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Nonlinear Instability of Dispersive Waves[1]

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ABSTRACT

This paper is mainly concerned with *Nonlinear instability of dispersive waves.* It deals with some advanced studies of instabilities of linear and nonlinear dispersive waves in water of finite and infinite depths. The paper has been organized in the following manner: Section 1 contains the preliminary introduction of the hydrodynamic instability. These preliminaries of hydrodynamic stability are discussed in detail in section 2. The advanced instability studies are presented in the following four sections which are related to linear and nonlinear ocean waves. Section 3 contains the nonlinear instability of water waves due to Benjamin and Feir [1]. The Rayleigh-Taylor linear instability of dispersive waves is illustrated in section 4, whereas their nonlinear instability is discussed in section 5. This article concludes with the brief study of Kelvin-Helmholtz linear instability with two inviscid streams which is presented in section 6.

RESUMEN

Este paper tiene que ver principalmente con *Inestabilidad nolineal de ondas dispersivas.* Considera algunos estudios avanzados de inestabilidades de ondas dispersas lineales y no lineales en aguas de profundidades finitas e infinitas. Este

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paper ha sido organizado de la siguiente manera: Sección 1 contiene la introducción preliminar de inestabilidad hidrodinámica. Estos preliminares de estabilidad hidrodinámica son discutidos en detalle en sección 2. Los estudios de inestabilidad avanzada son presentados en las siguientes cuatro secciones las cuales están relacionadas ondas de océanos lineal y no lineal. La sección 3 contiene la inestabilidad no lineal de ondas de agua debidas a Benjamin y Feir[1]. La inestabilidad lineal de Rayleigh-Taylor de ondas dispersivas es ilustrada en sección 4, mientras su inestabilidad no lineal es discutida en sección 5. Este artículo concluye con el estudio breve de inestabilidad lineal de Kelvin-Helmholtz con dos inviscid streams el cual es presentado en sección 6.

Key words and phrases: *Hydrodynamic Stability, Nonlinear Theory, Dispersive Waves, Viscosity, Inviscid Flows, Rayleigh-Taylor Instability, Kelvin-Helmholtz Instability, Stokes Waves and Benjamin-Feir Instability.*

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1 Introduction

We observe in the large bodies of water such as oceans that the flow around submerged or floating bodies does not normally develop in a smooth, orderly fashion to a perfectly steady state as might be expected but that more or less violent, irregular fluctuations appear specially in the wake. These behaviors of the flow patterns are usually ascribed to "instability" of the flow, that is the tendency for small disturbances which might be caused by winds, noise, mechanical vibrations, surface roughness, non-uniformity of oncoming stream, etc. to be amplified into substantial fluctuations. As a result,the final, "turbulent" motion is at most "statistically steady", in the sense that the fluid velocity at each point varies about a constant mean value.

The instability defined above can be viewed as an instability of the vortex distribution, as this fixes the flow field. Thus it would be supposed, therefore, that a very slight displacement of some vorticity, may induce slight changes in the velocities of convection of existing vortex lines, such that resulting changes in the vorticity pattern a short time later induce alterations in the velocities of convection of that vorticity pattern, such that the whole altered process of convection, production and diffusion of the vorticity tending more and more to depart from what it would have been without the original disturbance. In fact, although turbulence is observed most frequently in wakes, it was found soon after the discovery of the boundary layer (Prandtl 1914) that parts of it are turbulent in a wide range of flows.

Since diffusion by itself is a stable process, the vorticity distribution is necessarily stable when it dominates sufficiently over convection. By comparing a diffusion rate of order $\frac{\nu\omega_0}{\delta}$, where ν is the kinematic viscosity, ω_0 is a typical vorticity and δ is the thickness of a layer across which ω varies from 0 to ω_0, with a convection rate of order

$\nu\omega_0$ (both per unit area), we see that stability may be expected if

$$R_\delta = \frac{u\delta}{\nu} \tag{1}$$

is small enough. This is one of the reasons why the wake is often observed to be unstable when the boundary layer is not.

The stability theory is a mathematical construction of great complexity and beauty. Tollmien [26], [27] was the first to recognize the complete pictures of instability of flows and after him many researchers contribute to this field. To understand the instability of flows in a layman's term, we may consider small sine-wave disturbances and investigate their distribution across the layer, their phase velocity c_r, and their rate of amplification (positive, negative or zero) with time. It is worth noting here that according to wave theory, even a localized disturbance can be considered as a combination of such sine waves, although the part of the disturbance with wavelength around the value λ will travel along, not at the speed c_r of its individual crests, but at its "group velocity" $c_g = c_r - \lambda \frac{dc_r}{d\lambda}$. The most important results are as follows:

I. If, as the first approximation, diffusion and production of new vorticity are neglected (as in the inviscid flow theory), then the waves can have positive rate of amplification only if the undisturbed vorticity distribution has a maximum in the midst of the layer (see Tollmien [27] and Görtler [5]).

II. This simplified theory stated in (I) is not accurate enough; in particular, for the boundary layers in accelerating flow, the actual predicted wave system with zero amplification rate, has some seriously unrealistic features, due to the total neglect of the production and diffusion of vorticity.

III. At lower values of R_δ, the main effect of diffusion is stabilizing or smoothing as expected. However, a stronger diffusive effect is necessary to remove the instability of layers with vorticity maxima, which is longer and almost independent of R_δ than to remove the weaker instability of layers with monotonic vorticity.

IV. Although the above theory is for two-dimensional vortex layers, it can be applied to three-dimensional boundary if we consider separately the stability of "cross-stream" and "streamwise" vorticity distributions. Now, since the total stream wise vorticity (integrated across the layer) is zero, it must have a maximum somewhere. It may happen, therefore, if the external flow is accelerating that R_δ has a value for which the cross stream vorticity is stable but the streamwise vorticity unstable, leading to concentration of the latter vorticity in "streaks" which may be visible on oil flow photographs (see Gregory, Stuart and Walker [6].

2 Hydrodynamic stability

The Navier-Stoke's equations play a very important part in the field of hydrodynamics to study the flow stability. In real situation, a turbulent form of motion is often more likely to occur than the appropriate laminar form, the question of relative stability of two types of flow naturally arises. It has been observed experimentally that laminar flow occurs at low Reynolds numbers and in this range viscosity damped out any deviation from laminar flow. Clearly, at such Reynolds numbers, this type of flow is more stable than the turbulent form. On the other hand, at high Reynolds numbers, turbulent motion occurs. In this situation, laminar flow can be realized only by excluding all possible disturbances, however small. At such Reynolds numbers, therefore, turbulent flow appears to be the more stable form.

Because of the mathematical simplifications associated with linearization and because linearized theory is able to give the critical conditions for the occurrence of instability for infinitesimal disturbances, the stability theory has been largely devoted with this restriction. However, the role of non-linearity in flow instability has received considerable attention among scientists and engineers. We shall review these notions one after another.

The mathematical problem of hydrodynamics stability can be formulated by taking the given steady-state solution of motion and superimposing a disturbance of a suitable kind. This results in a set of nonlinear disturbance equations, which govern the behavior of the disturbance. If the disturbance ultimately decays to zero, the flow is said to be stable, but if a disturbance results which is permanently different from zero, the flow is said to be unstable. It does not follow that the instability leads to turbulent motion.

When the "disturbance differential equations" are linearized for small disturbances they become homogenous and it is possible to consider disturbances which contain an exponential time factor $e^{\sigma t}$, t being the time and σ the frequency. The boundary conditions on the disturbance equations require the vanishing at the boundaries of quantities like the disturbance velocity components. Consequently the boundary conditions are also homogenous, and we have an eigenvalue problem for the determination of the quantity σ. If it is possible for σ to have a positive real part, the flow said to be unstable according to the linearized theory; otherwise the flow is stable. The possible eigenvalues of σ depend, of course, on quantities such as flow speed, kinematic viscosity, thermal diffusivity, and disturbance wavelength. There are two main types of instability. They are

I. The instability of curved flows due to centrifugal forces, as for example, the flow between two concentric rotating cylinders.

II. The instability of two dimensional parallel flows, as illustrated by Poiseuille flow between parallel planes, in which viscosity may play itself a destabilizing role.

In each of these case, instability occurs when a certain parameters, representing the ratio of destabilizing to stabilizing forces, reaches a critical value. As for example, in case (II), the parameter is the Reynolds number. In discussing the mechanics of instability, the concept of the Reynolds stress is almost indispensable. To know about this concept the reader is referred to the book by Lin [13], and in particular, to papers by Lees and Lin [10], Gazley [4], and Dunn and Lin [3].

3 Instability of water waves

The study of nonlinear instability phenomena goes back to the days of Landau [9] when he first described these phenomena of certain class of flow by nonlinear ordinary differential equation for the amplitude of a wave mode. The theory of Landau [9] has been found later to have suffered serious weaknesses, but his work has served a fundamental basis for all subsequent developments in the modern theories of nonlinear instabilities of fluid motions. In 1847, Stokes [23] first proposed that the free surface elevation of the plane wave train in deep water can be expected in power of wave amplitude. The convergence of the Stokes expansion was proved by Levi-Civita in 1925 considering the wave amplitude was very small compared to the wavelength. The study of stability remained unattended until the 1960's expect for an isolated investigation by Korteweg and de Vries in 1895 on long surface waves of finite depth. One of the most striking discoveries made in the 1960's is that the periodic Stokes waves in sufficiently deep water appears to be unstable. This result has been verified by many pioneering researchers including Lighthill [11, 12], Whitham [28, 29], Benjamin and Feir [1] and Zakharov [30].

In the following we shall describe in some detail the instability of water waves as outlined by Benjamin and Feir [1]. The velocity potential, $\phi(\mathbf{x}, t)$, and vertical displacement of water surface above mean water level, $\eta(\mathbf{x}, t)$, of the second-order wave may written as Rahman [16, 17]

$$\begin{aligned} \phi = \phi_0 &= -\frac{Ag}{\sigma}\frac{\cosh k(z+h)}{\cosh kh}\cos(kx - \sigma t) \\ &- \frac{3A^2\sigma}{8}\frac{\cosh 2k(z+h)}{\sinh^4 kh}\cos 2(kx - \sigma t) \end{aligned} \tag{2}$$

$$\begin{aligned} \eta = \eta_0 &= A\sin(kx - \sigma t) - \frac{A^2 k}{4}\frac{\cosh kh}{\sinh^3 kh}(2 + \cosh 2kh)\cos 2(kx - \sigma t) \\ &- \frac{A^2 k}{2\sinh 2kh} \end{aligned} \tag{3}$$

where A is the first order wave amplitude, h is the water depth, k is the wave number $\frac{2\pi}{L}$, σ angular frequency and z is positive upward and is measured from the mean water level. The celerity of the wave up to the third-order can be determined as (see Stoker [22])

$$C^2 = \frac{g}{k}\tanh kh\left[1 + (kA)^2\left\{\frac{5 + 2\cosh 2kh + 2\cosh^2 2kh}{8\sinh^4 kh}\right\}\right] \tag{4}$$

This approximation is valid provided $kA << 1$ or alternatively $kA << (kh)^3$ for small

kh. For deep water waves, these results reduce to the following:

$$\begin{aligned} \phi &= \phi_0 = -\frac{Ag}{\sigma} e^{kz} \cos(kx - \sigma t) \\ \eta &= \eta_0 = A \sin(kx - \sigma t) - \frac{A^2 k}{2} \cos 2(kx - \sigma t) \quad (5) \\ \text{and} \quad C^2 &= \frac{g}{k}(1 + A^2 k^2) \\ \text{or} \quad \sigma^2 &= gk(1 + A^2 k^2). \quad (6) \end{aligned}$$

These describe the steady wave motion in deep water when $kA << 1$.

To investigate the stability of the steady wave motion described by (2) and (3), we write

$$\begin{aligned} \phi &= \phi_0 + \phi' \\ \eta &= \eta_0 + \eta' \quad (7) \end{aligned}$$

where ϕ' and η' are the perturbed solutions from their steady motions. For convenience we rewrite the governing equation with the boundary conditions

$$\phi_{xx} + \phi_{zz} = 0 \qquad -h \leq z \leq 0 \qquad -\infty < x < \infty \quad (8)$$

$$\phi_z = 0 \qquad \text{on} \qquad z = -h \quad (9)$$

$$\begin{aligned} \eta_t + \eta_x \phi_x - \phi_z &= 0 \\ g\eta + \phi_t + \frac{1}{2}[\phi_x^2 + \phi_z^2] &= 0 \quad \text{on} \quad z = \eta(x,t) \quad (10) \end{aligned}$$

The solution (7) must satisfy these equations and we have already seen the solutions for ϕ_0 and η_0 given by (3). The solutions ϕ' and η' must satisfy the following equations:

$$\phi'_{xx} + \phi'_{zz} = 0 \qquad -h \leq 0 \qquad -\infty < x < \infty \quad (11)$$

$$\phi'_z = 0 \qquad \text{on} \qquad z = 0 \quad (12)$$

$$\begin{aligned} \eta'_t + \eta_{0x}\phi'_x + \eta'\phi_{0x} - \phi'_z &= 0 \\ g\eta'_x + \phi'_t + [\phi_{0x}\phi'_x + \phi_{0z}\phi'_z] &= 0 \quad \text{on} \quad z = 0 \quad (13) \end{aligned}$$

The solutions ϕ', η' are assumed to consist of sideband modes, together with the products of their interaction with the basic wave train, whose fundamental simple harmonic component has amplitude A and phase $\theta = kx - \sigma t$. Since the system is nonlinear, harmonics with phases 2θ, 3θ $\cdots$ are travelling with the same phase velocity $C = \frac{\sigma}{k}$ as the fundamental, and their amplitudes are assumed to decrease in relative order of magnitude like successive integral powers of $Ak << 1$. A disturbance is introduced consisting of a pair of progressive wave modes with sideband frequencies and wave numbers close to σ and k so that their phases are expressed in the form

$$\theta_1 = (k + \Delta k)x - (\sigma + \Delta\sigma)t - \gamma_1(t) \quad (14)$$

$$\theta_2 = (k - \Delta k)x - (\sigma - \Delta\sigma)t - \gamma_2(t) \quad (15)$$

where (14) is the upper sideband and (15) is the lower sideband, and Δk and $\Delta\sigma$ are small increments, the respective amplitudes are $\delta_1(t)$ and $\delta_2(t)$ which are much smaller than A. Two particular products arise from the nonlinear interaction between the disturbances and the basic wave train as evidenced from (12), and these are the differences components produced between the sidebands and the second harmonic. This is clear by adding (14) and (15) as follows:

$$\theta_1 + \theta_2 = 2(kx - \sigma t) - (\gamma_1 + \gamma_2) = 2\theta - (\gamma_1 + \gamma_2) \quad (16)$$

Thus the components generated have phases

$$\begin{aligned} 2\theta - \theta_1 &= \theta_2 + (\gamma_1 + \gamma_2) \\ \text{and} \quad 2\theta - \theta_2 &= \theta_1 + (\gamma_1 + \gamma_2) \end{aligned} \quad (17)$$

and amplitudes $\delta_1(t)A^2k^2$ and $\delta_2(t)A^2k^2$, respectively. Therefore, if it happens that

$$\gamma(t) = \gamma_1 + \gamma_2 = constant \quad (18)$$

as the nonlinear process develops in time, each mode will produce effects that become resonant with the other. Subsequently, if $\gamma \neq 0$ or π, each mode will suffer a synchronous forcing effect proportional to the amplitude of the other so that the two amplitudes can grow exponentially. Thus, the basic wave train becomes **unstable** to this form of disturbance.

Bejamin and Feir [1], has obtained expressions for η_1', η_2' and ϕ_1', ϕ_2' and hence $\eta' = \eta_1' + \eta_2'$ and $\phi' = \phi_1' + \phi_2'$. Substituting these results into (12) leads to four equations with known parameters for the functions $\delta_1(t)$, δ_2 and $\gamma_1(t)$, $\gamma_2(t)$. These functions satisfy the following simultaneous ordinary differential equations.

$$\begin{aligned} \frac{d\delta_1}{dt} &= \left\{\frac{1}{2}\sigma(Ak)^2 f(kh)\sin\gamma\right\}\delta_2 \\ \frac{d\delta_2}{dt} &= \left\{\frac{1}{2}\sigma(Ak)^2 f(kh)\sin\gamma\right\}\delta_1 \end{aligned} \quad (19)$$

$$\text{and} \quad \frac{d\delta}{dt} = \sigma(Ak)^2 f(kh)\left\{1 + \frac{\delta_1^2 + \delta_2^2}{2\delta_1\delta_2}\cos\gamma\right\} - \sigma\left(\frac{\Delta\sigma}{\sigma}\right)g(kh) \quad (20)$$

where $\gamma = \gamma_1 + \gamma_2$ and

$$f(kh) = \frac{9 - 10\tanh^2 kh + 9\tanh^4 kh}{8\tanh^4 kh} + \frac{4 + 2sech^2kh + 3(kh)\coth khsech^4kh}{1 - 2kh\tanh khsech^2kh\cosh 2kh + (kh)^2 sech^4kh} \quad (21)$$

$$g(kh) = 1 - \frac{4(kh)(1 - kh\tanh kh)cosech2kh}{1 + 2(kh)cosech2kh} \quad (22)$$

The solutions of the pair of equations (18) are

$$\begin{aligned} \delta_1(t) &= \delta_1(0)\cosh\left\{\frac{\sigma}{2}(Ak)^2 f\int_0^t \sin\gamma dt\right\} \\ &+ \delta_2(0)\sinh\left\{\frac{\sigma}{2}(Ak)^2 f\int_0^t sin\gamma dt\right\} \\ \delta_2(t) &= \delta_2(0)\cosh\left\{\frac{\sigma}{2}(Ak)^2 f\int_0^t \sin\gamma dt\right\} \\ &+ \delta_2(0)\sinh\left\{\frac{\sigma}{2}(Ak)^2 f\int_0^t sin\gamma dt\right\} \end{aligned} \quad (23)$$

it can be easily seen from (23) that even though γ is yet unknown function of time, the amplitudes of the sideband modes undergo unbounded amplification if γ tend to constants other than 0 and π. It has found by Benjamin and Feir [1] that the solutions $\delta_1(t)$, $\delta_2(t)$ are periodic and finitely bounded if

$$2(Ak)^2 f(kh) < \left(\frac{\Delta\sigma}{\sigma}\right) g(kh) \quad (24)$$

which is the required condition of stability of the basic wave trains.

On the other hand, if

$$2(Ak)^2 f(kh) \geq \left(\frac{\Delta\sigma}{\sigma}\right) g(kh) \quad (25)$$

then the basic wave trains are unstable. It can be easily verified that $g(kh) \to 0$ as $kh \to 0$, and $g(kh) \to 1$ when $(kh) \to \infty$. So the values of $g(kh)$ are always positive. Thus the basic wave train will be **stable** or **unstable** entirely depends upon the sign of $f(kh)$. If $f(kh) > 0$, there exists a range of values of $(\frac{\Delta\sigma}{\sigma})$ so that the condition of instability (25) is satisfied. However. if $f(kh) < 0$, the stability condition (24) remains valid for all values of other parameters. Direct evaluation of $f(kh)$ from (21) shows that $f(kh)$ is positive or negative according to whether $kh > 1.363$ or < 1.363.

Thus in the case of infinitely deep water, the condition of instability becomes

$$\sqrt{2}(Ak) \geq \left(\frac{\Delta\sigma}{\sigma}\right) > 0 \quad (26)$$

The Stokes wave trains in infinitely deep water stable or unstable according to the wavelength $L > (\frac{2\pi h}{1.363}$, or $L < (\frac{2\pi h}{1.363})$. It is important to note that if $kh < 1.363$, there is a critical value $(\frac{\Delta\sigma}{\sigma})_c$, of $(\frac{\Delta\sigma}{\sigma})$ which is given by

$$\left(\frac{\Delta\sigma}{\sigma}\right)_c = (Ak)\left(\frac{2f}{g}\right)^{\frac{1}{2}} \tag{27}$$

in which there is no unbounded growth of the sideband amplitudes.

4 The Rayleigh-Taylor instability of dispersive waves

This section is concern with the classical linear instability problem of two semi-infinite homogenous inviscid fluids of densities, ρ_1 in the region $z < 0$ and ρ_2 in the region $z > 0$ ($\rho_1 > \rho_2$) and these two fluids are separated by a horizontal interface boundary $z = 0$.

This problem was first considered by Lord Rayleigh and G. I. Taylor in 1917 to investigate the stability of flow. The density of the lower fluid is ρ_1 and that of the upper fluid is ρ_2 when $\rho_1 > \rho_2$ with an interfacial surface tension T at the interface $z = 0$. The flow is considered to be irrotational which implies that the scalar velocity potentials exist such that the velocity potential of the lower fluid is ϕ_1 and that of the upper fluid is ϕ_2, and they satisfy the Laplace's equations

$$\begin{aligned} \nabla^2\phi_1 &= 0 \quad \text{in} \quad -\infty < z < 0 \\ \nabla^2\phi_2 &= 0 \quad \text{in} \quad 0 < z < \infty \end{aligned} \tag{28}$$

let us consider that the interfacial surface elevation is $\eta = \eta(x, y, t)$. The boundary conditions satisfied by the quantities ϕ_1, ϕ_2 and η are the following:

The boundary conditions at $z = \pm\infty$ are

$$\begin{aligned} \frac{\partial\phi_1}{\partial z} &= 0 \quad \text{at} \quad z = -\infty \\ \frac{\partial\phi_2}{\partial z} &= 0 \quad \text{at} \quad z = \infty \end{aligned} \tag{29}$$

The kinematic boundary conditions are:

$$\begin{aligned} \frac{\partial\phi_1}{\partial z} &= \frac{\partial\eta}{\partial t} \quad \text{on} \quad z = 0 \\ \frac{\partial\phi_2}{\partial z} &= \frac{\partial\eta}{\partial t} \quad \text{on} \quad z = 0 \end{aligned} \tag{30}$$

The dynamic boundary conditions are

$$\rho_1(\frac{\partial\phi_1}{\partial t} + g\eta) = \rho_2(\frac{\partial\phi_2}{\partial t} + g\eta) + T\nabla^2\eta \quad \text{on} \quad z = 0 \tag{31}$$

The solution of this moving boundary problem can be obtained by considering a typical Forrier component of interfacial elevation η which has the form

$$z = \eta(x, y, t) = A \exp\{i(\mathbf{k} \bullet \mathbf{x} - \sigma t)\} \tag{32}$$

where A is an arbitrary constant, $\mathbf{x} = (x, y)$, $\mathbf{k} = (k_1, k_2)$ is the horizontal wave number vector, k_1 and k_2 are real, and $\sigma(k_1, k_2)$ is a possible complex frequency. The associated velocity potentials satisfying the boundary conditions (29) are

$$\phi_1 = A_1 \exp\{i(\mathbf{k} \bullet \mathbf{x} - \sigma t) + kz\} \tag{33}$$

$$\phi_2 = A_2 \exp\{i(\mathbf{k} \bullet \mathbf{x} - \sigma t) + kz\} \tag{34}$$

where $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$ and constant A, A_1 and A_2 can readily be determined. The kinematic conditions (30) yield

$$kA_1 = -kA_2 = -i\sigma A \tag{35}$$

The dynamic boundary conditions (31) gives the linearized eigenvalue for σ as

$$\sigma^2 = gk\left[\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} + \frac{Tk^2}{g(\rho_1 + \rho_2)}\right] \tag{36}$$

This quadratic equation for σ has either real roots or complex roots. For real roots, there are two wave models that propagate with constant amplitude. For complex conjugate roots, $\sigma = \sigma_r \pm i\sigma_i$ and in this situation if $\sigma_i > 0$, the wave model with the positive imaginary part decays exponentially with time t, while with the negative imaginary part grows exponentially. If there exists such an exponentially growing model for some wave numbers (k_1, k_2), the primary flow is unstable. On the other hand, if there is no such a model for any (k_1, k_2) the flow is regarded as stable to linearized disturbances.

Let us define the horizontal coordinates $x = (\frac{1}{k})(\mathbf{k} \bullet \mathbf{x}) = \frac{k_1 x + k_2 y}{\sqrt{k_1^2 + k_2^2}}$ in the direction of the wave number vector $\mathbf{k}$ (that is normal to the wave crests). Clearly, waves propagate in the direction of increasing x with phase velocity $\frac{\sigma_1}{k}$. Also the value of $(\frac{\sigma}{k})$ depends on k_1 and k_2, and hence the waves are dispersive.

Result (36) is called the complex dispersive relation for interfacial waves and gives

$$\sigma = \pm\sqrt{gk}\left[\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} + \frac{Tk^2}{g(\rho_1 + \rho_2)}\right]^{1/2} \tag{37}$$

These roots are real provided $g(\rho_1 - \rho_2) + Tk^2$ is positive. Clearly, when $\rho_1 > \rho_2$, the associated models describes interfacial capillary-gravity waves. When the heavier fluid $\rho_1 (< \rho_2)$ is on the top of the lighter fluid, the system is unstable for all wave number with $k^2 < (\frac{g}{T})(\rho_2 - \rho_1)$, that is sufficiently long waves are unstable in the range $0 < k < k_c$, when

$$k_c = \left[\frac{g}{T}(\rho_2 - \rho_1)\right]^{1/2} \tag{38}$$

However, in the letter case, the system is stable for all disturbances with $k > k_e$. Thus the effect of surface tension is to stabilize a potentially unstable system for all sufficiently large wave numbers (or small wave lengths), where the system remains unstable for all sufficiently small wave numbers (or long wavelengths). This is universally known as the Rayleigh-Taylor instability.

Special cases:

I. In the absence of the surface tension ($T \equiv 0$), the eigenvalue equation (36) becomes

$$\sigma^2 = gk\left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right) \tag{39}$$

If $\rho_1 < \rho_2$, the value of σ is purely imaginary which implies that the system is unstable. This is a very obvious instability, when the heavier fluid is on the top of the lighter fluid.

On the other hand, when $\rho_1 > \rho_2$, σ is real, and hence gravity waves occur at the air-water interface. The phase velocity and group velocities of the waves are given by

$$\begin{aligned} c_p &= \frac{\sigma}{k} = \pm\sqrt{\left(\frac{g}{k}\right)\left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right)} \\ \mathbf{c}_g &= \nabla_{\mathbf{k}}\sigma \end{aligned} \tag{40}$$

II. In the absence of the upper fluid ($\rho_2 = 0$), there exists stable gravity waves on deep water. The waves are characterized by the famous dispersion relation $\sigma^2 = gk = g\sqrt{k_1^2 + k_2^2}$. It is observed from (39) that σ^2 depend critically on the density variation. For large ρ_1/ρ_2, $\sigma^2 = gk$, which is independent of density, and hence it describe the classical water waves.

III. If the two fluid are confined between two rigid horizontal planes at $z = -h$, and $z = h_2$ separated by a well-defined interface at $z = 0$, we assume the solutions for ϕ_1 and ϕ_2 instead of (33) and (34) in the following forms

$$\phi_1 = A_1 \cosh k(z + h_1) \exp\{i(kx - \sigma t)\} \tag{41}$$

$$\phi_2 = A_2 \cosh k(z + h_2) \exp\{i(kx - \sigma t)\} \tag{42}$$

so that the conditions at the lower and upper boundaries are satisfied. We assume that the interfacial wave equation function is

$$z = \eta(x, t) = A \exp\{i(kx - \sigma t)\} \tag{43}$$

The kinematic boundary conditions (30) require that

$$A_1 k \sinh kh_1 = -A_2 k \sinh kh_2 = -i\sigma A \tag{44}$$

Then the dynamic boundary condition (31) yield the eigenvalue

$$\sigma^2 = \frac{gk(\rho_1 - \rho_2) + Tk^3}{\rho_1 \coth kh_1 + \rho_2 \coth kh_2} \tag{45}$$

It can be easily noticed that in the limit as $kh_1 \to \infty$ and $kh_2 \to \infty$, this result reduced to the form (36) with $\mathbf{k} = (k, 0)$. This is the case for the deep water ocean.

On the other hand, for shallow water, when $kh_1 \to 0$ and $kh_2 \to 0$ (long wave limit), (45) becomes

$$\begin{aligned} \sigma^2 &= \frac{gk(\rho_1 - \rho_2) + Tk^3}{\frac{\rho_1}{kh_1} + \frac{\rho_2}{kh_2}} \\ &= \frac{gk^2[(\rho_1 - \rho_2) + Tk^2]h_1 h_2}{\rho_1 h_2 + \rho_2 h_1} \end{aligned} \tag{46}$$

Similarly, if kh_1 is very small and kh_2 is very large, then the eigenvalue equation is

$$\sigma^2 = k^2 \left[(1 - \frac{\rho_1}{\rho_2}) + \frac{Tk^2}{g\rho_1} \right] gh_1 \tag{47}$$

In the preceding analysis we has illustrated most of the fundamental ideas relied to free interfacial waves in inviscid fluids.

5 The Rayleigh-Taylor non-linear instability of dispersive waves

The Rayleigh-Taylor linear instability of dispersive waves problem has been discussed by Lamb [8] in which the problem was described in two inviscid fluids separated by a well-defined horizontal interface at $z = 0$. He predicted the unstable behavior of the interface of this fluid model in a gravitational field directed from the heavier to the lighter fluid. In fluid flow problem, viscosity and surface tension play a very important role in stabilizing the flow field. Lamb has proved that surface tension stabilizes or destabilizes the flow according to $k > \sqrt{(g/T)(\rho_1 - \rho_2)}$ or $k < \sqrt{(g/T)(\rho_1 - \rho_2)}$ where k is the wave number, g is the gravitational acceleration, T is the surface tension and ρ_1 and ρ_2 are the densities of the fluids. Here $k_c = \sqrt{(g/T)(\rho_1 - \rho_2)}$ is called the Rayleigh-Taylor critical value for the instability.

Following Nayfeh [15], Debnath [2] has given a lucid description of the nonlinear instability of the interface of a semi-infinite air and inviscid liquid of finite depth h. The nonlinear moving boundary conditions are used and solutions are obtained using a Fourier perturbation technique. The non-dimensional forms of the boundary value problem are given by:

- Laplace's equation:

$$\phi_{xx} + \phi_{zz} = 0 \qquad -\infty < x < \infty, \quad -h \le z \le \eta, \quad t > 0 \tag{48}$$

- Bottom boundary condition:

$$\phi_z = 0 \quad \text{at} \quad z = -h \tag{49}$$

- Kinematic boundary condition at free surface

$$\eta_t - \eta_x\phi_x + \eta_z = 0 \quad \text{on} \quad z = \eta \tag{50}$$

- Dynamic boundary condition at the free surface

$$(\eta + \phi_t) - \frac{1}{2}(\nabla\phi)^2 + k'^2\eta_{xx}(1+\eta_x^2)^{-\frac{3}{2}} = 0 \quad \text{on} \quad z = \eta \tag{51}$$

where $k' = \frac{k}{k_c}$, $k_c = \left(\frac{\rho g}{T}\right)^{\frac{1}{2}}$ is the linearized cut off wave number and represents the ratio of the generalized force to the surface tension force. The initial conditions are

$$\begin{aligned} \eta(x, t0) &= \epsilon\cos x \\ \eta_t(x,t) &= 0 \qquad \text{at} \quad t = 0 \end{aligned} \tag{52}$$

where $\epsilon = ak$ is the wave steepness parameter.

In order to obtain the appropriate solutions for small ϵ, we use the Fourier perturbation analysis and write the solutions in consistent with 49), (50) in the forms

$$\begin{aligned} \phi(x,z,t) &= \epsilon[\phi_1(t)e^{ix} + c.c.]\cosh(z+h) \\ &+ \epsilon^2[\phi_1(t)e^{2ix} + c.c.]\cosh 2(z+h) \\ &+ \cdots + \epsilon^2\tilde{\phi}_2(t) + \cdots \end{aligned} \tag{53}$$

$$\eta(x,t) = \epsilon[\eta_1(t)e^{ix} + c.c.] + \epsilon^2[\phi_1(t)e^{2ix} + c.c.] + \epsilon^2\tilde{\eta}_2(t) + \cdots \tag{54}$$

In this section we avoided the lengthy calculations to get the solutions for the potential ϕ and wave elevation η.

Note:

There are several important feature of the nonlinear instability of Rayleigh-Taylor problem. In this analysis, it has been found that the cut off wave number given by

$$k = k_c[1 + \frac{3}{8}(ak_c)^2 + \frac{52}{512}(ak_c)^4]^{\frac{1}{2}} + O(\epsilon^6) \tag{55}$$

Depend on the wave amplitude.

Both travelling and standing wave solutions with wave number greater than the cut off value oscillate with time-independent amplitudes and therefore are stable. However, the phase velocity of the travelling waves and the frequency of the standing waves depend on the amplitude. Below this cutoff value the disturbances grow in amplitude. The cutoff wave number is independent of the depth h of the lower liquid. However, the grown rate of unstable disturbances decrease as h decrease, leading to stability.

6 The Kelvin-Helmholtz linear instability with two inviscid streams

In this section we discuss the classical Kelvin-Helmholtz linear instability problem of two semi-infinite homogenous inviscid fluids of densities ρ_1 and ρ_2 where $\rho_1 > \rho_2$ such that the ρ_1 fluid below and ρ_2 fluid above the interfacial boundary $z = 0$. We assume that the lower and upper fluids have velocities u_1 and u_2 respectively in the x-direction.

The Forrier component of fluid elevation of the disturbance interface is described by

$$z = \eta(x, y, t) = A \exp\left[i(\mathbf{k} \bullet \mathbf{x} - \sigma t)\right] \tag{56}$$

where A is an arbitrary constant. The perturbations of the flows below and above the interface are assumed to be irrotational so that the velocity potentials of the lower and upper fluids are ϕ_1 and ϕ_2, respectively, which can be written as

$$\begin{aligned} \phi_1(x, y, z, t) &= u_1 x + \phi_1' \quad \text{in} \quad -\infty < z < \eta \\ \phi_2(x, y, z, t) &= u_2 x + \phi_2' \quad \text{in} \quad \eta < z < \infty \end{aligned} \tag{57}$$

where all the ϕ's satisfy the Laplace's equation, and products of ϕ_1', ϕ_2' and η are neglected.

Including the effect of surface tension T and invoking the linearization, the perturbed quantities satisfy the following equations, boundary and interfacial conditions:

- Laplace's equations

$$\begin{aligned} \nabla^2 \phi_1' &= 0 \quad \text{in} \quad z < 0 \\ \nabla^2 \phi_1' &= 0 \quad \text{in} \quad z > 0 \end{aligned} \tag{58}$$

- Bottom boundary conditions

$$\begin{aligned} |\nabla \phi_1'| &\to 0 \quad \text{as} \quad z \to -\infty \\ |\nabla \phi_2'| &\to 0 \quad \text{as} \quad z \to \infty \end{aligned} \tag{59}$$

- Kinematic boundary conditions

$$\begin{aligned} \frac{\partial \phi_1'}{\partial z} &= \frac{\partial \eta}{\partial t} + u_1 \frac{\partial \eta}{\partial x} \quad \text{on} \quad z = 0 \\ \frac{\partial \phi_2'}{\partial z} &= \frac{\partial \eta}{\partial t} + u_2 \frac{\partial \eta}{\partial x} \quad \text{on} \quad z = 0 \end{aligned} \tag{60}$$

- Dynamic boundary condition

$$\rho_1\left(u_1 \frac{\partial \phi_1'}{\partial x} + \frac{\partial \phi_1'}{\partial t} + g\eta\right) = \rho_2\left(u_2 \frac{\partial \phi_2'}{\partial x} + \frac{\partial \phi_2'}{\partial t} + g\eta\right) + T\nabla^2 \eta \tag{61}$$

A set of suitable solutions, namely the plane-wave solution of the perturbed quantities can be written as

$$\phi_1' = A_1 \exp\left[i(\mathbf{k} \bullet \mathbf{x} - \sigma t) + kz\right] \tag{62}$$

$$\phi_2' = A_2 \exp\left[i(\mathbf{k} \bullet \mathbf{x} - \sigma t) + kz\right] \tag{63}$$

$$\eta = A \exp\left[i(\mathbf{k} \bullet \mathbf{x} - \sigma t)\right] \tag{64}$$

where the wave numbers k_1 and k_2 are real and $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$. The kinematic conditions (60) require that

$$A_1 k = Ai(-\sigma + u_1 k_1)$$
$$A_2 k = Ai(-\sigma + u_1 k_1) \tag{65}$$

eliminating of A_1, A_2 and A from (60) and (61) gives the eigenvalue equation for σ as

$$\rho_1(u_1 k_1 - \sigma)^2 + \rho_2(u_2 k_1 - \sigma)^2 = gk(\rho_1 - \rho_2) + Tk^3 \tag{66}$$

The solution of the quadratic equation gives two roots for σ as

$$\sigma = k(\alpha_1 u_1 + \alpha_2 u_2) \pm \left[gk(\alpha_1 - \alpha_2) + \frac{Tk^3}{\rho_1 + \rho_2} - \alpha_1\alpha_2 k_1^2(u_1 - u_2)^2\right]^{\frac{1}{2}} \tag{67}$$

where

$$\alpha_1 = \frac{\rho_1}{\rho_1 + \rho_2} \quad , \quad \alpha_2 = \frac{\rho_2}{\rho_1 + \rho_2} \tag{68}$$

The first term on the right-hand side of (67) is always real and represents convection of any wave at velocity $(\alpha_1 u_1 + \alpha_2 u_2)$ in the x-direction. Thus the flow is stable or unstable according to whether

$$\alpha_1\alpha_2 k_1^2(u_1 - u_2)^2 \leq or > gk(\alpha_1 - \alpha_2) + \frac{Tk^3}{\rho_1 + \rho_2} \tag{69}$$

The equality corresponds to marginal stability. For any unstable wave with given k_1, its growth rate $Re(i\sigma) > 0$ is maximum when $k_2 = 0$ (or $k = k_1$) and the wave is in the direction parallel to the basic flow. Then the condition for instability becomes

$$\rho_1\rho_2(u_1 - u_2)^2 > (\rho_1^2 - \rho_2^2)\left(\frac{g}{k} + \frac{Tk}{\rho_1 - \rho_2}\right) \tag{70}$$

The right-hand side attains its maximum for$k^2 = (g/T)(\rho_1 - \rho_2)$, and the corresponding condition for instability of the basic flow is

$$(u_1 - u_2)^2 > \frac{2(\rho_1 + \rho_2)}{\rho_1\rho_2}\left[Tg(\rho_1 - \rho_2)\right]^{\frac{1}{2}} \tag{71}$$

This result was first derived by Kelvin.

In the absence of surface tension ($T \equiv 0$), the condition for instability (69) reduce to

$$\alpha_1\alpha_2(u_1 - u_2)^2 k_1^2 > gk(\alpha_1 - \alpha_2) \tag{72}$$

This means that, for a given velocity difference $u_1 - u_2$, and for a given direction of the wave number vector **k**, instability occurs for all wave numbers

$$k > \frac{g(\alpha_1 - \alpha_2)}{\alpha_1\alpha_2(u_1 - u_2)^2 \cos^2\theta} \tag{73}$$

where θ is the angle between **k** and **u**. For a given $u_1 - u_2$, result (75) shows that instabilty occurs for the small wave number, k_{min}, when **k** is the direction of **u**, where is given by

$$k_{min} = \frac{g(\alpha_1 - \alpha_2)}{\alpha_1\alpha_2(u_1 - u_2)^2} \tag{74}$$

Thus the flow is always unstable when $k > k_{min}$. The most remarkable feature of result (73) and (74) is that flow becomes unstable no matter how small the difference $(u_1 - u_2)$ may be. This instability is known as the Kelvin-Helmholtz instability. This instability occurs when stratified layers of fluid are in relative motion.

Special cases

I. When $\rho_1 = \rho_2$ and $u_1 \neq u_2$, (67) becomes for $T \equiv 0$

$$\sigma = \frac{k}{2}(u_1 + u_2) \pm i\frac{k}{2}|u_1 - u_2| \tag{75}$$

for all non-zero values of k, with exponential growth rate $\frac{1}{2}|(u_1 - u_2)k|$. This is the well-known Helmholtz instability for a vortex sheet. Likewise many result can be derived from (67) using many assumptions.

II. When the basic flow is at rate ($u_1 = u_2 = 0$), result (67) with $T \equiv 0$ yields

$$\sigma = \pm\left[\frac{g(\rho_1 - \rho_2)k}{\rho_1 + \rho_2}\right]^{\frac{1}{2}} \tag{76}$$

In this case, instability occurs if and if $\rho_1 < \rho_2$, that is, heavier fluid is above the lighter fluid. The phase velocity of the stable waves is given by

$$c_p = \frac{\sigma}{k} = \pm\left[\frac{g(\rho_1 - \rho_2)}{k(\rho_1 + \rho_2)}\right]^{\frac{1}{2}} \tag{77}$$

III. Another special case ($\rho_2 = 0$ and $u_1 = u_2 = 0$) corresponding to the surface waves on deep water. Result (66) gives the famous dispersion relation

$$\sigma^2 = gk + \frac{Tk^3}{\rho^2} \tag{78}$$

This describe the capillary gravity waves on deep water.

IV. When $u_1 = u_2 = 0$, result (66) yields

$$\sigma^2 = gk\left[\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} + \frac{Tk^2}{g(\rho_1 + \rho_2)}\right] \tag{79}$$

which corresponds to Rayleigh-Taylor instability as discussed in section 7.7.

The Kelvin-Helmholtz nonlinear instability will not be discussed here but the interested reader is referred to the book by Debnath [2].

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