Cubo Matemática Educacional
Vol. 5., N° 1, Enero 2003

A New Expansion Formula

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2000 AMS Subject Classification: 26A42, 26A99, 26D15

Key Words and Phrases: Taylor like expansion, Riemann–Stieltjes integral, inequality for integral and sum.

Abstract

A new Taylor like expansion formula is established. Here the Riemann–Stieltjes integral of a function is expanded into a finite sum form which involves the derivatives of the function evaluated at the right end point of the interval of integration. The error of the approximation is given in an integral form involving the nth derivative of the function. Implications and applications of the formula follow.

1. Results

We give our first and main result:

Theorem 1. *Let g_0 be a Lebesgue integrable and of bounded variation function on $[a,b]$, $a<b$. We form*

$$g_1(x) := \int_a^x g_0(t)dt, \ldots \tag{1}$$

$$g_n(x) := \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} g_0(t)dt, \quad n \in \mathbb{N}, \ x \in [a,b]. \tag{2}$$

Let f be such that $f^{(n-1)}$ is a absolutely continuous function on $[a,b]$. Then

$$\int_a^b f dg_0 = \sum_{k=0}^{n-1}(-1)^k f^{(k)}(b) g_k(b) - f(a)g_0(a) + (-1)^n \int_a^b g_{n-1}(t) f^{(n)}(t)dt. \tag{3}$$

Proof. We apply integration by parts repeatedly (see [1], p. 195):

$$\int_a^b f dg_0 = f(b)g_0(b) - f(a)g_0(a) - \int_a^b g_0 f' dt,$$

and

$$\int_a^b g_0 f' dt = \int_a^b f' dg_1 = f'(b)g_1(b) - f'(a)g_1(a) - \int_a^b g_1 f'' dt = f'(b)g_1(b) - \int_a^b g_1 f'' dt.$$

Furthermore

$$\int_a^b f'' g_1 dt = \int_a^b f'' dg_2 = f''(b)g_2(b) - f''(a)g_2(a) - \int_a^b g_2 f''' dt = f''(b)g_2(b) - \int_a^b g_2 f''' dt.$$

So far we have got

$$\int_a^b f dg_0 = f(b)g_0(b) - f(a)g_0(a) - f'(b)g_1(b) + f''(b)g_2(b) - \int_a^b g_2 f''' dt.$$

Similarly we find

$$\int_a^b g_2 f''' dt = \int_a^b f''' dg_3 = f'''(b)g_3(b) - \int_a^b g_3 f^{(4)} dt.$$

That is,

$$\int_a^b f dg_0 = f(b)g_0(b) - f(a)g_0(a) - f'(b)g_1(b) + f''(b)g_2(b) - f'''(b)g_3(b) + \int_a^b g_3 f^{(4)} dt.$$

The validity of (3) is now clear. □

On Theorem 1 we have

Corollary 1. *Additionally assume that $f^{(n)}$ exists and is bounded. Then*

$$\left|\int_a^b f dg_0 - \sum_{k=0}^{n-1}(-1)^k f^{(k)}(b) g_k(b) + f(a) g_0(a)\right|$$

$$= \left|\int_a^b g_{n-1}(t) f^{(n)}(t) dt\right| \leq \|f^{(n)}\|_\infty \int_a^b |g_{n-1}(t)| dt. \tag{4}$$

As a continuation of Corollary 1 we have

Corollary 2. *Assuming that g_0 is bounded we obtain*

$$\left|\int_a^b f dg_0 - \sum_{k=0}^{n-1}(-1)^k f^{(k)}(b) g_k(b) + f(a) g_0(a)\right|$$

$$\leq \|f^{(n)}\|_\infty \|g_0\|_\infty \frac{(b-a)^n}{n!}. \tag{5}$$

Proof. Here we estimate the right-hand side of inequality (4).

We see that

$$\begin{aligned} |g_{n-1}(x)| &= \left|\int_a^x \frac{(x-t)^{n-2}}{(n-2)!} g_0(t) dt\right| \\ &\leq \int_a^x \frac{(x-t)^{n-2}}{(n-2)!} |g_0(t)| dt \\ &\leq \frac{\|g_0\|_\infty}{(n-2)!} \int_a^x (x-t)^{n-2} dt \\ &= \frac{\|g_0\|_\infty}{(n-1)!} (x-a)^{n-1}. \end{aligned}$$

That is

$$|g_{n-1}(t)| \leq \|g_0\|_\infty \frac{(t-a)^{n-1}}{(n-1)!}, \quad \text{all } t \in [a,b].$$

Consequently,

$$\int_a^b |g_{n-1}(t)| dt \leq \|g_0\|_\infty \frac{(b-a)^n}{n!}.$$ □

A refinement of (5) follows in

Corollary 3. *Here the assumptions are as in Corollary 2. We additionally assume that*

$$\|f^{(n)}\|_\infty \le K, \quad \forall n \ge 1,$$

i.e., f possess infinitely many derivatives and all are uniformly bounded. Then

$$\left| \int_a^b f dg_0 - \sum_{k=0}^{n-1} (-1)^k f^{(k)}(b) g_k(b) + f(a) g_a(a) \right|$$
$$\le K \|g_0\|_\infty \frac{(b-a)^n}{n!}, \quad \forall n \ge 1. \tag{6}$$

A consequence of (6) comes next.

Corollary 4. *Same assumptions as in Corollary 3. For some $0 < r < 1$ it holds that*

$$O(r^n) = K\|g_0\|_\infty \frac{(b-a)^n}{n!} \to 0, \quad \text{as } n \to +\infty. \tag{7}$$

That is

$$\lim_{n\to+\infty} \sum_{k=0}^{n-1} (-1)^k f^{(k)}(b) g_k(b) = \int_a^b f dg_0 + f(a) g_0(a). \tag{8}$$

Proof. Call $A := b - a > 0$, we want to prove that $\frac{A^n}{n!} \to 0$ as $n \to +\infty$. Set

$$x_n := \frac{A^n}{n!}, \quad n = 1, 2, \dots .$$

See that

$$x_{n+1} = \frac{A}{n+1} \cdot x_n, \quad n = 1, 2, \dots .$$

But there exists $n_0 \in \mathbb{N}$: $n > A - 1$, i.e., $A < n_0 + 1$, that is, $r := \frac{A}{n_0+1} < 1$ (in fact take $n_0 := \lceil A - 1 \rceil + 1$, where $\lceil \cdot \rceil$ is the ceiling of the number). Thus

$$x_{n_0+1} = rc,$$

where

$$c := x_{n_0} > 0.$$

Therefore

$$x_{n_0+2} = \frac{A}{n_0+2} x_{n_0+1} = \frac{A}{n_0+2} rc < r^2 c,$$

i.e.,

$$x_{n_0+2} < r^2 c.$$

Likewise we get

$$x_{n_0+3} < r^3 c.$$

And in general we obtain

$$0 < x_{n_0+k} < r^k \cdot c = c^* \cdot r^{n_0+k}, \quad k \in \mathbb{N}$$

where $c^* := \frac{c}{r^{n_0}}$. That is

$$0 < x_N < c^* r^N, \quad \forall N \geq n_0 + 1.$$

Since $r^N \to 0$ as $N \to +\infty$, we obtain that $x_N \to 0$ as $N \to +\infty$. I.e., $\frac{(b-a)^n}{n!} \to 0$, as $n \to +\infty$. □

Remark 1. (On Theorem 1) Furthermore we see that (here $f \in C^n([a,b])$)

$$\begin{aligned} &\left| \int_a^b f dg_0 + f(a) g_0(a) \right| \\ &\stackrel{(3)}{=} \left| \sum_{k=0}^{n-1} (-1)^k f^{(k)}(b) g_k(b) + (-1)^n \int_a^b g_{n-1}(t) f^{(n)}(t) dt \right| \\ &\leq \sum_{k=0}^{n-1} \|f^{(k)}\|_\infty |g_k(b)| + \|f^{(n)}\|_\infty \int_a^b |g_{n-1}(t)| dt. \end{aligned}$$

Call

$$L := \max\{\|f\|_\infty, \|f'\|_\infty, \ldots, \|f^{(n)}\|_\infty\}. \tag{9}$$

Then

$$\begin{aligned} &\left| \int_a^b f dg_0 + f(a) g_0(a) \right| \\ &\leq L \cdot \left\{ \sum_{k=0}^{n-1} |g_k(b)| + \int_a^b |g_{n-1}(t)| dt \right\}, \quad n \in \mathbb{N} \text{ fixed.} \end{aligned} \tag{10}$$

2. Applications

(I) Let $\{\mu_m\}_{m\in\mathbb{N}}$ be a sequence of Borel finite signed measures on $[a,b]$. Consider the distribution functions $g_{o,m}(x) := \mu_m[a,x]$, $x \in [a,b]$, $m \in \mathbb{N}$, which are of bounded variation and Lebesgue integrable. Clearly (here $f \in C^n[a,b]$)

$$\int_a^b f d\mu_m = \int_a^b f dg_{o,m} + g_{o,m}(a) \cdot f(a). \tag{11}$$

We would like to study the weak convergence of μ_m to zero, as $m \to +\infty$. From (10) and (11) we get

$$\left|\int_a^b f d\mu_m\right| \le L \cdot \left\{\sum_{k=0}^{n-1} |g_{k,m}(b)| + \int_a^b |g_{n-1,m}(t)|dt\right\}, \quad \forall m \in \mathbb{N}. \tag{12}$$

Here we assume that

$$g_{o,m}(b) \to 0,$$

and

$$\int_a^b |g_{o,m}(t)|dt \to 0, \quad \text{as } m \to +\infty. \tag{13}$$

Let $k \in \mathbb{N}$, then

$$g_{k,m}(b) = \int_a^b \frac{(b-t)^{k-1}}{(k-1)!} g_{o,m}(t)dt$$

by (2).

Thus

$$|g_{k,m}(b)| \le \left(\int_a^b |g_{o,m}(t)|dt\right) \frac{(b-a)^{k-1}}{(k-1)!} \to 0.$$

That is,

$$|g_{k,m}(b)| \to 0, \quad \forall k \in \mathbb{N}, \ m \to +\infty. \tag{14}$$

Next we have

$$g_{n-1,m}(x) = \int_a^x \frac{(x-t)^{n-2}}{(n-2)!} g_{o,m}(t)dt.$$

Hence

$$\begin{aligned} |g_{n-1,m}(x)| &\le \int_a^x \frac{(x-t)^{n-2}}{(n-2)!} |g_{o,m}(t)|dt \\ &\le \frac{(x-a)^{n-2}}{(n-2)!} \int_a^x |g_{o,m}(t)|dt \\ &\le \frac{(b-a)^{n-2}}{(n-2)!} \int_a^b |g_{o,m}(t)|dt. \end{aligned}$$

Consequently we get

$$\int_a^b |g_{n-1,m}(x)|dx \le \frac{(b-a)^{n-1}}{(n-2)!} \int_a^b |g_{o,m}(t)|dt \to 0,$$

as $m \to +\infty$. I.e.,

$$\int_a^b |g_{n-1,m}(t)|dt \to 0, \quad \text{as } m \to +\infty. \tag{15}$$

Finally from (12), (13), (14) and (15) we derive that

$$\int_a^b f d\mu_m \to 0, \quad \text{as } m \to +\infty.$$

That is, μ_m converges weakly to zero as $m \to +\infty$.

The last result was first proved (case of $n = 0$) in [2] and then in [3].

(II) Formula (3) is expected to have applications to Numerical Integration.

(III) Let $[a, b] = [0, 1]$ and $g_0(t) = t$. Let f be such that $f^{(n-1)}$ is a absolutely continuous function on $[0, 1]$. Then by Theorem 1 we find $g_n(x) = \frac{x^{n+1}}{(n+1)!}$, all $n \in \mathbb{N}$. Furthermore we get

$$\int_0^1 f dt = \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}(1)}{(k+1)!} + \frac{(-1)^n}{n!} \int_0^1 t^n f^{(n)}(t) dt. \tag{16}$$

One can derive other formulas like (16) for various basic g_0's.

References

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