see also [1] for generalizations concerning global hypo-ellipticity.

In the present paper we analyze at first the role of (1.5) in the definition of the Cohen class, showing that we can replace $Wig(f)$ in (1.4) by $Wig_\tau(f)$, for an arbitrary fixed $\tau \in [0,1]$, getting equivalent definitions of the Cohen class. In the second part of the paper, we propose a new form based on the two window spectrogram and the τ-Wigner representation. The *two window spectrogram* was studied in [3]-[4] (called there *generalized spectrogram*) and is defined by

$$Sp_{\phi,\psi}(f,g)(x,w) = V_\phi f(x,w)\overline{V_\psi g}(x,w). \tag{1.6}$$

Using τ-Wigner distribution, we generalize here definition (1.6) by replacing the classical Wigner distribution with τ-Wigner distributions. We obtain new representations that we shall call *parameterized two window spectrograms* and we study some of their basic properties such as positivity, support properties and boundedness in the L^p context. We show that our definition is motivated by the fact that the parameterized two window spectrograms show in some basic cases reduced interference phenomena with respect to (1.6) without a loss in the quality of the time-frequency localization. Finally we prove that among the variety of time-frequency representations they constitute a peculiarity as they do not belong to the Cohen class [1].

2 τ-Wigner Representations and the Cohen Class

In the definition (1.4) of the Cohen class the Wigner representation plays a special role and one natural question is if it can be replaced by another representation. In general this can be achieved under some additional conditions. More precisely suppose

$$C_0(f) = \sigma_0 * Wig(f)$$

is a fixed representation in the Cohen class; then, as long as $\hat{C}_0(f)/\hat{\sigma}_0$ belongs to $\mathscr{S}'(\mathbb{R}^{2d})$ for every signal $f \in \mathscr{S}(\mathbb{R}^d)$, we have

$$Wig(f) = \mathscr{F}^{-1}(\hat{C}_0(f)/\hat{\sigma}_0).$$

But even under this somewhat restrictive condition it does not necessarily happen that $C_0 \to \mathscr{F}^{-1}(\hat{C}_0(f)/\hat{\sigma}_0)$ is a convolution. Actually only if this were the case we could write

$$\mathscr{F}^{-1}(\hat{C}_0(f)/\hat{\sigma}_0) = \sigma' * C_0(f)$$

for a suitable fixed $\sigma' \in \mathscr{S}'(\mathbb{R}^{2d})$, and then for any generic representation in the Cohen class $C = \sigma * Wig$, (with $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$), we would obtain

$$C(f) = \sigma * Wig(f) = (\sigma * \sigma') * C_0(f).$$

In this case, under the further condition that $\sigma * \sigma' \in \mathscr{S}'(\mathbb{R}^{2d})$, we would have that every element in the Cohen class could be expressed in terms of C_0 instead of Wig.

In view of these observations it is interesting, even if not surprising, that any Wig_τ representation can replace the Wigner representation in the expression of the Cohen class.

In order to prove this assertion we need the explicit expression of Wig_τ as a member of the Cohen class. We recall then from [2] the following result.

Proposition 1. *The representation $Wig_\tau(f)$ belongs to the Cohen class for every $\tau \in [0,1]$, in particular*

$$Wig_\tau(f)(x,\omega) = \big(\sigma_\tau * Wig(f)\big)(x,\omega), \tag{2.1}$$

for every $f \in \mathscr{S}(\mathbb{R}^d)$, where

$$\sigma_\tau = \begin{cases} \frac{2^d}{|2\tau-1|^d}\, e^{2\pi i \frac{2}{2\tau-1} x\omega} & \text{for } \tau \neq \frac{1}{2} \\ \delta & \text{for } \tau = \frac{1}{2} \end{cases} \tag{2.2}$$

and δ is the Dirac distribution.

[1] According to (1.4) we only consider signal independent kernels σ

We have now the following Proposition:

Proposition 2. *Let* $\tau \in [0,1]$ *be fixed, then* Wig_τ *can be used to express the entire Cohen class, i.e. every representation* C *in the Cohen class can be written in the form*

$$C(f) = \sigma' * Wig_\tau(f)$$

for a suitable $\sigma' \in \mathscr{S}'(\mathbb{R}^{2d})$.

Proof. Let

$$C(f) = \sigma * Wig(f) \tag{2.3}$$

with $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$, be the expression of $C(f)$ in the Cohen class. From the previous proposition we have

$$Wig_\tau(f) = \sigma_\tau * Wig(f)$$

and a straightforward computation yields:

$$\sigma_\tau * \sigma_{1-\tau} = \delta.$$

We have therefore

$$\sigma_{1-\tau} * Wig_\tau(f) = Wig(f)$$

and substituting in (2.3) we get formally:

$$C(f) = (\sigma * \sigma_{1-\tau}) * Wig_\tau(f)$$

This expression has actually a meaning if we show that $\sigma * \sigma_{1-\tau}$ is a well defined tempered distribution. As $\sigma * \sigma_{1-\tau} = \mathscr{F}^{-1}(\hat{\sigma}\hat{\sigma}_{1-\tau})$ and $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$, this is equivalent to prove that $\hat{\sigma}_{1-\tau}$ is a multiplier of $\mathscr{S}'(\mathbb{R}^{2d})$. Since $\int e^{2\pi i y \rho}\, dy\, d\rho = 1$ we have

$$\mathscr{F}\sigma_{1-\tau}(\xi, t) = e^{-\pi i(1-2\tau)t\xi} \tag{2.4}$$

which is a C^∞ function with derivatives with polynomial growth and therefore our assertion is proved. The thesis is then satisfied with $\sigma' = \sigma * \sigma_{1-\tau}$. □

We turn now our attention to the spectrograms with the aim of describing how the general context above applies to this specific case.

As already pointed out in the Introduction, the classical spectrogram, defined by

$$Sp_g(f)(x,w) = |V_g f(x,w)|^2, \tag{2.5}$$

is a way to represent the energy of a signal f simultaneously with respect to time and frequency; $V_g f$ is the short-time Fourier transform, or Gabor transform, with window g, see for reference [13], [16], [10]. In [3], the two window spectrogram has been introduced and studied: it depends on two windows and it is defined by the skew-linear form

$$Sp_{\phi,\psi}(f,g)(x,w) = V_\phi(f)\overline{V_\psi(g)}(x,w); \tag{2.6}$$

when $\phi = \psi, f = g$, formula (2.6) becomes the classical spectrogram.

The following relationship between Wigner distribution and two window spectrogram holds (see [3]):

$$Sp_{\phi,\psi}(f,g)(x,w) = Wig(\tilde{\psi},\tilde{\phi}) * Wig(f,g)(x,w), \tag{2.7}$$

where $\tilde{\phi}(s) := \phi(-s)$ and $\tilde{\psi}(s) := \psi(-s)$. Relation (2.7), valid in suitable functional settings, for example when $f,g,\phi,\psi \in \mathscr{S}(\mathbb{R}^d)$, gives us the expression of the two window spectrogram as an element of the Cohen class, where σ in (1.4) is given now by $Wig(\tilde{\psi},\tilde{\phi})$. As proved in Proposition 2, we can re-write $Sp_{\phi,\psi}(f,g)$ through the τ-Wigner transform. In the special case of the two window spectrogram this can be made more explicit as showed by the following result.

Proposition 3. *For every $f,g,\phi,\psi \in \mathscr{S}(\mathbb{R}^d)$ and for every $\tau \in [0,1]$, we have*

$$Sp_{\phi,\psi}(f,g) = Wig_{1-\tau}(\tilde{\psi},\tilde{\phi}) * Wig_\tau(f,g)(x,w).$$

Proof. Since

$$Wig_{1-\tau}(\tilde{\psi},\tilde{\phi}) = \overline{Wig_\tau(\tilde{\phi},\tilde{\psi})}, \tag{2.8}$$

we have to prove that

$$Sp_{\phi,\psi}(f,g) = \overline{Wig_\tau(\tilde{\phi},\tilde{\psi})} * Wig_\tau(f,g)(x,w). \tag{2.9}$$

Let us observe that, by a simple change of variables, we can write

$$Wig_\tau(f,g)(x-y,w-\eta) = \mathscr{F}_{t\to\eta}\Big(e^{2\pi i\omega t} f(x-y-\tau t)\overline{g(x-y+(1-\tau)t)}\Big).$$

Since

$$Wig_\tau(\tilde{\phi},\tilde{\psi})(y,\eta) = \mathscr{F}_{t\to\eta}\Big(\tilde{\phi}(y+\tau t)\overline{\tilde{\psi}(y-(1-\tau)t)}\Big),$$

by the standard properties of the Fourier transform we get

$$\begin{aligned}
&\overline{Wig_\tau(\tilde{\phi},\tilde{\psi})} * Wig_\tau(f,g)(x,w) \\
&= \Big(\tilde{\phi}(y+\tau t)\overline{\tilde{\psi}(y-(1-\tau)t)}, e^{2\pi i\omega t} f(x-y-\tau t)\overline{g(x-y+(1-\tau)t)}\Big)_{L^2(\mathbb{R}^{2d}_{y,t})}.
\end{aligned}$$

Finally, by the change of variables

$$\begin{cases} y+\tau t = Y \\ y-(1-\tau)t = T \end{cases}$$

in the L^2-product, we have

$$\overline{Wig_\tau(\tilde{\phi},\tilde{\psi})} * Wig_\tau(f,g)(x,w) = \left(\tilde{\phi}(Y)\overline{\tilde{\psi}(T)}, e^{2\pi i\omega(Y-T)} f(x-Y)\overline{g(x-T)}\right)_{L^2(\mathbb{R}^{2d}_{Y,T})}.$$

This shows that $\overline{Wig_\tau(\tilde{\phi},\tilde{\psi})} * Wig_\tau(f,g)(x,w)$ is independent of $\tau \in [0,1]$, and so for every $\tau \in [0,1]$,

$$\overline{Wig_\tau(\tilde{\phi},\tilde{\psi})} * Wig_\tau(f,g)(x,w) = \overline{Wig(\tilde{\phi},\tilde{\psi})} * Wig(f,g)(x,w).$$

From (2.8), (2.7) and this last identity, we get (2.9). □

3 The Parameterized Two Window Spectrogram: Definition and Motivations

So far we have been concerned with relationships between $\tau-$Wigner and spectrograms representations within the frame of the Cohen class. In this section we want to consider relationships between these two types of representations under another point of view which will bring us to the definition of a further representation. We start with some preliminary remarks. It is well-known that the Wigner transform can be expressed in function of the spectrogram by the following equality

$$Wig(f,g)(x,w) = 2^d e^{4\pi ixw} V_{\tilde{g}} f(2x,2w), \tag{3.1}$$

and viceversa we have

$$V_g f(x,w) = 2^{-d} e^{-\pi ixw} Wig(f,\tilde{g})(\frac{x}{2},\frac{w}{2}). \tag{3.2}$$

From (2.6) it is then clear that we can then rewrite the two window spectrogram as

$$Sp_{\phi,\psi}(f,g)(x,w) = 4^{-d} Wig(f,\tilde{\phi})(\frac{x}{2},\frac{w}{2})\overline{Wig(g,\tilde{\psi})(\frac{x}{2},\frac{w}{2})}. \tag{3.3}$$

In view of this equality it is natural to introduce the following generalization of the spectrogram:

Definiton 4. Let $\tau_1,\tau_2 \in [0,1]$ be two parameters, the *parameterized two window spectrogram*, denoted $Sp^{(\tau_1,\tau_2)}_{\phi,\psi}(f,g)$, is defined by

$$Sp^{(\tau_1,\tau_2)}_{\phi,\psi}(f,g)(x,w) = 4^{-d} Wig_{\tau_1}(f,\tilde{\phi})(\frac{x}{2},\frac{w}{2})\overline{Wig_{\tau_2}(g,\tilde{\psi})(\frac{x}{2},\frac{w}{2})}, \tag{3.4}$$

where ϕ,ψ are window functions and f,g are signals in suitable functional or distributional spaces.

Remark 5. *When* $\tau_1 = \tau_2 = 1/2$*, the parameterized two window spectrogram becomes the two window spectrogram*

$$Sp^{(\tau_1,\tau_2)}_{\phi,\psi}(f,g)(x,w) = Sp_{\phi,\psi}(f,g)(x,w).$$

The introduction of this new family of parameterized representations is not due to pure search of mathematical generality. Actually, as we describe next, the form $Sp^{(\tau_1,\tau_2)}_{\phi,\psi}(f,g)$ shows an interesting behavior for what concerns localization properties and reduction of interference disturbances in particular in the cases where frequencies occur in time intervals very close to one another. To this aim let us consider a signal f containing the frequency $\omega = 2$ in the time interval $[-4,0]$ and the frequency $\omega = 3$ in the time interval $[0,4]$; we fix the window functions $\phi = \chi_{[-10,10]}$ and $\psi = \chi_{[-\frac{1}{10},\frac{1}{10}]}$, where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$ and we compare the pictures of the parameterized two window spectrograms $Sp^{(\tau_1,\tau_2)}_{\phi,\psi}(f,g)$ for different values of τ_1 and τ_2. The two window spectrogram $Sp_{\phi,\psi}(f,f)$, corresponding to case $\tau_1 = \tau_2 = \frac{1}{2}$, is visualized in Figure 1:

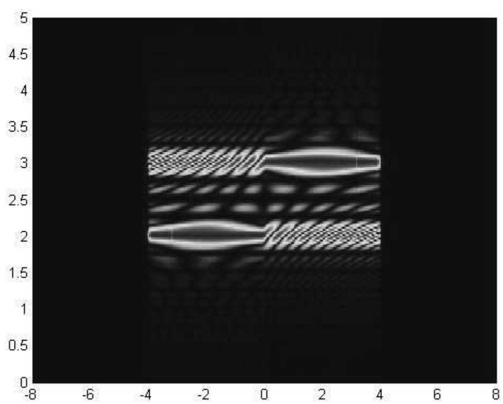


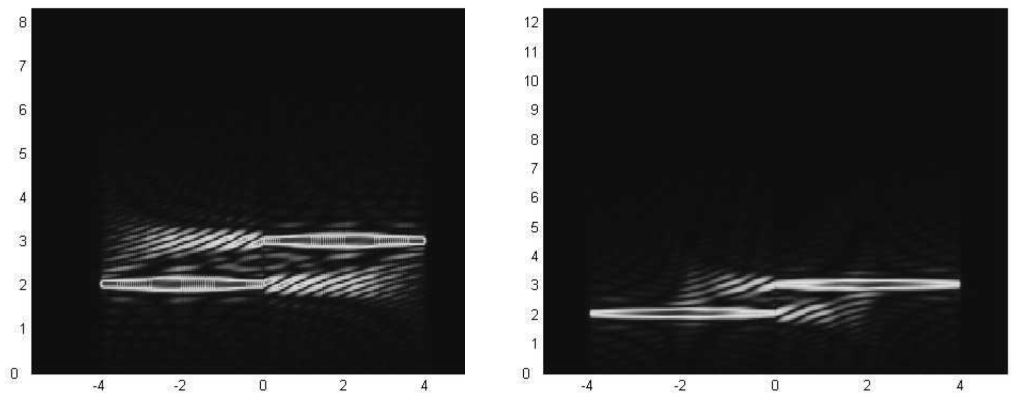
Figure 1: $Sp^{(\frac{1}{2},\frac{1}{2})}_{\phi,\psi}(f,f) = Sp_{\phi,\psi}(f,f)$

As we can see, although the localization is good both in time and in frequency, the picture presents disturbing interference patterns. The explanation of this fact is the following. The Gabor transform $V_\phi f$ with a large window ϕ gives better information regarding frequencies, and the Gabor transform $V_\psi f$ with a narrow window ψ gives better information concerning time. When we consider the two window spectrogram

$$Sp_{\phi,\psi}(f,g) = V_\phi f \overline{V_\psi g}$$

we take a product of one Gabor transform well localized in time and another one well localized in frequency, and so the reciprocal cut-off effect yields good localization both in time and frequency, see [4] for a detailed discussion on this subject. It could seem therefore that we have overcome the Heisenberg uncertainty principle but of course it is not so. Actually what is obtained in good localization, is "paid" terms of interference. More precisely, the fact that each Gabor transform is well localized in one variable and, consequently, badly localized in the other, implies that the supports of the two Gabor transforms also intersects in places where no frequency is present. This is what is observed in Figure 1 and clearly represents a considerable drawback in the use of the classical two window spectrogram.

Let us consider now the parameterized two window spectrogram, with the same windows and signal as above. In Picture 2 we have a representation of $Sp_{\phi,\psi}^{(0.3,0.3)}(f,f)$ and $Sp_{\phi,\psi}^{(0.2,0.2)}(f,f)$ (for simplicity we take here $\tau_1 = \tau_2$).



$Sp_{\phi,\psi}^{(0.3,0.3)}(f,f)$ $Sp_{\phi,\psi}^{(0.2,0.2)}(f,f)$

Figure 2: Parameterized two window spectrogram for different values of τ_1, τ_2.

As we observe from the pictures, although the windows ϕ and ψ are kept fixed, the interference between the two frequencies is considerably reduced when the parameter τ in $Sp_{\phi,\psi}^{(\tau,\tau)}(f,f)$ becomes small, keeping on the other hand the good level of localization. Incidentally we also remark that the improvement of frequency localization is only apparent as it is essentially the consequence of an effect of vertical contraction and horizontal dilation compensated in the picture by a relabeling of the axis.

4 Properties of the Parameterized Two Window Spectrogram

In this section we analyze some properties of the representation $Sp_{\phi,\psi}^{(\tau_1,\tau_2)}(f,g)$ with $\tau_1, \tau_2 \in [0,1]$. More precisely we consider positivity, L^p−boundedness and support property, we conclude then our investigations by showing that the parameterized two window spectrogram does not belong to the Cohen class.

For what positivity is concerned we limit ourself to the following basic fact, we have

$$Sp_{\phi}^{(\tau)}(f)(x,w) := Sp_{\phi,\phi}^{(\tau,\tau)}(f,f)(x,w) = 4^{-d}|Wig_{\tau}(f,\tilde{\phi})(x,w)|^2 \geq 0.$$

and therefore the following property holds:

Proposition 6. *For $\tau_1 = \tau_2, f = g$ and $\phi = \psi$ the parameterized two window spectrogram is a positive time-frequency representation.*

We consider next the parameterized two window spectrogram in the context of the L^p spaces. For this purpose we shall need the following Proposition, which is proved in [2].

Proposition 7. *Let us fix q and p satisfying $q \geq 2$ and $q' \leq p \leq q, (\frac{1}{q} + \frac{1}{q'} = 1)$. Then:*

i) *For each $\tau \in (0,1)$, $Wig_{\tau} : L^{p'}(\mathbb{R}) \times L^p(\mathbb{R}) \to L^q(\mathbb{R}^{2d})$ is continuous, in particular:*

$$\|Wig_{\tau}(g,f)\|_{L^q} \leq \frac{1}{|1-\tau|^{d(\frac{1}{p}-\frac{1}{q})}} \frac{1}{|\tau|^{d(1-\frac{1}{p}-\frac{1}{q})}} \|g\|_{L^{p'}} \|f\|_{L^p}. \tag{4.1}$$

ii) *For $\tau = 0$, $Wig_0(g,f)(x,w) = R(g,f)(x,w)$ and $Wig_0 : L^q(\mathbb{R}) \times L^{q'}(\mathbb{R}) \to L^q(\mathbb{R}^{2d})$ is continuous, in particular*

$$\|Wig_0(g,f)\|_{L^q} \leq \|g\|_{L^{q'}} \|f\|_{L^q}. \tag{4.2}$$

iii) *For $\tau = 1$, $Wig_1(g,f)(x,w) = \overline{R(g,f)}(x,w)$ and $Wig_1 : L^{q'}(\mathbb{R}) \times L^q(\mathbb{R}) \to L^q(\mathbb{R}^{2d})$ is continuous, in particular*

$$\|Wig_1(g,f)\|_{L^q} \leq \|g\|_{L^q} \|f\|_{L^{q'}}. \tag{4.3}$$

Furthermore for p,q in the remaining cases the τ-Wigner transform is not bounded as sesquilinear map: $L^{p'}(\mathbb{R}) \times L^p(\mathbb{R}) \to L^q(\mathbb{R}^{2d})$.

The L^p behavior of the parameterized two window spectrogram is specified by the following proposition.

Theorem 8. *Let $q \geq 1$, $q_j \geq 2, p_j \geq 1, (j = 1,2)$ satisfy the following conditions: $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}; q_j' \leq p_j \leq q_j, (j = 1,2)$, where $\frac{1}{q_j} + \frac{1}{q_j'} = 1$. Then*

i) *The parameterized two window spectrogram $Sp^{(\tau_1,\tau_2)} : L^{p_1'} \times L^{p_1} \times L^{p_2'} \times L^{p_2} \to L^q$ is continuous $(0 < \tau_1, \tau_2 < 1)$, in particular*

$$\|Sp_{\phi,\psi}^{(\tau_1,\tau_2)}(f,g)\|_{L^q} \leq C\|f\|_{L^{p_1'}} \|\phi\|_{L^{p_1}} \|g\|_{L^{p_2'}} \|\psi\|_{L^{p_2}}, \tag{4.4}$$

where $C = C_1 C_2$ with $C_j = \frac{1}{|1-\tau|^{d(\frac{1}{p_j}-\frac{1}{q_j})}} \frac{1}{|\tau|^{d(1-\frac{1}{p_j}-\frac{1}{q_j})}}, j = 1,2.$

ii) *When* $\tau_1 = 1, \tau_2 = 0$ *then* $Sp^{(1,0)} : L^{q_1} \times L^{q_1'} \times L^{q_2'} \times L^{q_2} \to L^q$ *is continuous, in particular*

$$\|Sp^{(1,0)}_{\phi,\psi}(f,g)\|_{L^q} \le \|f\|_{L^{q_1}} \|\phi\|_{L^{q_1'}} \|g\|_{L^{q_2'}} \|\psi\|_{L^{q_2}}. \tag{4.5}$$

iii) *When* $\tau_1 = 0, \tau_2 = 1$ *then* $Sp^{(0,1)} : L^{q_1'} \times L^{q_1} \times L^{q_2} \times L^{q_2'} \to L^q$ *is continuous, in particular*

$$\|Sp^{(0,1)}_{\phi,\psi}(f,g)\|_{L^q} \le \|f\|_{L^{q_1'}} \|\phi\|_{L^{q_1}} \|g\|_{L^{q_2}} \|\psi\|_{L^{q_2'}}. \tag{4.6}$$

iv) *When* $\tau_1 = \tau_2 = 1$ *then* $Sp^{(1,1)} : L^{q_1} \times L^{q_1'} \times L^{q_2} \times L^{q_2'} \to L^q$ *is continuous, in particular*

$$\|Sp^{(1,1)}_{\phi,\psi}(f,g)\|_{L^q} \le \|f\|_{L^{q_1}} \|\phi\|_{L^{q_1'}} \|g\|_{L^{q_2}} \|\psi\|_{L^{q_2'}}. \tag{4.7}$$

v) *When* $\tau_1 = \tau_2 = 0$ *then* $Sp^{(0,0)} : L^{q_1'} \times L^{q_1} \times L^{q_2'} \times L^{q_2} \to L^q$ *is continuous, in particular*

$$\|Sp^{(0,0)}_{\phi,\psi}(f,g)\|_{L^q} \le \|f\|_{L^{q_1'}} \|\phi\|_{L^{q_1}} \|g\|_{L^{q_2'}} \|\psi\|_{L^{q_2}}. \tag{4.8}$$

Proof. It is an easy consequence of Proposition 7 and the generalized Hölder's inequality

$$\|fg\|_{L^q} \le \|f\|_{L^{q_1}} \|f\|_{L^{q_2}} \text{ for } \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}, \quad q_1 \ge q,$$

□

We recall now some notations. We indicate with $H(\text{supp} f)$ the convex hull of $\text{supp} f$ and with Π_x, Π_w the orthogonal projections on the first and the second factor in $\mathbb{R}^d_x \times \mathbb{R}^d_w$ respectively. Properties on the support of time-frequency representations is a widely studied subject because too large projections Π_x and Π_w of the support of a representation in comparison with the supports of the signal itself and its Fourier transform respectively would indicate a "spreading" of the energy that is seen as disturbance in the applications, see for instance [12]. We have the following basic results.

Lemma 9. *Let* $Wig_\tau(f,g)$ *be the* τ*-Wigner representation defined by* (1.5)*; then*

$$\Pi_x(\text{supp} Wig_\tau(f,g)) \subset H(\text{supp} f + \text{supp} g). \tag{4.9}$$

and

$$\Pi_w(\text{supp} Wig_\tau(f,g)) \subset H(\text{supp} \hat{f} + \text{supp} \hat{g}). \tag{4.10}$$

Proof. Suppose that $Wig_\tau(f,g)(x,\omega) \ne 0$, then there exists $t \in \mathbb{R}^d$ such that $f(y_1) \ne 0$ and $g(y_2) \ne 0$ with $y_1 = x + \tau t$ and $y_2 = x - (1-\tau)t$. On the other hand $x = \lambda y_1 + \mu y_2$ with $\lambda = 1 - \tau$ and $\mu = \tau$, i.e. x can be written as convex linear combination of y_1 and y_2. We have therefore that x belongs to the segment $[y_1, y_2]$ and (4.9) follows then immediately. To obtain (4.10) we just need to recall that

$$Wig_\tau(f,g)(x,w) = Wig_\tau(\hat{f},\hat{g})(w,-x).$$

and repeat the argument above with x replaced by ω. □

From (4.9), (4.10), and the equality $\mathrm{supp}(fg) = \mathrm{supp} f \cap \mathrm{supp} g$, we obtain the "support" property of the parameterized two window spectrogram.

Proposition 10. *The support of the parameterized two window spectrogram satisfies the following properties:*

$$\Pi_x(\mathrm{supp} Sp_{\phi,\psi}^{(\tau_1,\tau_2)}(f,g)) \subset H(\mathrm{supp} f + \mathrm{supp}\tilde{\phi}) \cap H(\mathrm{supp} g + \mathrm{supp}\tilde{\psi}) \tag{4.11}$$

and

$$\Pi_w(\mathrm{supp} Sp_{\phi,\psi}^{(\tau_1,\tau_2)}(f,g)) \subset H(\mathrm{supp}\hat{f} + \mathrm{supp}\hat{\tilde{\phi}}) \cap H(\mathrm{supp}\hat{g} + \mathrm{supp}\hat{\tilde{\psi}}). \tag{4.12}$$

Remark 11. *The meaning of the Proposition 10 becomes even more evident if we consider the case where $f = g$ is a signal and we suppose that one window is well localized in time and the other one in frequency. Assume for example that* $\mathrm{supp}\ \phi \subset B^\delta$ *and* $\mathrm{supp}\ \hat{\psi} \subset B^\delta$*, with B^δ ball of radius $\delta > 0$, then Proposition 10 implies that*

$$\mathrm{supp}\ Sp_{\phi,\psi}^{(\tau_1,\tau_2)}(f,f) \subset H(\mathrm{supp} f + B^\delta) \times H(\mathrm{supp}\ \hat{f} + B^\delta),$$

i.e. we have good localization both in time and in frequency, having reduced the spread of the energy to a ball of radius δ with respect to each variable.

Finally we prove that the parameterized two window spectrogram, in general, does not belong to the Cohen class. Let us consider for simplicity the case $\tau_1 = \tau_2 := \tau$ in Definition 4, with $\tau \neq \frac{1}{2}$ (actually for $\tau = \frac{1}{2}$, the representation $Sp_{\phi,\psi}^{(\frac{1}{2},\frac{1}{2})}(f,g)$ belongs to the Cohen class, since, as proved in [3], it coincides with $Sp_{\phi,\psi}(f,g)$). We denote for shortness $Sp_{\phi,\psi}^{\tau}(f,g) := Sp_{\phi,\psi}^{(\tau,\tau)}(f,g)$; the following proposition holds.

Proposition 12. *For $\tau \neq \frac{1}{2}$ there does not exist a tempered distribution $\sigma = \sigma_{\tau,\phi,\psi} \in \mathscr{S}'(\mathbb{R}^{2d})$ such that*

$$Sp_{\phi,\psi}^{\tau} = \sigma * Wig, \tag{4.13}$$

i.e. $Sp_{\phi,\psi}^{\tau}(f,g) = \sigma * Wig(f,g)$ for every $f,g \in \mathscr{S}(\mathbb{R}^d)$.

Proof. By Definition 4 and simple changes of variables we have:

$$\begin{aligned}
Sp^{\tau}_{\phi,\psi}(f,g) &= 4^{-d}\int e^{-2\pi it\frac{\omega}{2}} f\Big(\frac{x}{2}+\tau t\Big)\overline{\tilde{\phi}\Big(\frac{x}{2}-(1-\tau)t\Big)}\,dt \\
&\qquad \int e^{2\pi it\frac{\omega}{2}}\overline{g\Big(\frac{x}{2}+\tau t\Big)}\tilde{\psi}\Big(\frac{x}{2}-(1-\tau)t\Big)\,dt \\
&= \int e^{-2\pi is\omega} f(2\tau s)\overline{\phi\Big(2(1-\tau)s-\frac{x}{2\tau}\Big)}\,ds \\
&\qquad \int e^{-2\pi is\omega}\overline{g(-2\tau s)}\psi\Big(-2(1-\tau)s-\frac{x}{2\tau}\Big)\,ds.
\end{aligned}$$

By standard properties of the Fourier transform we can write the inverse Fourier transform of $Sp^{\tau}_{\phi,\psi}(f,g)(x,\omega)$ in the following way:

$$\begin{aligned}
&\mathscr{F}^{-1}_{\substack{x\to t\\ \omega\to\xi}}\Big(Sp^{\tau}_{\phi,\psi}(f,g)(x,\omega)\Big) = \\
&= \mathscr{F}^{-1}_{x\to t}\Big[f(2\tau\xi)\overline{\phi\Big(2(1-\tau)\xi-\frac{x}{2\tau}\Big)}\Big] * \mathscr{F}^{-1}_{x\to t}\Big[\overline{g(-2\tau\xi)}\psi\Big(-2(1-\tau)\xi-\frac{x}{2\tau}\Big)\Big] \\
&= (2\tau)^{2d}\Big[e^{2\pi i(4\tau(1-\tau))t\xi} f(2\tau\xi)\overline{\hat{\phi}}(2\tau t)\Big] * \Big[e^{-2\pi i(4\tau(1-\tau))t\xi}\overline{g(-2\tau\xi)}\,\hat{\psi}(2\tau t)\Big],
\end{aligned}$$

where the convolution is performed in both the variables (t,ξ). Finally, writing explicitly the convolution, we obtain

$$\begin{aligned}
\mathscr{F}^{-1}_{\substack{x\to t\\ \omega\to\xi}}\Big(Sp^{\tau}_{\phi,\psi}(f,g)(x,\omega)\Big) &= (2\tau)^{2d} e^{2\pi i(4\tau(1-\tau))t\xi} \\
&\int e^{-2\pi i(4\tau(1-\tau))tx} f(2\tau(\xi-x))\overline{g(-2\tau x)}\,dx \\
&\int e^{-2\pi i(4\tau(1-\tau))\xi s}\overline{\hat{\phi}}(2\tau(t-s))\hat{\psi}(2\tau s)\,ds.
\end{aligned} \tag{4.14}$$

We observe that, by the definition of the Wigner transform,

$$\begin{aligned}
\mathscr{F}^{-1}_{\substack{x\to t\\ \omega\to\xi}}\big(Wig(f,g)\big) &= \mathscr{F}^{-1}_{\substack{x\to t\\ \omega\to\xi}}\Big[\mathscr{F}_{s\to\omega}\Big(f\big(x+\frac{s}{2}\big)\overline{g\big(x-\frac{s}{2}\big)}\Big)\Big] \\
&= \int e^{2\pi ixt} f\big(x+\frac{\xi}{2}\big)\overline{g\big(x-\frac{\xi}{2}\big)}\,dx.
\end{aligned} \tag{4.15}$$

Now let us suppose that (4.13) holds for some tempered distribution σ; by taking the inverse Fourier transform and using (4.14) and (4.15), the following should be verified for every $f,g \in \mathscr{S}(\mathbb{R}^d)$:

$$\begin{aligned}
(2\tau)^{2d} e^{2\pi i(4\tau(1-\tau))t\xi} &\int e^{-2\pi i(4\tau(1-\tau))tx} f(2\tau(\xi-x))\overline{g(-2\tau x)}\,dx \\
&\int e^{-2\pi i(4\tau(1-\tau))\xi s}\overline{\hat{\phi}}(2\tau(t-s))\hat{\psi}(2\tau s)\,ds \\
= \check{\sigma}(t,\xi) &\int e^{2\pi ixt} f\big(x+\frac{\xi}{2}\big)\overline{g\big(x-\frac{\xi}{2}\big)}\,dx,
\end{aligned} \tag{4.16}$$

where $\check{\sigma}(t,\xi)$ is the inverse Fourier transform of σ. In particular, (4.16) should hold for f and g of the following type:

$$f(s) = e^{-\pi\lambda s^2}, \quad g(s) = e^{-\pi\mu s^2},$$

for every $\lambda, \mu > 0$. In this case we can compute explicitly the integrals involving f and g in (4.16) and we have:

$$\begin{aligned}
&\int e^{-2\pi i(4\tau(1-\tau))tx} e^{-\pi\lambda(2\tau\xi-2\tau x)^2} e^{-\pi\mu(-2\tau x)^2}\, dx = \\
&= e^{-4\pi\frac{\lambda\mu}{\lambda+\mu}\tau^2\xi^2} \int e^{-2\pi i(4\tau(1-\tau))tx} e^{-\pi\left(2(\lambda+\mu)^{1/2}\tau x - \frac{2\lambda\tau}{(\lambda+\mu)^{1/2}}\xi\right)^2} dx \\
&= (2\tau\sqrt{\lambda+\mu})^{-d} e^{-4\pi\frac{\lambda\mu}{\lambda+\mu}\tau^2\xi^2} e^{-2\pi i\frac{\lambda}{\lambda+\mu}4\tau(1-\tau)t\xi} \int e^{-2\pi i\frac{2(1-\tau)}{(\lambda+\mu)^{1/2}}ty} e^{-\pi y^2}\, dy \\
&= (2\tau\sqrt{\lambda+\mu})^{-d} e^{-2\pi i\frac{\lambda}{\lambda+\mu}4\tau(1-\tau)t\xi} e^{-4\pi\frac{\lambda\mu}{\lambda+\mu}\tau^2\xi^2} e^{-\pi\frac{4(1-\tau)^2}{\lambda+\mu}t^2}.
\end{aligned} \tag{4.17}$$

Similarly we obtain that

$$\int e^{2\pi i x t} f\left(x+\frac{\xi}{2}\right)\overline{g\left(x-\frac{\xi}{2}\right)}\, dx = (\sqrt{\lambda+\mu})^{-d} e^{-2\pi i\frac{\lambda}{\lambda+\mu}t\xi} e^{-\pi\frac{\lambda\mu}{\lambda+\mu}\xi^2} e^{-\pi\frac{1}{\lambda+\mu}t^2}. \tag{4.18}$$

Now, replacing (4.17) and (4.18) in (4.16) we have for $\check{\sigma}(t,\xi)$ the following expression

$$\begin{aligned}
\check{\sigma}(t,\xi) = (2\tau)^d & e^{2\pi i(4\tau(1-\tau))t\xi - \pi i t\xi - 2\pi i\frac{\lambda}{\lambda+\mu}(4\tau(1-\tau)-1)t\xi} \\
& e^{-\pi\frac{4\lambda\mu\tau^2-\lambda\mu}{\lambda+\mu}\xi^2} e^{-\pi\frac{4(1-\tau)^2-1}{\lambda+\mu}t^2} \int e^{-2\pi i(4\tau(1-\tau))\xi s}\widehat{\overline{\phi}}\big(2\tau(t-s)\big)\,\hat{\psi}(2\tau s)\, ds.
\end{aligned} \tag{4.19}$$

For $\tau \neq \frac{1}{2}$ we deduce then that $\check{\sigma}(t,\xi)$ necessarily depends on the two parameters λ and μ, and this is impossible since σ in (4.13) is independent of f and g. □

Remark 13. *We also observe that in the case $\tau = 1/2$ all terms in* (4.19) *involving the parameters λ and μ cancel, making σ independent of them, and confirming, as expected, that in this case the representation is in the Cohen class.*

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