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Some remarks on the non-real roots of polynomials

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ABSTRACT

Let $f \in \mathbb{R}(t)[x]$ be given by $f(t, x) = x^n + t \cdot g(x)$ and $\beta_1 < \cdots < \beta_m$ the distinct real roots of the discriminant $\Delta_{(f,x)}(t)$ of $f(t, x)$ with respect to x. Let γ be the number of real roots of $g(x) = \sum_{k=0}^{s} t_{s-k} x^{s-k}$. For any $\xi > |\beta_m|$, if $n-s$ is odd then the number of real roots of $f(\xi, x)$ is $\gamma + 1$, and if $n - s$ is even then the number of real roots of $f(\xi, x)$ is γ, $\gamma + 2$ if $t_s > 0$ or $t_s < 0$ respectively. A special case of the above result is constructing a family of degree $n \geq 3$ irreducible polynomials over $\mathbb{Q}$ with many non-real roots and automorphism group S_n.

RESUMEN

Sea $f \in \mathbb{R}(t)[x]$ dada por $f(t, x) = x^n + t \cdot g(x)$ y $\beta_1 < \cdots < \beta_m$ las diferentes raíces reales del discriminante $\Delta_{(f,x)}(t)$ de $f(t, x)$ con respecto de x. Sea γ el número de raíces reales de $g(x) = \sum_{k=0}^{s} t_{s-k} x^{s-k}$. Para todo $\xi > |\beta_m|$, si $n - s$ es impar entonces el número de raíces reales de $f(\xi, x)$ es $\gamma + 1$, y si $n - s$ es par entonces el número de raíces reales de $f(\xi, x)$ es γ, $\gamma + 2$ si $t_s > 0$ o $t_s < 0$, respectivamente. Un caso especial del resultado anterior es construyendo una familia de polinomios irreducibles sobre $\mathbb{Q}$ de grado $n \geq 3$ con muchas raíces no-reales y grupo de automorfismos S_n

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1 Introduction

Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree $n \geq 2$ and Gal (f) its Galois group over $\mathbb{Q}$. Let us assume that over $\mathbb{R}$, $f(x)$ is factored as

$$f(x) = a \prod_{j=1}^{r} (x - \alpha_j) \prod_{i=1}^{s} (x^2 + a_i x + b_i),$$

where $a_i^2 < 4b_i$, for all $i = 1, \ldots, s$. The pair (r, s) is called the *signature* of $f(x)$. Obviously $\deg f = 2s + r$. If $s = 0$ then $f(x)$ is called *totally real* and if $r = 0$ it is called *totally complex.* Equivalently the above terminology can be defined for binary forms $f(x, z)$. By a reordering of the roots we may assume that if $f(x)$ has $2s$ non-real roots then

$$\alpha := (1, 2)(3, 4) \cdots (2s - 1, 2s) \in \mathrm{Gal}(f).$$

In [4] it is proved that if $\deg f = p$, for a prime p, and s satisfies

$$s\,(s \log s + 2 \log s + 3) \leq p$$

then $\mathrm{Gal}(f) = A_p, S_p$. Moreover, a list of all possible groups for various values of r is given for $p \leq 29$; see [4, Thm. 2]. There are some follow up papers to [4].

In [1] the author proves that if $p \geq 4s + 1$, then the Galois group is either S_p or A_p. This improves the bound given in [4]. The author also studies when polynomials with non-real roots are solvable by radicals, which are consequences of Table 2 and Theorem 2 in [4]. In [13] the author uses Bezoutians of a polynomial and its derivative to construct polynomials with real coefficients where the number of real roots can be counted explicitly. Thereby, irreducible polynomials in $\mathbb{Q}[x]$ of prime degree p are constructed for which the Galois group is either S_p or A_p.

In this paper we study a family of polynomials with non-real roots whose degree is not necessarily prime. Given a polynomial $g(x) = \sum_{i=0}^{s} t_i x^i$ and with γ number of non-real roots we construct a polynomial $f(t, x) = x^n + t\, g(x)$ which has $\gamma, \gamma + 1, \gamma + 2$ non-real roots for certain values of $t \in \mathbb{R}$; see Theorem 3.2. The values of $t \in \mathbb{R}$ are given in terms of the Bezoutian matrix of polynomials or equivalently the discriminant of $f(t, x)$ with respect to x. This is the focus of Section 3 in the paper.

While most of the efforts have been focusing on the case of irreducible polynomials over $\mathbb{Q}$ which have real roots, the case of polynomials with no real roots is equally interesting. How should an irreducible polynomial over $\mathbb{Q}$ with all non-real roots must look like? What can be said about the Galois group of such totally complex polynomials? In [5] is developed a reduction theory for such polynomials via the hyperbolic center of mass. A special case of Theorem 3.2 provides a class of totally complex polynomials.

Notation For any polynomial $f(x)$ we denote by $\Delta_{(f,x)}$ its discriminant with respect to x. If f is a univariate polynomial then Δ_f is used and the leading coefficient is denoted by led(f). Throughout this paper the ground field is a field of characteristic zero.

2 Preliminaries

Let $f_1(x)$, $f_2(x)$ be polynomials over a field F of characteristic zero and, let n be an integer which is greater than or equal to $\max\{\deg f_1, \deg f_2\}$. Then, we put

$$B_n(f_1, f_2) := \frac{f_1(x)f_2(y) - f_1(y)f_2(x)}{x - y} = \sum_{i,j=1}^{n} \alpha_{ij} x^{n-i} y^{n-j} \in F[x, y],$$
$$M_n(f_1, f_2) := (\alpha_{ij})_{1 \le i,j \le n}.$$

The matrix $M_n(f_1, f_2)$ is called the *Bezoutian* of f_1 and f_2. Clearly, $B_n(f_1, f_1) = 0$ and hence $M_n(f_1, f_1)$ is the zero matrix. The following properties hold true; see [6, Theorem 8.25] for details.

Proposition 1. *The following are true:*

(1) $M_n(f_1, f_2)$ *is an* $n \times n$ *symmetric matrix over* F.

(2) $B_n(f_1, f_2)$ *is linear in* f_1 *and* f_2*, separately.*

(3) $B_n(f_1, f_2) = -B_n(f_2, f_1)$.

When $f_2 = f_1'$, the formal derivative of f_1 (with respect to the indeterminate x), we often write $B_n(f_1) := B_n(f_1, f_1')$. From now on, for any degree $n \ge 2$ polynomial $f(x) \in \mathbb{R}[x]$ we will denote by $M_n(f) := M_n(f, f')$ as above. The matrix $M_n(f)$ is called the **Bezoutian matrix** of f.

Remark 2.1. *It is often the case that the matrix* $M'_n(f_1, f_2) = (\alpha'_{ij})_{1 \le i,j \le n}$ *defined by the generating function*

$$B'_n(f_1, f_2) := \frac{f_1(x)f_2(y) - f_1(y)f_2(x)}{x - y} = \sum_{i,j=1}^{n} \alpha'_{ij} x^{i-1} y^{j-1} \in F[x, y]$$

is called the Bezoutian of f_1 *and* f_2. *But no difference can be seen between these two definitions as far as we consider the corresponding quadratic forms*

$$\sum_{i,j=1}^{n} \alpha_{ij} x_i x_j \quad \text{and} \quad \sum_{i,j=1}^{n} \alpha'_{ij} x_i x_j.$$

In fact, these two quadratic forms are equivalent over the prime field $\mathbb{Q}$ $(\subset F)$ *since we have* $M'_n(f_1, f_2) = {}^t J_n M_n(f_1, f_2) J_n$, *where*

$$J_n = \begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & 0 \end{bmatrix}$$

is an $n \times n$ *anti-identity matrix. This implies that above two quadratic forms are equivalent over* $\mathbb{Q}$ *or more precisely, over the ring of rational integers* $\mathbb{Z}$.

Let $f(x) \in \mathbb{R}[x]$ be a degree $n \geq 2$ polynomial which is given by

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

Then over $\mathbb{R}$ this polynomial is factored as

$$f(x) = a \prod_{j=1}^{r} (x - \alpha_j) \prod_{i=1}^{s} (x^2 + a_i x + b_i)$$

for some $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ and $a_i, b_i, a \in \mathbb{R}$, where $a_i^2 < 4b_i$, for all $i = 1, \cdots, s$.

Throughout this paper, for a univariate polynomial f, its discriminant will be denoted by Δ_f. For any two polynomials $f_1(x)$, $f_2(x)$ the resultant with respect to x will be denoted by $\text{Res}(f_1, f_2, x)$. We notice the following elementary fact, its proof is elementary and we skip the details.

Remark 2.2. *For any polynomial $f(x)$, the determinant of the Bezoutian is the same as the discriminant up to a multiplication by a constant. More precisely,*

$$\Delta_f = \frac{1}{\mathit{led}(f)^2} \det M_n(f),$$

where $\mathit{led}(f)$ is the leading coefficient of $f(x)$.

If $f(x) \in \mathbb{Q}[x]$ is irreducible and its degree is a prime number, say $\deg f = p$, then there is enough known for the Galois group of polynomials with some non-real roots; see [4], [1], [13] for details. If the number of non-real roots is "small" enough with respect to the prime degree $\deg f = p$ of the polynomial, then the Galois group is A_p or S_p. Furthermore, using the classification of finite simple groups one can provide a complete list of possible Galois groups for every polynomial of prime degree p which has non-real roots; see [4] for details.

On the other extreme are the polynomials which have all roots non-real. We called them above, totally complex polynomials. We have the following:

Lemma 2.1. *The followings are equivalent:*

i) $f(x) \in \mathbb{R}[x]$ *is totally complex*

ii) $f(x)$ *can be written as*

$$f(x) = a \prod_{i=1}^{n} f_i$$

where $f_i = x^2 + a_i x + b_i$, for $i = 1, \ldots, n$ and $a_i, b_i, a \in \mathbb{R}$, where $a_i^2 < 4b_i$, for all $i = 1, \ldots, n$. Moreover, the determinant of the Bezoutian $M_n(f)$ is given by

$$\Delta_f = \frac{1}{\mathit{led}(f)^2} \det M_n(f) = \prod_{i=1}^{n} \Delta_{f_i} \cdot \prod_{i,j,i \neq j}^{n} \left(\mathit{Res}(f_i, f_j, x)\right)^2$$

where led(f) *is the leading coefficient of* $f(x)$.

ii) the index of inertia of Bezoutian $M(f)$ *is 0*

iii) if $\Delta_f \neq 0$ *then the equivalence class of* $M(f)$ *in the Witt ring* $W(R)$ *is 0.*

Proof. The equivalence between i), ii), and iii) can be found in [6]. ☐

It is not clear when such polynomials are irreducible over $\mathbb{Q}$. If that's the case, what is the Galois group Gal (f)? Clearly the group generated by the involution $(1,2)(3,4)\cdots(2n-1,2n)$ is embedded in Gal (f). Is Gal (f) larger in general?

3 On the number of real roots of polynomials

For any degree $n \geq 2$ polynomial $f(x) \in \mathbb{R}[x]$ and any symmetric matrix $M := M_n(f)$ with real entries, let N_f be the **number of distinct real roots** of f and $\sigma(M)$ be the index of inertia of M, respectively. The next result plays a fundamental role throughout this section ([6, Theorem 9.2]).

Proposition 2. *For any real polynomial* $f \in \mathbb{R}[x]$*, the number* N_f *of its distinct real roots is the index of inertia of the Bezoutian matrix* $M_n(f)$*. In other words,*

$$N_f = \sigma\left(M_n(f)\right).$$

Let us cite one more result which says that the roots of a polynomial depend continuously on its coefficients ([11, Theorem 1.4], [16, Theorem 1.3.1]).

Proposition 3. *Let be given a polynomial*

$$f(x) = \sum_{l=0}^{n} a_l x^l \in \mathbb{C}[x],$$

with distinct roots $\alpha_1, \ldots, \alpha_k$ *of multiplicities* $m_1, \ldots, m_k$ *respectively. Then, for any given a positive*

$$\varepsilon < \min_{1 \leq i < j \leq k} \left\{ \frac{|\alpha_i - \alpha_j|}{2} \right\},$$

there exists a real number $\delta > 0$ *such that any monic polynomial* $g(x) = \sum_{l=0}^{n} b_l x^l \in \mathbb{C}[x]$ *whose coefficients satisfy*

$$|b_l - a_l| < \delta,$$

for $l = 0, \cdots, n-1$*, has exactly* m_j *roots in the disk*

$$\mathcal{D}(\alpha_j; \varepsilon) = \{z \in \mathbb{C} \mid |z - \alpha_j| < \epsilon\} \ (j = 1, \cdots, k).$$

Let n, s be positive integers such that $n > s$ and let

$$\begin{aligned} g(t_0, \cdots, t_s; x) &= \sum_{k=0}^{s} t_{s-k} x^{s-k}, \\ f^{(n)}(t_0, \cdots, t_s, t; x) &= x^n + t \cdot g(t_0, \cdots, t_s; x) \end{aligned} \tag{3.1}$$

be polynomials in x over $E_1 = \mathbb{R}(t_0, \cdots, t_s)$, $E_2 = \mathbb{R}(t_0, \cdots, t_s, t)$, respectively. Here, E_1 (resp., E_2) is a rational function field with $s+1$ (resp., $(s+2)$) variables $t_0, \cdots, t_s$ (resp., $(t_0, \cdots, t_s, t)$). To ease notation, let us put

$$g(x) = g(t_0, \cdots, t_s; x), \ \ f(t; x) = f^{(n)}(t_0, \cdots, t_s, t; x)$$

and for any real vector $v = (v_0, \cdots, v_s) \in \mathbb{R}^{s+1}$, we put

$$g_v(x) = g(v_0, \cdots, v_s; x), \ \ f_v(t; x) = f^{(n)}(v_0, \cdots, v_s, t; x). \tag{3.2}$$

By using Proposition 2, we can prove the next theorem ([13, Main Theorem 1.3]).

Theorem 3.1. *Let $r = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be a vector such that $N_{g_r} = s$. Let us consider $f_r(t; x) = f^{(n)}(r_0, \cdots, r_s, t; x)$ as a polynomial over $\mathbb{R}(t)$ in x and put*

$$P_r(t) = \det M_n(f_r(t; x)) = \det M_n(f_r(t; x), f_r'(t; x)),$$

where $f_r'(t; x)$ is a derivative of $f_r(t; x)$ with respect to x. Then, for any real number $\xi > \alpha_r = \max\{\alpha \in \mathbb{R} \mid P_r(\alpha) = 0\}$, we have

$$N_{f_r(\xi; x)} = \begin{cases} s+1 & \text{if } n-s : \text{odd} \\ s & \text{if } n-s : \text{even}, \ r_s > 0 \\ s+2 & \text{if } n-s : \text{even}, \ r_s < 0. \end{cases}$$

By this theorem and a theorem of Oz Ben-Shimol [1, Theorem 2.6], we can obtain an algorithm to construct prime degree p polynomials with given number of real roots, and whose Galois groups are isomorphic to the symmetric group S_p or the alternating group A_p ([13, Corollary 1.6]).

In this section, we extend this theorem as follows;

Theorem 3.2. *Let $r = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be a vector such that $g_r(x)$ is a degree s separable polynomial satisfying $N_{g_r(x)} = \gamma$ $(0 \le \gamma \le s)$. Let us consider $f_r(t; x) = f^{(n)}(r_0, \cdots, r_s, t; x)$ as a polynomial over $\mathbb{R}(t)$ in x and put*

$$P_r(t) = \det M_n(f_r(t; x)) = \det M_n(f_r(t; x), f_r'(t; x)),$$

where $f_r'(t; x)$ is a derivative of $f_r(t; x)$ with respect to x. Then, for any real number $\xi > \alpha_r = \max\{\alpha \in \mathbb{R} \mid P_r(\alpha) = 0\}$, we have

$$N_{f_r(\xi; x)} = \begin{cases} \gamma+1 & \text{if } n-s : \text{odd} \\ \gamma & \text{if } n-s : \text{even}, \ r_s > 0 \\ \gamma+2 & \text{if } n-s : \text{even}, \ r_s < 0. \end{cases} \tag{3.3}$$

The above theorem can be restated as follows:

Corolary 1. *Let* $f \in \mathbb{R}(t)[x]$ *be given by*

$$f(t,x) = x^n + t \cdot \sum_{k=0}^{s} t_{s-k} x^{s-k}$$

and $\beta_1 < \cdots < \beta_m$ *the distinct real roots of the degree* s *polynomial*

$$P(t) := \frac{1}{t^{n-1}} \Delta_{(f,x)}(t).$$

For any $\xi > |\beta_m|$, *the number of real roots of* $f(\xi, x)$ *is*

$$N_{f(\xi,x)} = \begin{cases} \gamma + 1 & \text{if } n-s : \text{odd} \\ \gamma & \text{if } n-s : \text{even}, \ t_s > 0 \\ \gamma + 2 & \text{if } n-s : \text{even}, \ t_s < 0. \end{cases}$$

where γ *is the number or real roots of* $g(x) = \frac{f(x)-x^n}{t} \in \mathbb{R}[x]$.

The rest of the section is concerned with proving Thm. 3.2.

3.1 The Bezoutian of $f(t;x)$

First, let us put

$$A(t_0, \cdots, t_s, t) = (a_{ij}(t_0, \cdots, t_s, t))_{1 \le i,j \le n} = M_n(f(t;x)) \in \mathrm{Sym}_n(E_2),$$
$$B(t_0, \cdots, t_s) = (b_{ij}(t_0, \cdots, t_s))_{1 \le i,j \le s} = M_s(g(x)) \in \mathrm{Sym}_s(E_1).$$

For ease of notation, we also write

$$A(t_0, \cdots, t_s, t) = A(t) = (a_{ij}(t))_{1 \le i,j \le n}, \ B(t_0, \cdots, t_s) = B = (b_{ij})_{1 \le i,j \le s}$$

and we put $B(t) = (b_{ij}(t))_{1 \le i,j \le s} = t^2 B$. Then, by Proposition 1, we have

$$\begin{aligned} A(t) &= M_n(x^n + tg(x), nx^{n-1} + tg'(x)) \\ &= nM_n(x^n, x^{n-1}) - ntM_n(x^{n-1}, g(x)) + tM_n(x^n, g'(x)) + t^2 M_n(g(x), g'(x)) \\ &= nM_n(x^n, x^{n-1}) - nt \sum_{k=0}^{s} t_{s-k} M_n(x^{n-1}, x^{s-k}) \\ &\qquad + t \sum_{k=0}^{s-1} (s-k) t_{s-k} M_n(x^n, x^{s-k-1}) + t^2 M_n(g(x), g'(x)). \end{aligned}$$

Lemma 3.1. *Let* λ, μ, ν *be integers such that* $\lambda \ge \mu > \nu \ge 0$. *Then* $M_\lambda(x^\mu, x^\nu) = (m_{ij})_{1 \le i,j \le \lambda}$, *where*

$$m_{ij} = \begin{cases} 1 & i + j = 2\lambda - (\mu + \nu) + 1 \quad (\lambda - \mu + 1 \le i, j \le \lambda - \nu), \\ 0 & \text{otherwise}. \end{cases}$$

Proof. By definition, we have

$$B_\lambda(x^\mu, x^\nu) = \frac{x^\mu y^\nu - x^\nu y^\mu}{x-y}$$
$$= \sum_{k=1}^{\mu-\nu} x^{\mu-k} y^{\nu+k-1} = \sum_{k=1}^{\mu-\nu} x^{\lambda-(\lambda-\mu+k)} y^{\lambda-(\lambda-\nu-k+1)},$$

which implies

$$m_{ij} = \begin{cases} 1 & (i,j) = (\lambda-\mu+k, \lambda-\nu-k+1) \quad (1 \le k \le \mu-\nu) \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1 & i+j = 2\lambda-(\mu+\nu)+1 \quad (\lambda-\mu+1 \le i, j \le \lambda-\nu), \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof. □

Here, let us divide $A(t)$ into two parts $\hat{A}(t)$ and $\tilde{A}(t)$, where

$$\hat{A}(t) = (\hat{a}_{ij}(t))_{1\le i,j\le n} = nM_n(x^n, x^{n-1}) - nt\sum_{k=0}^{s} t_{s-k} M_n(x^{n-1}, x^{s-k})$$
$$+ t\sum_{k=0}^{s-1}(s-k)t_{s-k} M_n(x^n, x^{s-k-1}),$$
$$\tilde{A}(t) = (\tilde{a}_{ij}(t))_{1\le i,j\le n} = t^2 M_n(g(x), g'(x))$$

and put $l_k = n-s+k+2 \; (= 2n-(n+s-k-1)+1)$. Then, by lemma 3.1, we have

$$\begin{cases} \hat{a}_{11}(t) = n \\ \hat{a}_{1,l_k-1}(t) = \hat{a}_{l_k-1,1}(t) = (s-k)t_{s-k}t \; (0 \le k \le s-1). \end{cases}$$

Moreover, when $i+j = l_k$, we have

$$\hat{a}_{ij}(t) = -ntt_{s-k} + t(s-k)t_{s-k} = -(l_k-2)t_{s-k}t \quad (2 \le i,j \le l_k-2,\; 0 \le k \le s). \tag{3.4}$$

Remark 3.3. *Note that, if* $s = n-1$*, we have*

$$-nt\sum_{k=0}^{s} t_{s-k}M_n(x^{n-1}, x^{s-k}) = -nt\sum_{k=1}^{s} t_{s-k}M_n(x^{n-1}, x^{s-k}),$$

Thus, when $i+j = l_k$*, equation* (3.4) *should be modified by*

$$\hat{a}_{ij}(t) = -ntt_{s-k} + t(s-k)t_{s-k} = -(l_k-2)t_{s-k}t \quad (2 \le i,j \le l_k-2,\; 1 \le k \le s).$$

We avoid this minor defect by considering that there is no entries satisfying $2 \le i,j \le l_0-2$ *when* $s = n-1$ *since* $l_0 - 2 = n-s = 1$.

Proposition 4. *Put* $l_k = n - s + k + 2$. *Then*

$$\hat{a}_{ij}(t) = \begin{cases} n & (i,j) = (1,1) \\ (s-k)t_{s-k}t & (i,j) = (1, l_k - 1) \text{ or } (l_k - 1, 1) \quad (0 \le k \le s-1) \\ -(l_k - 2)t_{s-k}t & i + j = l_k,\ 2 \le i, j \le l_k - 2,\ (0 \le k \le s) \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{a}_{ij}(t) = \begin{cases} b_{i-(n-s),\, j-(n-s)}t^2 & n - s + 1 \le i, j \le n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The statement for $\hat{a}_{ij}(t)$ has just been proved. For $\tilde{a}_{ij}(t)$, it is enough to see that we can denote

$$M_s(g(x)) = \sum_{\ell=0}^{s} \sum_{m=1}^{s} m t_\ell t_m M_s(x^\ell, x^{m-1}),$$
$$M_n(g(x)) = \sum_{\ell=0}^{s} \sum_{m=1}^{s} m t_\ell t_m M_n(x^\ell, x^{m-1}),$$

that is, we can obtain $M_n(g(x))$ from $M_s(g(x))$ by just replacing s with n for all $M_s(x^\ell, x^m)$, which, by Lemma 3.1, means that $s \times s$ matrix $M_s(g(x))$ occupies the part $\{b^\dagger_{ij} \mid n - s + 1 \le i, j \le n\}$ of the matrix $M_n(g(x)) = (b^\dagger_{ij})_{1 \le i,j \le n}$. ◻

By Proposition 4, we can express the matrix $A(t)$ as follows;

$$A(t) = \left[\begin{array}{cccc|cccc} n & 0 & \dots & 0 & st_st & (s-1)t_{s-1}t & \dots & t_1t \\ 0 & & & -(n-s)t_st & -(n-s+1)t_{s-1}t & \dots & -(n-1)t_1t & -nt_0t \\ \vdots & & \iddots & \iddots & & \iddots & \iddots & 0 \\ 0 & -(n-s)t_st & \iddots & & & \iddots & 0 & 0 \\ \hline st_st & -(n-s+1)t_{s-1}t & & & & & & \\ (s-1)t_{s-1}t & \vdots & \iddots & \iddots & & & & \\ \vdots & -(n-1)t_1t & \iddots & 0 & & C(t) & & \\ t_1t & -nt_0t & 0 & 0 & & & & \end{array}\right]. \tag{3.5}$$

Here, $C(t) = (c_{ij}(t))_{1 \le i,j \le s} = C(t_0, \cdots, t_s, t) = (c_{ij}(t_0, \cdots, t_s, t))_{1 \le i,j \le s}$ is an $s \times s$ symmetric matrix whose entries are of the form

$$\begin{aligned} c_{ij}(t_0, \cdots, t_s, t) &= b_{ij}t^2 + \lambda_{ij}t \\ &= b_{ij}(t_0, \cdots, t_s)t^2 + \lambda_{ij}(t_0, \cdots, t_s)t \quad (\lambda_{ij} = \lambda_{ij}(t_0, \cdots, t_s) \in E_1). \end{aligned}$$

Next, let $A(t)_1 = (a_{ij}(t)_1)_{1 \le i,j \le n} = A(t_0, \cdots, t_s, t)_1 = (a_{ij}(t_0, \cdots, t_s, t)_1)_{1 \le i,j \le n}$ be the $n \times n$ symmetric matrix obtained from $A(t)$ by multiplying the first row and the first column by $1/\sqrt{n}$ and then sweeping out the entries of the first row and the first column by the $(1,1)$ entry 1. Here, let $Q_m(k; c) = (q_{ij})_{1 \le i,j \le m}$ and $R_m(k, l; c) = (r_{ij})_{1 \le i,j \le m}$ be $m \times m$ elementary matrices such that

$$
Q_m(k;c)=\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix},\ R_m(k,l;c)=\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & c & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix},
$$

where $q_{kk} = c$ and $r_{kl} = c$. Moreover, for any $m \times m$ matrices $M_1, M_2, \cdots, M_l$, put $\prod_{k=1}^{l} M_k = M_1 M_2 \cdots M_l$. Then, we have $A(t)_1 = {}^tS(t)_1 A(t) S(t)_1$, where

$$
S(t)_1 = Q_n(1; 1/\sqrt{n}) \prod_{k=0}^{s-1} R_n(1, l_k - 1; -a_{1,l_k-1}(t)/\sqrt{n}).
$$

The matrix $A(t)_1$ can be expressed as follows;

$$
A(t)_1 = \left[\begin{array}{cccc|cccc}
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & -(n-s)t_s t & -(n-s+1)t_{s-1}t & \dots & -(n-1)t_1 t & -nt_0 t \\
\vdots & \vdots & \iddots & \iddots & & \iddots & \iddots & 0 \\
0 & -(n-s)t_s t & \iddots & & & \iddots & 0 & 0 \\
\hline
0 & -(n-s+1)t_{s-1}t & & & & & & \\
0 & \vdots & \iddots & \iddots & & & & \\
\vdots & -(n-1)t_1 t & \iddots & 0 & & C(t)_1 & & \\
0 & -nt_0 t & 0 & 0 & & & &
\end{array}\right]. \tag{3.6}
$$

Here, $C(t)_1 = (c_{ij}(t)_1)_{1 \le i,j \le s} = C(t_0, \cdots, t_s, t)_1 = (c_{ij}(t_0, \cdots, t_s, t)_1)_{1 \le i,j \le s}$ is an $s \times s$ symmetric matrix whose entries are of the form

$$
c_{ij}(t_0, \cdots, t_s, t)_1 = \bar{b}_{ij}(t_0, \cdots, t_s)t^2 + \lambda_{ij}(t_0, \cdots, t_s)t \quad (\bar{b}_{ij}(t_0, \cdots, t_s) \in E_1),
$$

where

$$
\bar{b}_{ij}(t_0, \cdots, t_s) = b_{ij}(t_0, \cdots, t_s) - \frac{(s-i+1)(s-j+1)}{n} t_{s-i+1} t_{s-j+1} \tag{3.7}
$$

for any i, j $(1 \le i, j \le s)$. We put $\bar{b}_{ij}(t_0, \cdots, t_s) = \bar{b}_{ij}$ and $\bar{B} = (\bar{b}_{ij})_{1 \le i,j \le s}$.

3.2 Some results for the Bezoutian of $f_r(t; x)$

Let $r = (r_0, \cdots, r_s) \in \mathbb{R}^{s+1}$ be a vector as in Theorem 3.2. We put

$$
\begin{aligned}
A_r(t) &= (a_{ij}^{(r)}(t))_{1 \le i,j \le n} = A(r_0, \cdots, r_s, t) \in \mathrm{Sym}_n(\mathbb{R}(t)), \\
B_r &= (b_{ij}^{(r)})_{1 \le i,j \le s} = B(r_0, \cdots, r_s) \in \mathrm{Sym}_s(\mathbb{R})
\end{aligned}
$$

and $B_r(t) = t^2 B_r$. Let us also put $A_r(t)_1 = A(r_0, \cdots, r_s, t)_1$. By equation (3.6), the matrix $A_r(t)_1$ can be expressed as follows;

$$A_r(t)_1 = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -(n-s)r_s t & -(n-s+1)r_{s-1}t & \cdots & -(n-1)r_1 t & -nr_0 t \\ \vdots & \vdots & \iddots & \iddots & & \iddots & \iddots & 0 \\ 0 & -(n-s)r_s t & \iddots & & & \iddots & 0 & 0 \\ \hline 0 & -(n-s+1)r_{s-1}t & & & & & & \\ 0 & \vdots & \iddots & \iddots & & & & \\ \vdots & -(n-1)r_1 t & \iddots & 0 & & C_r(t)_1 & & \\ 0 & -nr_0 t & 0 & 0 & & & & \end{array}\right].$$

Here, $C_r(t)_1 = (c_{ij}^{(r)}(t)_1)_{1 \le i,j \le s} = C(r_0, \cdots, r_s, t)_1$ and

$$c_{ij}^{(r)}(t)_1 = \bar{b}_{ij}(r_0, \cdots, r_s)t^2 + \lambda_{ij}(r_0, \cdots, r_s)t \quad (\bar{b}_{ij}(r_0, \cdots, r_s), \lambda_{ij}(r_0, \cdots, r_s) \in \mathbb{R}).$$

Note that, by equation (3.7), we have

$$\bar{b}_{ij}(r_0, \cdots, r_s) = b_{ij}^{(r)} - \frac{(s-i+1)(s-j+1)}{n} r_{s-i+1} r_{s-j+1} \quad (1 \le i, j \le s).$$

To ease notation, we put $\bar{b}_{ij}(r_0, \cdots, r_s) = \bar{b}_{ij}^{(r)}$ and $\bar{B}_r = (\bar{b}_{ij}^{(r)})_{1 \le i,j \le s}$.

In particular, since

$$\begin{aligned} M_s(g_r) &= M_s\left(r_s x^s, \sum_{k=0}^{s-1}(s-k)r_{s-k}x^{s-k-1}\right) + M_s\left(\sum_{k=1}^{s} r_{s-k}x^{s-k}, g_r'\right) \\ &= \sum_{k=0}^{s-1}(s-k)r_s r_{s-k} M_s(x^s, x^{s-k-1}) + M_s\left(\sum_{k=1}^{s} r_{s-k}x^{s-k}, g_r'\right), \end{aligned}$$

we have

$$b_{1,k+1}^{(r)} = b_{k+1,1}^{(r)} = (s-k)r_s r_{s-k} \ (0 \le k \le s-1) \tag{3.8}$$

by Lemma 3.1 and hence

$$\begin{aligned} \bar{b}_{1j}^{(r)} &= (s-j+1)r_s r_{s-j+1} - \frac{s(s-j+1)}{n} r_s r_{s-j+1} \\ &= (s-j+1)\left(1 - \frac{s}{n}\right) r_s r_{s-j+1} \ (1 \le j \le s). \end{aligned} \tag{3.9}$$

Lemma 3.2. *Put* $\bar{B}_r(t) = t^2 \bar{B}_r$. *Then,* $B_r(\xi)$ *and* $\bar{B}_r(\xi)$ *are equivalent over* $\mathbb{R}$ *for any real number* ξ *and we have* $\sigma(\bar{B}_r(\xi)) = N_{g_r}$ *for any non-zero real number* ξ.

Proof. Let us denote by $B_r^* = (b_{ij}^{(r,*)})_{1 \le i,j \le s}$ $(\bar{B}_r^* = (\bar{b}_{ij}^{(r,*)})_{1 \le i,j \le s})$ the matrix obtained from B_r $(\bar{B}_r)$ by multiplying the first row and the first column by $1 \big/ \pm\sqrt{b_{11}^{(r)}}$ $\left(1 \big/ \pm\sqrt{\bar{b}_{11}^{(r)}}\right)$ (the sign before $\sqrt{b_{11}^{(r)}}$ $\left(\sqrt{\bar{b}_{11}^{(r)}}\right)$ are the same as the sign of r_s; see the definition of d $\left(\bar{d}\right)$ below) and then sweeping out the entries of the first row and the first column by the $(1,1)$ entry 1. Since $b_{11} = sr_s^2$ (> 0) and $\bar{b}_{11} = s(1 - s/n)r_s^2$ (> 0) by (3.8) and (3.9), we have

$$B_r^* = {}^t T B_r T, \ \bar{B}_r^* = {}^t \bar{T} \bar{B}_r \bar{T}, \tag{3.10}$$

where

$$T = Q_s(1; 1/d) \prod_{k=2}^{s} R_s(1, k; -b_{1k}^{(r)}/d) \ (d = \sqrt{s} \cdot r_s),$$
$$\bar{T} = Q_s(1; 1/\bar{d}) \prod_{k=2}^{s} R_s(1, k; -\bar{b}_{1k}^{(r)}/\bar{d}) \ (\bar{d} = \sqrt{s(1 - s/n)} \cdot r_s).$$

Note that in [13, Lemma 3.3], we have proved $b_{ij}^{(r,*)} = \bar{b}_{ij}^{(r,*)}$ $(1 \le i, j \le s)$ and hence $t^2 B_r^* = t^2 \bar{B}_r^*$, which, by (3.10), implies that symmetric matrices $B_r(\xi)$ and $\bar{B}_r(\xi)$ are equivalent over $\mathbb{R}$ for any real number ξ. Then, since $N_{g_r} = \sigma(B_r) = \sigma(B_r(\xi))$ for any $\xi \in \mathbb{R} \setminus \{0\}$, the latter half of the statement have also been proved. □

3.3 Nonvanishingness of some coefficients

In this subsection, we prove the next lemma.

Lemma 3.3. *Let*

$$\Phi(x) = \Phi(t_0, \cdots, t_s; x) = \sum_{k=0}^{s} h_{s-k}(t_0, \cdots, t_s) x^{s-k} \in E_1[x] \tag{3.11}$$

be the characteristic polynomial of $\bar{B}$*. Then,* $h_{s-k}(t_0, \cdots, t_s)$ *is a non-zero polynomial in* E_1 *for any* k $(1 \le k \le s)$.

Proof. Lemma 3.3 is clear for $s = 1$, since we have

$$B = M_1(t_1 x + t_0) = \left[\ t_1^2 \ \right]$$

and hence, by equation (3.7),

$$\bar{B} = \left[\ t_1^2 - \frac{1}{n} t_1^2 \ \right] = \left[\ \frac{n-1}{n} t_1^2 \ \right].$$

Next, suppose $s \ge 2$. Then, by equation (3.7) and the definition of the Bezoutian, we have $h_{s-k}(t_0, \cdots, t_s) \in \mathbb{R}[t_0, \cdots, t_s]$ for any k $(1 \le k \le s)$. Thus, we have only to prove that $h_{s-k}(t_0, \cdots, t_s) \neq 0$ for any k $(1 \le k \le s)$, which is clear from the next Lemma 3.4. □

Lemma 3.4. *Suppose* $s \ge 2$ *and put* $u_0 = u_s = 1$, $u_1 = t_1$ *and* $u_k = 0$ $(2 \le k \le s-1)$*. Then,* $h_{s-k}(u_0, \cdots, u_s)$ *is a non-constant polynomial in* $\mathbb{R}(t_1)$ *for any* k $(1 \le k \le s)$*, i.e.,* $h_{s-k}(u_0, \cdots, u_s) \in \mathbb{R}[t_1] \setminus \mathbb{R}$ $(1 \le k \le s)$.

To prove lemma 3.4, let us put $u = (u_0, \cdots, u_s)$ and

$$
\begin{aligned}
&g_u(x) = g(u_0, \cdots, u_s; x) = x^s + t_1 x + 1 \in \mathbb{R}(t_1)[x],\\
&f_u(t;x) = x^n + t g_u(x) \in \mathbb{R}(t_1, t)[x] \ (n > s),\\
&A_u(t) = (a_{ij}^{(u)}(t))_{1\le i,j\le n} = A(u_0, \cdots, u_s, t) \in \mathrm{Sym}_n(\mathbb{R}(t_1, t)),\\
&B_u = (b_{ij}^{(u)})_{1\le i,j\le s} = B(t_0, \cdots, u_s) \in \mathrm{Sym}_s(\mathbb{R}(t_1)),\ B_u(t) = t^2 B_u.
\end{aligned}
$$

Then, by equation (3.5), we have

$$
A_u(t) = \left[\begin{array}{cccc|cccc}
n & 0 & \dots & 0 & st & 0 & \dots & t_1 t\\
0 & & & -(n-s)t & 0 & \dots & -(n-1)t_1 t & -nt\\
\vdots & & \iddots & \iddots & \iddots & \iddots & & 0\\
0 & -(n-s)t & \iddots & & \iddots & & 0 & 0\\
\hline
st & 0 & & & & & & \\
0 & \vdots & \iddots & \iddots & & & & \\
\vdots & -(n-1)t_1 t & \iddots & 0 & & C_u(t) & & \\
t_1 t & -nt & 0 & 0 & & & &
\end{array}\right],
$$

where $C_u(t) = (c_{ij}^{(u)}(t))_{1\le i,j\le s} = C(u_0, \cdots, u_s, t)$ and

$$
c_{ij}^{(u)}(t) = b_{ij}(u_0, \cdots, u_s)t^2 + \lambda_{ij}(u_0, \cdots, u_s)t \quad (\lambda_{ij}(u_0, \cdots, u_s) \in \mathbb{R}(t_1)).
$$

Moreover, by equation (3.6), we also have

$$
A_u(t)_1 = \left[\begin{array}{cccc|cccc}
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0\\
0 & 0 & \dots & -(n-s)t & 0 & \dots & -(n-1)t_1 t & -nt\\
\vdots & \vdots & \iddots & \iddots & \iddots & \iddots & & 0\\
0 & -(n-s)t & \iddots & & \iddots & & 0 & 0\\
\hline
0 & 0 & & & & & & \\
0 & \vdots & \iddots & \iddots & & & & \\
\vdots & -(n-1)t_1 t & \iddots & 0 & & C_u(t)_1 & & \\
0 & -nt & 0 & 0 & & & &
\end{array}\right].
$$

Here, $C_u(t)_1 = (c_{ij}^{(u)}(t)_1)_{1\le i,j\le s} = C(u_0, \cdots, u_s, t)_1$ and

$$
c_{ij}^{(u)}(t)_1 = \bar{b}_{ij}(u_0, \cdots, u_s)t^2 + \lambda_{ij}(u_0, \cdots, u_s)t \quad (\bar{b}_{ij}(u_0, \cdots, u_s) \in \mathbb{R}).
$$

Note that, by equation (3.7), we have

$$
\bar{b}_{ij}^{(u)} = \begin{cases}
b_{11}^{(u)} - (s^2/n) & (i,j) = (1,1)\\
b_{1s}^{(u)} - (s/n)t_1 & (i,j) = (1,s) \text{ or } (s,1)\\
b_{ss}^{(u)} - (1/n)t_1^2 & (i,j) = (s,s)\\
b_{ij}^{(u)} & \text{otherwise.}
\end{cases} \tag{3.12}
$$

Let us put $\bar{B}_u = (\bar{b}_{ij}^{(u)})_{1\le i,j\le s}$ and $\bar{B}_u(t) = t^2\bar{B}_u$. Then, since

$$\begin{aligned} M_s(g_u) &= M_s(x^s + t_1x + 1, sx^{s-1} + t_1) \\ &= sM_s(x^s, x^{s-1}) + t_1M_s(x^s, 1) - st_1M_s(x^{s-1}, x) - sM_s(x^{s-1}, 1) \\ &\qquad + t_1^2M_s(x, 1) + t_1M_s(1, 1), \end{aligned}$$

we have

(a) if $s = 2$,

$$B_u = \begin{bmatrix} 2 & t_1 \\ t_1 & t_1^2 - 2 \end{bmatrix},$$

(b) if $s \ge 3$,

$$b_{ij}^{(u)} = \begin{cases} s & (i,j) = (1,1) \\ t_1 & (i,j) = (1,s) \text{ or } (s,1) \\ (1-s)t_1 & i+j = s+1,\ 2 \le i,j \le s-1 \\ -s & i+j = s+2 \\ t_1^2 & (i,j) = (s,s), \\ 0 & \text{otherwise,} \end{cases}$$

which, by equation (3.12), implies

(a′) if $s = 2$,

$$\bar{B}_u = \begin{bmatrix} 2(n-2)/n & (n-2)t_1/n \\ (n-2)t_1/n & (n-1)t_1^2/n - 2 \end{bmatrix},$$

(b′) if $s \ge 3$,

$$\bar{b}_{ij}^{(u)} = \begin{cases} s(n-s)/n & (i,j) = (1,1) \\ (n-s)t_1/n & (i,j) = (1,s) \text{ or } (s,1) \\ (1-s)t_1 & i+j = s+1,\ 2 \le i,j \le s-1 \\ -s & i+j = s+2 \\ (n-1)t_1^2/n & (i,j) = (s,s), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $s \ge 3$, the matrix $\bar{B}_u = (\bar{b}_{ij}^{(u)})_{1\le i,j\le s}$ has the expression of the form

$$\begin{bmatrix} s(n-s)/n & 0 & 0 & 0 & \cdots & 0 & (n-s)t_1/n \\ 0 & 0 & 0 & \cdots & 0 & (1-s)t_1 & -s \\ 0 & 0 & & \iddots & (1-s)t_1 & -s & 0 \\ 0 & \vdots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & 0 & (1-s)t_1 & \iddots & \iddots & & 0 \\ 0 & (1-s)t_1 & -s & \iddots & & & 0 \\ (n-s)t_1/n & -s & 0 & \cdots & 0 & 0 & (n-1)t_1^2/n \end{bmatrix}.$$

Here, let us denote by

$$\Phi_u(x) = \sum_{k=0}^{s} h_{s-k}^{(u)} x^{s-k} = \Phi(u_0, \cdots, u_s; x) \ \left(= \sum_{k=0}^{s} h_{s-k}(u_0, \cdots, u_s) x^{s-k} \right)$$

the characteristic polynomial of $\bar{B}_u$. Note that since we have $h_{s-k}^{(u)} \in \mathbb{R}[t_1]$ by the proof of Lemma 3.3, we have only to prove $h_{s-k}^{(u)}$ is non-constant for any k $(1 \leq k \leq s)$.

By the above expression of $\bar{B}_u$, we have

(a'') if $s = 2$,

$$\Phi_u(x) = x^2 - \frac{(n-1)t_1^2 - 4}{n} x + \frac{(n-2)t_1^2 - 4n + 8}{n},$$

(b'') if $s \geq 3$,

$$\Phi_u(x) = \begin{cases} \begin{vmatrix} x - s(n-s)/n & & & & & & & & -(n-s)t_1/n \\ & x & & & & & & (s-1)t_1 & s \\ & & \ddots & & & & \iddots & s & \\ & & & \ddots & & \iddots & \iddots & & \\ & & & & x + (s-1)t_1 & s & & & \\ & & & \iddots & s & x & & & \\ & & \iddots & \iddots & & & \ddots & & \\ & (s-1)t_1 & s & & & & & x & \\ -(n-s)t_1/n & s & & & & & & & x - (n-1)t_1^2/n \end{vmatrix} & (s \text{ is odd}), \\ \begin{vmatrix} x - s(n-s)/n & & & & & & & -(n-s)t_1/n \\ & x & & & & & (s-1)t_1 & s \\ & & \ddots & & & \iddots & s & \\ & & & x & (s-1)t_1 & \iddots & & \\ & & & (s-1)t_1 & x + s & & & \\ & & \iddots & \iddots & & \ddots & & \\ & (s-1)t_1 & s & & & & x & \\ -(n-s)t_1/n & s & & & & & & x - (n-1)t_1^2/n \end{vmatrix} & (s \text{ is even}). \end{cases}$$

Example 3.1. (1) *Put* $s = 7$ *and* $n = 10$. *Then, we have*

$$g_u(x) = x^7 + t_1 x + 1, \ \ f_u(t; x) = x^{10} + t(x^7 + t_1 x + 1),$$

$$\Phi_u(x) = \begin{vmatrix} x-21/10 & 0 & 0 & 0 & 0 & 0 & -3t_1/10 \\ 0 & x & 0 & 0 & 0 & 6t_1 & 7 \\ 0 & 0 & x & 0 & 6t_1 & 7 & 0 \\ 0 & 0 & 0 & x+6t_1 & 7 & 0 & 0 \\ 0 & 0 & 6t_1 & 7 & x & 0 & 0 \\ 0 & 6t_1 & 7 & 0 & 0 & x & 0 \\ -3t_1/10 & 7 & 0 & 0 & 0 & 0 & x-9t_1^2/10 \end{vmatrix}$$

$$\begin{aligned} =&x^7 + \left(-\frac{9}{10}t_1^2 + 6t_1 - \frac{21}{10}\right)x^6 + \left(-\frac{27}{5}t_1^3 - \frac{351}{5}t_1^2 - \frac{63}{5}t_1 - 147\right)x^5 \\ &+ \left(\frac{324}{5}t_1^4 - \frac{2106}{5}t_1^3 + \frac{1197}{5}t_1^2 - 588t_1 + \frac{3087}{10}\right)x^4 \\ &+ \left(\frac{1944}{5}t_1^5 + \frac{5832}{5}t_1^4 + \frac{5859}{5}t_1^3 + \frac{16758}{5}t_1^2 + \frac{6174}{5}t_1 + 7203\right)x^3 \\ &+ \left(-\frac{5832}{5}t_1^6 + \frac{34992}{5}t_1^5 - \frac{21546}{5}t_1^4 + \frac{50274}{5}t_1^3 - \frac{95697}{10}t_1^2 + 14406t_1 - \frac{151263}{10}\right)x^2 \end{aligned}$$

$$\begin{aligned} &+ \left(-\frac{34992}{5}t_1^7 + \frac{11664}{5}t_1^6 - \frac{81648}{5}t_1^5 + \frac{15876}{5}t_1^4 - \frac{111132}{5}t_1^3 + \frac{21609}{5}t_1^2 - \frac{151263}{5}t_1 \right. \\ &\left. - 117649\right)x + \frac{69984}{5}t_1^7 + \frac{2470629}{10}. \end{aligned}$$

(2) *Put* $s = 8$ *and* $n = 12$*. Then, we have*

$$g_u(x) = x^8 + t_1x + 1,\ f_u(t;x) = x^{12} + t(x^8 + t_1x + 1)$$

and

$$\Phi_u(x) = \begin{vmatrix} x-8/3 & 0 & 0 & 0 & 0 & 0 & 0 & -t_1/3 \\ 0 & x & 0 & 0 & 0 & 0 & 7t_1 & 8 \\ 0 & 0 & x & 0 & 0 & 7t_1 & 8 & 0 \\ 0 & 0 & 0 & x & 7t_1 & 8 & 0 & 0 \\ 0 & 0 & 0 & 7t_1 & x+8 & 0 & 0 & 0 \\ 0 & 0 & 7t_1 & 8 & 0 & x & 0 & 0 \\ 0 & 7t_1 & 8 & 0 & 0 & 0 & x & 0 \\ -t_1/3 & 8 & 0 & 0 & 0 & 0 & 0 & x-11t_1^2/12 \end{vmatrix}$$

$$\begin{aligned} =&x^8 + \left(-\frac{11}{12}t_1^2 + \frac{16}{3}\right)x^7 + \left(-152t_1^2 - \frac{640}{3}\right)x^6 + \left(\frac{539}{4}t_1^4 - 256t_1^2 - 1024\right)x^5 \\ &+ \left(\frac{22736}{3}t_1^4 + \frac{45824}{3}t_1^2 + 16384\right)x^4 + \left(-\frac{26411}{4}t_1^6 - \frac{22736}{3}t_1^4 + \frac{31744}{3}t_1^2 + 65536\right)x^3 \\ &+ \left(-\frac{355348}{3}t_1^6 - 213248t_1^4 - \frac{1064960}{3}t_1^2 - 524288\right)x^2 + \left(\frac{1294139}{12}t_1^8 + \frac{1075648}{3}t_1^6 \right. \\ &\left. + \frac{1404928}{3}t_1^4 + \frac{1835008}{3}t_1^2 - \frac{4194304}{3}\right)x - \frac{823543}{3}t_1^8 + \frac{16777216}{3}. \end{aligned}$$

Proof of Lemma 3.4. To prove Lemma 3.4, it is enough to prove $\deg h_{s-k}^{(u)} \geq 1$ for any k $(1 \leq k \leq s)$. This is clear for $s = 2$ by (a'') and we suppose $s \geq 3$ hereafter. To prove $\deg h_{s-k}^{(u)} \geq 1$ $(1 \leq k \leq s)$, let us compute the leading term of $h^{(u)}_{s-k}$ ($\in \mathbb{R}[t_1]$). Then, since $h^{(u)}_{s-k}$ is the coefficient of the term $h^{(u)}_{s-k}x^{s-k}$ of the characteristic polynomial $\Phi_u(x)$, we need to maximize the degree in t_1 when we take '$s-k$' x and the remaining k elements from $\mathbb{R}[t_1]$.

(a) Suppose s is odd. Let us divide the case into three other sub-cases.

$(a1)$ Suppose k is odd and $1 \le k \le s-2$.
In this case, the degree of the leading term of $h^{(u)}_{s-k}$ is $k+1$. In fact, it is obtained by taking

$(a11)$ $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

$(a12)$ '$k-1$' $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le s-1)$.

First, suppose we take the (s,s) entry $x-(n-1)t_1^2/n$ from the s-th row. Then we must take the $(1,1)$ entry from the first row. Next, let us proceed to the $(s-1)$-th row. If we take the $(s-1,s-1)$ entry x from the $(s-1)$-th row, then we must also take x from the second row, while if we take $(s-1)t_1$ from the $(s-1)$-th row, then we must also take $(s-1)t_1$ from the second row. The situation is the same for the $(s-2)$-th row, the $(s-3)$-th row ... and so on, which implies that $(s-1)t_1$ must occur in pair.

Hence, the leading term of $h^{(u)}_{s-k}$ is

$$-\frac{n-1}{n}t_1^2 \cdot \binom{(s-3)/2}{(k-1)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-1)/2} \qquad \left(\binom{n}{0}=1\ (n\ge 0)\right)$$

and the degree of this term is $k+1$ (≥ 2).
$(a2)$ Suppose k is odd and $k=s$.
If $k=s$, $h^{(u)}_{s-k}=h^{(u)}_0$ is the constant term of $\Phi_u(x)$. In this case, the degree of the leading term of $h^{(u)}_0$ is s. In fact, it is obtained by taking

$(a21)$ $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

$(a22)$ If $s \ge 5$ ($\Leftrightarrow (s,k) \ne (3,3)$), '$(s-3)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le (s-1)/2,\ (s+3)/2 \le i \le s-1)$,

$(a23)$ $(s-1)t_1$ from the $((s+1)/2,(s+1)/2)$ entry $x+(s-1)t_1$,

$(a24)$ $-s(n-s)/n$ from the $(1,1)$ entry $x-s(n-s)/n$

or by taking

$(a25)$ all anti-diagonal entries.

Therefore, the leading term of $h_0^{(u)}$ is

$$\begin{aligned}
&-\frac{n-1}{n}t_1^2 \cdot \{(-1)\cdot(s-1)^2t_1^2\}^{(s-3)/2}\cdot(s-1)t_1\cdot\left(-\frac{s(n-s)}{n}\right)\\
&\qquad\qquad + (-1)\cdot\left(-\frac{n-s}{n}t_1\right)^2\cdot\{(-1)\cdot(s-1)^2t_1^2\}^{(s-3)/2}\cdot(s-1)t_1\\
&=\frac{(n-s)(s-1)}{n}\cdot(-1)^{(s-3)/2}(s-1)^{s-2}t_1^s\\
&=(-1)^{(s-3)/2}\frac{(n-s)(s-1)^{s-1}}{n}t_1^s
\end{aligned}$$

for any s $(s \geq 3)$ and the degree of this term is s.

$(a3)$ Suppose k is even.
In this case, we have $2 \leq k \leq s-1$ and the degree of the leading term of $h_{s-k}^{(u)}$ is $k+1$. In fact, it is obtained by taking

$(a31)$ $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

$(a32)$ If $s \geq 5$ $(\Leftrightarrow (s,k) \neq (3,2))$, '$(k-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \leq i \leq (s-1)/2,\ (s+3)/2 \leq i \leq s-1)$,

$(a33)$ $(s-1)t_1$ from the $((s+1)/2,(s+1)/2)$ entry $x+(s-1)t_1$.

Therefore, the leading term of $h_{s-k}^{(u)}$ is

$$-\frac{n-1}{n}t_1^2\cdot\binom{(s-3)/2}{(k-2)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2}\cdot(s-1)t_1$$

for any s $(s \geq 3)$ and the degree of this term is $k+1$ (≥ 3).

(b) Suppose s is even $(s \geq 4)$. We also divide this case into three other sub-cases.

$(b1)$ Suppose k is odd.
In this case, we have $1 \leq k \leq s-1$ and the degree of the leading term of $h_{s-k}^{(u)}$ is $k+1$. In fact, it is obtained by taking

$(b11)$ $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

$(b12)$ '$(k-1)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \leq i \leq s-1)$.

Therefore, the leading term of $h_{s-k}^{(u)}$ is

$$-\frac{n-1}{n}t_1^2\cdot\binom{(s-2)/2}{(k-1)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-1)/2}$$

and the degree of this term is $k+1$ (≥ 2).

$(b2)$ Suppose k is even and $2 \leq k \leq s-2$.
In this case, the degree of the leading term of $h_{s-k}^{(u)}$ is k. In fact, it is obtained by taking

(b21) $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

(b22) '$(k-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le s-1)$,

(b23) $-s(n-s)/n$ from the $(1,1)$ entry $x - s(n-s)/n$

or by taking

(b24) $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

(b25) I f $s \ge 6$ ($\Leftrightarrow (s,k) \ne (4.2)$), '$(k-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le (s-2)/2,\ (s+4)/2 \le i \le s-1)$,

(b26) s from the $((s+2)/2, (s+2)/2)$ entry $x+s$

or by taking

(b27) '$k/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le s-1)$

or by taking

(b28) One pair of $-(n-s)t_1/n$ from the $(1,s)$ and the $(s,1)$ entry,

(b29) '$(k-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \le i \le s-1)$.

Here, note that if we take the $(s,1)$ entry $-(n-s)t_1/n$ from the s-th row, we must also take the $(1,s)$ entry $-(n-s)t_1/n$ from the first row.

Therefore, the leading term of $h_{s-k}^{(u)}$ is

$$
\begin{aligned}
&-\frac{n-1}{n}t_1^2 \cdot \binom{(s-2)/2}{(k-2)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2}\cdot\left(-\frac{s(n-s)}{n}\right)\\
&\quad -\frac{n-1}{n}t_1^2 \cdot \binom{(s-4)/2}{(k-2)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2}\cdot s + \binom{(s-2)/2}{k/2}\{(-1)\cdot(s-1)^2t_1^2\}^{k/2}\\
&\quad +\left((-1)\cdot\frac{\{-(n-s)\}^2}{n^2}t_1^2\right)\cdot\binom{(s-2)/2}{(k-2)/2}\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2}\\
&=\left(\frac{s(n-s)(n-1)}{n^2}\binom{(s-2)/2}{(k-2)/2} - \frac{s(n-1)}{n}\binom{(s-4)/2}{(k-2)/2}\right.\\
&\qquad\qquad \left. -(s-1)^2\binom{(s-2)/2}{k/2} - \frac{(n-s)^2}{n^2}\binom{(s-2)/2}{(k-2)/2}\right)\{(-1)\cdot(s-1)^2t_1^2\}^{(k-2)/2}t_1^2.
\end{aligned}
$$

for any s $(s \ge 4)$. Then, since

$$
\binom{(s-4)/2}{(k-2)/2} = \frac{s-k}{s-2}\binom{(s-2)/2}{(k-2)/2},\quad \binom{(s-2)/2}{k/2} = \frac{s-k}{k}\binom{(s-2)/2}{(k-2)/2},
$$

we have

$$\begin{aligned} &\frac{s(n-s)(n-1)}{n^2}\binom{(s-2)/2}{(k-2)/2} - \frac{s(n-1)}{n}\binom{(s-4)/2}{(k-2)/2} \\ &\qquad - (s-1)^2\binom{(s-2)/2}{k/2} - \frac{(n-s)^2}{n^2}\binom{(s-2)/2}{(k-2)/2} \\ &= \left(\frac{s(n-s)(n-1)}{n^2} - \frac{s(s-k)(n-1)}{n(s-2)} - \frac{(s-1)^2(s-k)}{k} - \frac{(n-s)^2}{n^2}\right)\binom{(s-2)/2}{(k-2)/2} \\ &= \frac{s\{\big(k(k+s^2-4s+2)-s^3+4s^2-5s+2\big)n - k(k+s^2-4s+2)\}}{nk(s-2)}\binom{(s-2)/2}{(k-2)/2}. \end{aligned} \tag{3.13}$$

Hence, if the above value becomes zero, we have

$$\big(k(k+s^2-4s+2)-s^3+4s^2-5s+2\big)n - k(k+s^2-4s+2) = 0,$$

which implies

$$k(k+s^2-4s+2) = 0, \ -s^3+4s^2-5s+2 = 0 \tag{3.14}$$

or

$$n = \frac{k(k+s^2-4s+2)}{k(k+s^2-4s+2)-s^3+4s^2-5s+2}. \tag{3.15}$$

Here, (3.14) is impossible since $-s^3+4s^2-5s+2 = -(s-1)^2(s-2)$ and $s \geq 4$. Also, (3.15) is impossible since, for any $s \geq 4$ and $2 \leq k \leq s-2$, we have

$$k(k+s^2-4s+2) \geq 2(2+s^2-4s+2) \geq 2(s-2)^2 > 0$$

and

$$\begin{aligned} &k(k+s^2-4s+2)-s^3+4s^2-5s+2 \\ &\leq (s-2)\{(s-2)+s^2-4s+2\}-s^3+4s^2-5s+2 \\ &= -s^2+s+2 \\ &= -(s+1)(s-2) < 0, \end{aligned}$$

which implies $n < 0$, a contradiction. Thus, the above value (3.13) is non-zero and the degree of the leading term of $h_{s-k}^{(u)}$ is k.

(b3) Suppose k is even and $k = s$.

If $k = s$, $h_{s-k}^{(u)} = h_0^{(u)}$ is the constant term of $\Phi_u(x)$. In this case, the degree of the leading term of $h_0^{(u)}$ is s. In fact, it is obtained by taking

(b31) $-(n-1)t_1^2/n$ from the (s,s) entry $x-(n-1)t_1^2/n$,

(b32) '$(s-2)/2$' pairs of $(s-1)t_1$ from entries of the form $(i, s+1-i)$ $(2 \leq i \leq s-1)$,

(b33) $-s(n-s)/n$ from the $(1,1)$ entry $x-s(n-s)/n$

or by taking

(b34) all anti-diagonal entries.

Therefore, the leading term of $h_0^{(u)}$ is

$$\begin{aligned} &-\frac{n-1}{n}t_1^2\cdot\{(-1)\cdot(s-1)^2t_1^2\}^{(s-2)/2}\cdot\left(-\frac{s(n-s)}{n}\right)\\ &\qquad\qquad + (-1)\cdot\left(-\frac{n-s}{n}t_1\right)^2\cdot\{(-1)\cdot(s-1)^2t_1^2\}^{(s-2)/2}\\ &=(-1)^{(s-2)/2}\frac{(n-s)(s-1)^{s-1}}{n}t_1^s \end{aligned}$$

and the degree of this term is s ($s\geq 4$). ☐

Lemma 3.5. *Let* $v=(v_0,\cdots,v_s)\in\mathbb{R}^{s+1}$ *be a real vector and* n $(>s)$ *be an integer. Put*

$$P_v(t)=\det M_n(f_v(t;x))=\det M_n(f^{(n)}(v_0,\cdots,v_s,t;x))$$

and $\alpha_v=\max\{\alpha\in\mathbb{R}\mid P_v(\alpha)=0\}$*. If there exists a real number* ρ_0 $(>\alpha_v)$ *such that* $N_{f_v(\xi;x)}=\gamma_0$ *for any* $\xi>\rho_0$*, we have* $N_{f_v(\xi;x)}=\gamma_0$ *for any* $\xi>\alpha_v$*.*

Proof. Put $A_v(t)=M_n(f_v(t;x))$. Then, by Proposition 2, we have $\gamma_0=\sigma(A_v(\xi))$ for any $\xi>\rho_0$. Let us also put

$$R=\{\rho\in\mathbb{R}\mid\rho>\alpha_v,\ \sigma(A_v(\xi))=\gamma_0 \text{ for any } \xi>\rho\}.$$

Since R is a nonempty set ($\rho_0\in R$) having a lower bound α_v, R has the infimum ρ_v; $\rho_v=\inf R$. Then, it is enough to prove $\rho_v=\alpha_v$. Here, suppose to the contrary that $\rho_v>\alpha_v$ and we denote by

$$\Omega_v(t;x)=\sum_{k=0}^{n}\omega_k(t)x^k\in\mathbb{R}(t)[x]$$

the characteristic polynomial of $A_v(t)$. Note that $\omega_k(t)\in\mathbb{R}[t]$ $(0\leq k\leq n)$ and for any $\xi>\alpha_v$, $\Omega_v(\xi;x)$ has n non-zero real roots (counted with multiplicity) since $A_v(\xi)$ is symmetric and $\det A_v(\xi)\neq 0$. Then, by Proposition 3, there exists a positive real number δ such that $\rho_v-\delta>\alpha_v$ and for any $\xi\in[\rho_v-\delta,\rho_v+\delta]$, $\Omega_v(\xi;x)$ has the same number of positive and hence negative real roots with $\Omega_v(\rho_v;x)$. On the other hand, since $\rho_v=\inf R$, there exist real numbers ξ_+ $(\rho_v<\xi_+<\rho_v+\delta)$ and ξ_- $(\rho_v-\delta<\xi_-<\rho_v)$ such that $\sigma(A_v(\xi_+))\neq\sigma(A_v(\xi_-))$, which implies $\Omega_v(\xi_+;x)$ and $\Omega_v(\xi_-;x)$ have different number of positive and hence negative real roots. This is a contradiction and we have $\rho_v=\alpha_v$. ☐

3.4 Proof of Theorem 3.2

Let $r=(r_0,\cdots,r_s)\in\mathbb{R}^{s+1}$ be the vector as in Theorem 3.2 and put

$$n_0=\begin{cases}(n-s+1)/2, & n-s-1 : \text{even}\\ (n-s+2)/2, & n-s-1 : \text{odd}.\end{cases}$$

When $n-s \geq 2$, we inductively define the matrix $A_r(t)_k = (a_{ij}^{(r)}(t)_k)_{1\leq i,j\leq n}$ $(2 \leq k \leq n-s)$ as the matrix obtained from $A_r(t)_{k-1}$ by sweeping out the entries of the k-th row (k-th column) by the (k, l_0-k) entry $-(n-s)r_s t$ ((l_0-k,k) entry $-(n-s)r_s t$). That is, we define $A_r(t)_k = {}^t S_r(t)_k A_r(t)_{k-1} S_r(t)_k$, where

$$S_r(t)_k = \begin{cases} \displaystyle\prod_{m=l_0-k+1}^{n} R_n\left(l_0-k, m; -\frac{a_{km}^{(r)}(t)_{k-1}}{-(n-s)r_s t}\right) & (2 \leq k \leq n_0) \\ \displaystyle R_n\left(l_0-k, k; -\frac{a_{kk}^{(r)}(t)_{k-1}}{-2(n-s)r_s t}\right) \prod_{m=k+1}^{n} R_n\left(l_0-k, m; -\frac{a_{km}^{(r)}(t)_{k-1}}{-(n-s)r_s t}\right) & \\ & (n_0 < k \leq n-s). \end{cases}$$

Then, if $n-s \geq 1$, we can express the matrix $A_r(t)_{n-s}$ as follows;

$$A_r(t)_{n-s} = \left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & \\ 0 & 0 & \dots & -(n-s)r_s t & \\ \vdots & \vdots & \iddots & 0 & O \\ 0 & -(n-s)r_s t & 0 & 0 & \\ \hline & & O & & C_r(t)_{n-s} \end{array}\right].$$

Note that $a_{km}^{(r)}(t)_{k-1}$ and $a_{kk}^{(r)}(t)_{k-1}$ appearing in $S_r(t)_k$ are degree 1 monomials in t and hence the numbers $-a_{km}^{(r)}(t)_{k-1}/(-(n-s)r_s t)$, $-a_{kk}^{(r)}(t)_{k-1}/(-2(n-s)r_s t)$ appearing in $S_r(t)_k$ are just real numbers. Therefore, the entries of the $s \times s$ symmetric matrix $C_r(t)_{n-s} = (c_{ij}^{(r)}(t)_{n-s})_{1\leq i,j\leq s}$ $(n-s \geq 1)$ are of the form

$$c_{ij}^{(r)}(t)_{n-s} = \bar{b}_{ij}^{(r)} t^2 + \bar{\lambda}_{ij}^{(r)} t \quad (\bar{\lambda}_{ij}^{(r)} \in \mathbb{R}). \tag{3.16}$$

Moreover, since the matrix

$$D_r(t)_{n-s} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & -(n-s)r_s t \\ \vdots & \vdots & \iddots & 0 \\ 0 & -(n-s)r_s t & 0 & 0 \end{bmatrix}$$

is equivalent to the matrix

$$\bar{D}_r(t)_{n-s} = \begin{cases} \begin{bmatrix} 1 & & & \\ & \begin{matrix} 0 & -(n-s)r_s t \\ -(n-s)r_s t & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & -(n-s)r_s t \\ -(n-s)r_s t & 0 \end{matrix} \end{bmatrix} & (n-s : \text{odd}) \\ \begin{bmatrix} 1 & & & & \\ & -(n-s)r_s t & & & \\ & & \begin{matrix} 0 & -(n-s)r_s t \\ -(n-s)r_s t & 0 \end{matrix} & & \\ & & & \ddots & \\ & & & & \begin{matrix} 0 & -(n-s)r_s t \\ -(n-s)r_s t & 0 \end{matrix} \end{bmatrix} & (n-s : \text{even}) \end{cases}$$

over $\mathbb{R}$, we have

$$\sigma(D_r(\xi)_{n-s}) = \sigma(\bar{D}_r(\xi)_{n-s}) = \begin{cases} 1 & n-s : \text{odd} \\ 0 & n-s : \text{even}, \ r_s > 0 \\ 2 & n-s : \text{even}, \ r_s < 0 \end{cases} \tag{3.17}$$

for any real number $\xi > \alpha_r$ (≥ 0). Here, note that since $P_r(0) = 0$, we have $\alpha_r \geq 0$.

Next, let $\Phi_r(t;x)$, $\Psi_r(t;x)$ be characteristic polynomials of $\bar{B}_r(t)$, $C_r(t)_{n-s}$, respectively. Then, by equations (3.11) and (3.16), we have

$$\begin{aligned} \Phi_r(t;x) &= x^s + h_{s-1}^{(r)} t^2 x^{s-1} + \cdots + h_1^{(r)} t^{2s-2} x + h_0^{(r)} t^{2s} \\ &\qquad \left(h_{s-k}^{(r)} = h_{s-k}(r_0, \cdots, r_s) \in \mathbb{R} \ (1 \leq k \leq s) \right), \\ \Psi_r(t;x) &= x^s + \left(h_{s-1}^{(r)} t^2 + \psi_{s-1}(t) \right) x^{s-1} + \cdots \\ &\qquad + \left(h_1^{(r)} t^{2s-2} + \psi_1(t) \right) x + \left(h_0^{(r)} t^{2s} + \psi_0(t) \right) \\ &\qquad (\psi_0(t), \cdots, \psi_{s-1}(t) \in \mathbb{R}[t], \deg \psi_{s-k}(t) < 2k \ (1 \leq k \leq s)). \end{aligned}$$

Here, let us divide the proof into next two cases.

(i) The case $h_0^{(r)} h_1^{(r)} \cdots h_{s-1}^{(r)} \neq 0$.

In this case, we have

$$\begin{aligned} \Psi_r(t;x) &= x^s + h_{s-1}^{(r)} t^2 \left(1 + \frac{\psi_{s-1}(t)}{h_{s-1}^{(r)} t^2} \right) x^{s-1} + \cdots \\ &\qquad + h_1^{(r)} t^{2s-2} \left(1 + \frac{\psi_1(t)}{h_1^{(r)} t^{2s-2}} \right) x + h_0^{(r)} t^{2s} \left(1 + \frac{\psi_0(t)}{h_0^{(r)} t^{2s}} \right) \end{aligned}$$

and $1+\psi_{s-k}(t)/h^{(r)}_{s-k}t^{2k} \to 1$ $(t \to \infty)$ for any k $(1 \le k \le s)$. Moreover, since $h^{(r)}_0 h^{(r)}_1 \cdots h^{(r)}_{s-1} \neq 0$, we have $h^{(r)}_0 \neq 0$, which implies that for any non-zero real number ξ, $\Phi_r(\xi; x)$ have s non-zero real roots (counted with multiplicity). Thus, there exists a real number ρ_0 $(> \alpha_r)$ such that for any real number $\xi > \rho_0$, $\Psi_r(\xi; x)$ have the same number of positive (hence also negative) real roots with $\Phi_r(\xi; x)$ by Proposition 3, which implies $\sigma(C_r(\xi)_{n-s}) = \sigma(\bar{B}_r(\xi))$ and hence $\sigma(C_r(\xi)_{n-s}) = N_{g_r} = \gamma$ $(\xi > \rho_0)$ by Lemma 3.2. Then, by the equation (3.17), we have

$$\sigma(A_r(\xi)_{n-s}) = \begin{cases} \gamma + 1 & n - s : \text{odd} \\ \gamma & n - s : \text{even, } r_s > 0 \\ \gamma + 2 & n - s : \text{even, } r_s < 0 \end{cases}$$

for any $\xi > \rho_0$, which implies

$$N_{f_r(\xi;x)} = \sigma(A_r(\xi)) = \begin{cases} \gamma + 1 & n - s : \text{odd} \\ \gamma & n - s : \text{even, } r_s > 0 \\ \gamma + 2 & n - s : \text{even, } r_s < 0 \end{cases}$$

for any $\xi > \rho_0$ since $A_r(\xi)$ and $A_r(\xi)_{n-s}$ are equivalent over $\mathbb{R}$. Hence, by Lemma 3.5, we have

$$N_{f_r(\xi;x)} = \begin{cases} \gamma + 1 & n - s : \text{odd} \\ \gamma & n - s : \text{even, } r_s > 0 \\ \gamma + 2 & n - s : \text{even, } r_s < 0 \end{cases}$$

for any $\xi > \alpha_r$.

(ii) General case.

Let ε_0 be a positive real number and for any vector $\nu \in \mathbb{R}^{s+1}$, set

$$\alpha'_\nu = \max\{|\alpha| \mid \alpha \in \mathbb{C}, P_\nu(\alpha) = 0\}.$$

Clearly, we have $\alpha'_\nu \ge \alpha_\nu$ for any $\nu \in \mathbb{R}^{s+1}$. Here, let us put $\rho'_0 = \alpha'_r + \varepsilon_0$. Then, by Lemma 3.5, it is enough to prove the next claim.

Claim 1. *For any real number $\xi > \rho'_0$, we have*

$$N_{f_r(\xi;x)} = \begin{cases} \gamma + 1 & n - s : \textit{odd} \\ \gamma & n - s : \textit{even, } r_s > 0 \\ \gamma + 2 & n - s : \textit{even, } r_s < 0. \end{cases}$$

Proof. By the assumption that $g_r(x)$ is a separable polynomial of degree s and the fact that the non-real roots must occur in pair with its complex conjugate, there exists a real number δ_0 such that for any vector $\nu = (\nu_0, \cdots, \nu_s) \in \mathbb{R}^{s+1}$ satisfying $|r - \nu|_0 = \max_{0 \le k \le s}\{|r_k - \nu_k|\} < \delta_0$, $g_\nu(x)$ is also a degree s separable polynomial satisfying $N_{g_\nu} = N_{g_r} = \gamma$ by Proposition 3.

(S1) If a vector $\nu \in \mathbb{R}^{s+1}$ satisfies $|r - \nu|_0 < \delta_0$, then $g_\nu(x)$ is also a degree s separable polynomial satisfying $N_{g_\nu} = N_{g_r} = \gamma$.

Next, we put

$$P(t) = \sum_{k\geq 0} x_k(t_0, \cdots, t_s)t^k = \det A(t) \ (A(t) = A(t_0, \cdots, t_s, t))$$

and let us consider $P(t)$ as a polynomial over $E_1 = \mathbb{R}(t_0, \cdots, t_s)$ in t. Then, since $x_k(t_0, \cdots, t_s) \in \mathbb{R}[t_0, \cdots, t_s]$ for any $k \geq 0$, there exists a real number $\delta_1 > 0$ such that for any vector $v \in \mathbb{R}^{s+1}$ satisfying $|r - v|_0 < \delta_1$, we have $|\alpha'_r - \alpha'_v| < \varepsilon_0$ by Proposition 3;

(S2) If a vector $v \in \mathbb{R}^{s+1}$ satisfies $|r - v|_0 < \delta_1$, we have $|\alpha'_r - \alpha'_v| < \varepsilon_0$.

Here, let ξ be any real number such that $\xi > \rho'_0 = \alpha'_r + \varepsilon_0$ and let

$$\Omega(t_0, \cdots, t_s, \xi; x) = \sum_{k=0}^{n} y_k(t_0, \cdots, t_s)x^k \in E_1[x]$$

be the characteristic polynomial of the Bezoutian

$$A(t_0, \cdots, t_s, \xi; x) = M_n(f^{(n)}(t_0, \cdots, t_s, \xi; x), f^{(n)}(t_0, \cdots, t_s, \xi; x)').$$

Here, $f^{(n)}(t_0, \cdots, t_s, \xi; x)'$ is the derivative of

$$f^{(n)}(t_0, \cdots, t_s, \xi; x) = \sum_{k=0}^{n} z_k(t_0, \cdots, t_s)x^k \in E_1[x]$$

with respect to x. Then, since $z_k(t_0, \cdots, t_s) \in \mathbb{R}[t_0, \cdots, t_s]$ $(0 \leq k \leq n)$, we also have $y_k(t_0, \cdots, t_s) \in \mathbb{R}[t_0, \cdots, t_s]$ $(0 \leq k \leq n)$. Moreover, since $\xi > \rho'_0 > \alpha_r$, we have $\det A_r(\xi) = \det A(r_0, \cdots, r_s, \xi) \neq 0$.

By these arguments, we can also deduce that there exists a positive real number δ_2 such that for any vector $v \in \mathbb{R}^{s+1}$ satisfying $|r - v|_0 < \delta_2$, the characteristic polynomial $\Omega_v(\xi; x)$ have the same number of positive and hence negative real roots with $\Omega_r(\xi; x)$ (counted with multiplicity), which implies $N_{f_r(\xi;x)} = \sigma(A_r(\xi)) = \sigma(A_v(\xi)) = N_{f_v(\xi;x)}$.

(S3) If a vector $v \in \mathbb{R}^{s+1}$ satisfies $|r - v|_0 < \delta_2$, we have $N_{f_r(\xi;x)} = N_{f_v(\xi;x)}$.

Put $\delta = \min\{\delta_0, \delta_1, \delta_2\} > 0$. Then, there exists a vector $w = (w_0, \cdots, w_s) \in \mathbb{R}^{s+1}$ such that

$$(a)\ |r - w|_0 < \delta,\ (b)\ h_0^{(w)} h_1^{(w)} \cdots h_{s-1}^{(w)} \neq 0.$$

Here, we put $h_{s-k}^{(w)} = h_{s-k}(w_0, \cdots, w_s)$ for any k $(1 \leq k \leq s)$. In fact, since $h_{s-k}(t_0, \cdots, t_s)$ is a non-zero polynomial for any k $(1 \leq k \leq s)$ by Lemma 3.3, the product $\prod_{k=1}^{s} h_{s-k}(t_0, \cdots, t_s)$ is also non-zero, which implies that there exists a vector $w \in \mathbb{R}^{s+1}$ satisfying (a) and (b).

Let $w \in \mathbb{R}^{s+1}$ be the vector as above. Then, since $|r - w|_0 < \delta \leq \delta_0$, $g_w(x)$ is a degree s separable polynomial satisfying $N_{g_w} = \gamma$ by (S1) and also, by (S2), we have $\alpha_w \leq \alpha'_w < \alpha'_r + \varepsilon_0 = \rho'_0 < \xi$. Thus, by (b) and the case (i), we have

$$N_{f_w(\xi;x)} = \begin{cases} \gamma + 1 & n - s : \text{odd} \\ \gamma & n - s : \text{even},\ r_s > 0 \\ \gamma + 2 & n - s : \text{even},\ r_s < 0, \end{cases}$$

which, by (S3), implies

$$N_{f_r(\xi;x)} = \begin{cases} \gamma+1 & n-s : \text{odd} \\ \gamma & n-s : \text{even}, \ r_s > 0 \\ \gamma+2 & n-s : \text{even}, \ r_s < 0. \end{cases}$$

Since ξ is any real number such that $\xi > \rho'_0$, this completes the proof of Claim and hence the proof of Theorem 3.2. □

Proposition 5. *Let* $g(x) = \sum_{i=0}^{s} a_i x^i$ *be a polynomial in* $\mathbb{R}[x]$ *such that* $\Delta_g \neq 0$ *and*

$$f(t,x) = x^n + t \cdot g(x) \tag{3.18}$$

If $g(x)$ *is totally complex,* $(n-s)$ *is even, and* $a_s > 0$ *then* $f(\beta, x)$ *is totally complex for all* $\beta > \max\{\alpha \,|\, \Delta_{(f,x)}(\alpha) = 0\}$.

Proof. We have to show that $f(\beta, x)$ has no real roots. Since $g(x)$ is totally complex we have that $\gamma = 0$. $N_{f(\beta,x)} = \gamma$ as $\beta > \max\{\alpha \,|\, \Delta_{(f,x)}(\alpha) = 0\}$ and $a_s > 0$, so $N_{f(\beta,x)} = \gamma = 0$. Hence, $f(\beta, x)$ is totally complex.

□

Let $K := \mathbb{Q}(t, a_0, \ldots, a_s)$ be the field of transcendental degree $s+1$ and $g(x) = \sum_{i=0}^{s} a_i x^i$. Then we have the following.

Corolary 2. *Let* $K := \mathbb{Q}(t, a_0, \ldots, a_s)$ *be the field of transcendental degree* $s+1$, $g(x) = \sum_{i=0}^{s} a_i x^i$ *and*

$$f(t,x) = x^n + t \cdot g(x)$$

For any value of $(\lambda_0, \ldots, \lambda_s) \in \mathbb{Z}^{s+1}$, *if* $g(\lambda_0, \ldots, \lambda_s, x) \in \mathbb{Z}[x]$ *is irreducible and satisfies the conditions of the Eisenstein criteria, then* $f(x)$ *is irreducible, over* $\mathbb{Q}$.

We also note:

Remark 3.4. *It can be verified computationally by Maple that if* $n \leq 9$ *and* $1 \leq s < n$ *then the Galois group Gal* $_K(f, x)$ *is isomorphic to* S_n.

Remark 3.5. *Polynomials in Eq.* (3.18) *for* $s = 1$ *and* $t = 1$ *has been treated by Y. Zarhin in* [18] *while studying Mori trinomials. It is shown there that the Galois group of* $f(x)$ *over* $\mathbb{Q}$ *is isomorphic to* S_n*; see* [18, Cor. 3.5] *for details.*

In general, if we let $K := \mathbb{Q}(t, a_0, \ldots, a_s)$ be the field of transcendental degree $s+1$, for $1 \leq s < n$, then we expect that Gal $_K(f) \cong S_n$ for all $n \geq 1$. If true, this would generalize Zarhin's result to a more general class of polynomials.

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