CUBO, A Mathematical Journal
Vol. 25, no. 1, pp. 89–101, April 2023
DOI: 10.56754/0719-0646.2501.089

Cubic and quartic series with the tail of $\ln 2$

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ABSTRACT

In this paper we calculate some remarkable cubic and quartic series involving the tail of $\ln 2$. We also evaluate several linear and quadratic series with the tail of $\ln 2$.

RESUMEN

En este artículo calculamos algunas series cúbicas y cuárticas notables que involucran la cola de $\ln 2$. También evaluamos varias series lineales y cuadráticas con la cola de $\ln 2$.

Keywords and Phrases: Abel's summation formula, cubic series, quartic series, tail of $\ln 2$.

2020 AMS Mathematics Subject Classification: 40A05, 40C10.

Accepted: 13 January, 2023
Received: 09 May, 2022

1 Introduction and the main results

In this paper we calculate several remarkable cubic and quartic series involving the term $\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots$. The goal of this paper is to extend, to the case of cubic and quartic series, the results recorded in [3], in problems 3.15, 3.29 and 3.45, concerning the calculation of some quadratic series involving the tail of $\ln 2$. Our results are new in the literature and they are obtained based on a combination of techniques involving Abel's summation formula and shifting the index of summation, which allow us to reduce the calculation of a cubic or a quartic series to a linear or a quadratic series, respectively. We also solve an open problem posed in [5, Open problem, p. 107].

The main results of this paper are Theorems 1.1 and 1.2 below.

Theorem 1.1 (Remarkable cubic series with the tail of $\ln 2$)**.** *The following identities hold:*

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3=\frac{5}{16}\zeta(3);$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3=\frac{\zeta(2)}{4}-\frac{3}{2}\ln^2 2;$

(c) $\displaystyle\sum_{n=1}^{\infty}n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3=-\frac{\zeta(2)}{4}-\frac{3}{4}\ln^2 2+\frac{3}{2}\ln 2+\frac{5}{32}\zeta(3).$

We mention that the alternating version of the series in part (c) of Theorem 1.1

$$\sum_{n=1}^{\infty}(-1)^n n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3=\frac{\ln 2}{4}-\frac{3\ln^2 2}{4}-\frac{\zeta(2)}{16}$$

was calculated in [4]. The results in parts (a) and (b) of Theorem 1.1 are due to C. I. Vălean, who communicated them to the first author, without proof, in an equivalent form in 2015.

Theorem 1.2 (Quartic series with the tail of $\ln 2$)**.** *The following identities hold:*

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^4=2\ln^3 2+2\zeta(2)\ln 2-\frac{9}{4}\zeta(3);$

(b) $\displaystyle\sum_{n=1}^{\infty}n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^4=\ln^3 2-\frac{1}{2}\ln^2 2+\zeta(2)\ln 2-\frac{13}{16}\zeta(3).$

We collect, in the next lemma, some results we need in proving Theorems 1.1 and 1.2.

Lemma 1.3 (A mosaic of linear and quadratic series with the tail of $\ln 2$)**.** *The following identities hold:*

Linear series

$$(a)\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)=\frac{\ln^2 2}{2}-\frac{\zeta(2)}{2};$$

$$(b)\quad \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)=\frac{\ln^2 2}{2}+\frac{\zeta(2)}{2};$$

$$(c)\quad \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{2n}-\frac{1}{2n+1}+\frac{1}{2n+2}-\cdots\right)=\ln^2 2;$$

$$(d)\quad \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)=\frac{13}{8}\zeta(3)-\zeta(2)\ln 2;$$

$$(e)\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)=\frac{\ln 2}{2}\zeta(2)-\zeta(3);$$

Quadratic series

$$(f)\quad \sum_{n=1}^{\infty} (-1)^n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2=-\frac{\zeta(2)}{4};$$

$$(g)\quad \sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2=\ln 2;$$

$$(h)\quad \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2=\frac{3}{2}\zeta(3)-\zeta(2)\ln 2-\frac{\ln^3 2}{3};$$

$$(i)\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2=\frac{\zeta(2)}{2}\ln 2-\frac{3}{4}\zeta(3)-\frac{\ln^3 2}{3}.$$

Since

$$\ln 2-\left[1+\frac{(-1)^1}{2}+\frac{(-1)^2}{3}+\cdots+\frac{(-1)^{n-2}}{n-1}\right]=(-1)^{n-1}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right),$$

we have that all the series in this paper involve the tail of $\ln 2$.

Before we prove the lemma, we observe that

$$\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots=\int_0^1\left(x^{n-1}-x^n+x^{n+1}-\cdots\right)\mathrm{d}x=\int_0^1\frac{x^{n-1}}{1+x}\,\mathrm{d}x. \tag{1.1}$$

This implies

$$\lim_{n\to\infty} n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)=\frac{1}{2} \tag{1.2}$$

and it follows that

$$\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \sim \frac{1}{2n}.$$

This shows that the series in Theorems 1.1, 1.2 and Lemma 1.3 are all convergent.

We also need in our analysis Abel's summation formula, which states that: if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of real numbers and $A_n = \sum_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}),$$

or, the infinite version

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} A_n b_{n+1} + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}). \tag{1.3}$$

Now we are ready to prove Lemma 1.3.

2 Proof of Lemma 1.3

Proof. (a) We have, based on (1.1), that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^1 \frac{x^{n-1}}{1+x}\,\mathrm{d}x = \int_0^1 \frac{1}{x(1+x)} \left(\sum_{n=1}^{\infty} \frac{(-x)^n}{n} \right) \mathrm{d}x \\
&= -\int_0^1 \frac{\ln(1+x)}{x(1+x)}\,\mathrm{d}x = \int_0^1 \frac{\ln(1+x)}{1+x}\,\mathrm{d}x - \int_0^1 \frac{\ln(1+x)}{x}\,\mathrm{d}x \\
&= \frac{\ln^2 2}{2} - \int_0^1 \frac{\ln(1+x)}{x}\,\mathrm{d}x \\
&= \frac{\ln^2 2}{2} - \frac{\zeta(2)}{2}.
\end{aligned}$$

We used that $\int_0^1 \frac{\ln(1+x)}{x}\,\mathrm{d}x = \frac{\zeta(2)}{2}$.

(b) We have, based on (1.1), that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right) &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 \frac{x^{n-1}}{1+x}\,\mathrm{d}x = \int_0^1 \frac{1}{x(1+x)} \left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right) \mathrm{d}x \\
&= -\int_0^1 \frac{\ln(1-x)}{x(1+x)}\,\mathrm{d}x = \int_0^1 \frac{\ln(1-x)}{1+x}\,\mathrm{d}x - \int_0^1 \frac{\ln(1-x)}{x}\,\mathrm{d}x \\
&= \frac{\ln^2 2}{2} + \frac{\zeta(2)}{2}.
\end{aligned}$$

We used that $\int_0^1 \frac{\ln(1-x)}{x}\,\mathrm{d}x = -\zeta(2)$ and $\int_0^1 \frac{\ln(1-x)}{1+x}\,\mathrm{d}x = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}$ (see [5, p. 203]).

(c) This nice result, which may be of independent interest, is obtained by adding the series in parts (a) and (b) of the lemma.

(d) We have, based on (1.1), that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right) &= \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1\frac{x^{n-1}}{1+x}\,\mathrm{d}x\\ &= \int_0^1\frac{1}{x(1+x)}\sum_{n=1}^{\infty}\frac{x^n}{n^2}\,\mathrm{d}x = \int_0^1\frac{\mathrm{Li}_2(x)}{x(1+x)}\,\mathrm{d}x \\ &= \int_0^1\frac{\mathrm{Li}_2(x)}{x}\,\mathrm{d}x - \int_0^1\frac{\mathrm{Li}_2(x)}{1+x}\,\mathrm{d}x.\end{aligned} \tag{2.1}$$

We calculate the first integral in (2.1) and we have that

$$\int_0^1\frac{\mathrm{Li}_2(x)}{x}\,\mathrm{d}x = \int_0^1\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^n}{n^2}\,\mathrm{d}x = \sum_{n=1}^{\infty}\frac{1}{n^3} = \zeta(3). \tag{2.2}$$

We calculate the second integral in (2.1). We integrate by parts, with $f(x) = \mathrm{Li}_2(x)$, $f'(x) = -\frac{\ln(1-x)}{x}$, $g'(x) = \frac{1}{1+x}$ and $g(x) = \ln(1+x)$, and we have that

$$\int_0^1\frac{\mathrm{Li}_2(x)}{1+x}\,\mathrm{d}x = \ln(1+x)\mathrm{Li}_2(x)\Big|_0^1 + \int_0^1\frac{\ln(1-x)\ln(1+x)}{x}\,\mathrm{d}x = \zeta(2)\ln 2 - \frac{5}{8}\zeta(3), \tag{2.3}$$

since $\int_0^1\frac{\ln(1-x)\ln(1+x)}{x}\,\mathrm{d}x = -\frac{5}{8}\zeta(3)$. For a proof of this result see [5, p. 328]. Combining (2.1), (2.2) and (2.3), the desired result holds and part (d) of the lemma is proved.

(e) We have, based on (1.1), that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1\frac{x^{n-1}}{1+x}\,\mathrm{d}x\\ &= \int_0^1\frac{1}{x(1+x)}\sum_{n=1}^{\infty}\frac{(-x)^n}{n^2}\,\mathrm{d}x\\ &= \int_0^1\frac{\mathrm{Li}_2(-x)}{x(1+x)}\,\mathrm{d}x\\ &= \int_0^1\frac{\mathrm{Li}_2(-x)}{x}\,\mathrm{d}x - \int_0^1\frac{\mathrm{Li}_2(-x)}{1+x}\,\mathrm{d}x.\end{aligned} \tag{2.4}$$

We calculate the first integral in (2.4) and we have that

$$\int_0^1\frac{\mathrm{Li}_2(-x)}{x}\,\mathrm{d}x = \int_0^1\frac{1}{x}\sum_{n=1}^{\infty}\frac{(-x)^n}{n^2}\,\mathrm{d}x = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} = -\frac{3}{4}\zeta(3). \tag{2.5}$$

We calculate the second integral in (2.4). We integrate by parts, with $f(x) = \operatorname{Li}_2(-x)$, $f'(x) = -\dfrac{\ln(1+x)}{x}$, $g'(x) = \dfrac{1}{1+x}$, $g(x) = \ln(1+x)$, and we have that

$$\int_0^1 \frac{\operatorname{Li}_2(-x)}{1+x}\,\mathrm{d}x = \ln(1+x)\operatorname{Li}_2(-x)\Big|_0^1 + \int_0^1 \frac{\ln^2(1+x)}{x}\,\mathrm{d}x = -\frac{\ln 2}{2}\zeta(2) + \frac{\zeta(3)}{4}, \tag{2.6}$$

since $\displaystyle\int_0^1 \frac{\ln^2(1+x)}{x}\,\mathrm{d}x = \frac{\zeta(3)}{4}$ (see [1, pp. 291–292]).

Combining (2.4), (2.5) and (2.6), the desired result holds and part (e) of the lemma is proved.

(f) We calculate the series by shifting the index of summation. We have

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)^2\\
&= \ln^2 2 + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)^2\\
&\overset{n-1=m}{=} \ln^2 2 + \sum_{m=1}^{\infty}(-1)^{m}\left(\frac{1}{m+1} - \frac{1}{m+2} + \frac{1}{m+3} - \cdots\right)^2\\
&= \ln^2 2 - \sum_{m=1}^{\infty}(-1)^{m-1}\left[\frac{1}{m} - \left(\frac{1}{m} - \frac{1}{m+1} + \frac{1}{m+2} - \cdots\right)\right]^2\\
&= \ln^2 2 + \sum_{m=1}^{\infty}\frac{(-1)^m}{m^2} + 2\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\frac{1}{m} - \frac{1}{m+1} + \frac{1}{m+2} - \cdots\right)\\
&\quad - \sum_{m=1}^{\infty}(-1)^{m-1}\left(\frac{1}{m} - \frac{1}{m+1} + \frac{1}{m+2} - \cdots\right)^2
\end{aligned}$$

and it follows that

$$\begin{aligned}
&2\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)^2\\
&= \ln^2 2 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)\\
&\overset{(a)}{=} \ln^2 2 - \frac{\zeta(2)}{2} + 2\left(\frac{\zeta(2)}{2} - \frac{\ln^2 2}{2}\right) = \frac{\zeta(2)}{2}.
\end{aligned}$$

We mention that this series was calculated by a different method in [3, problem 3.45].

(g) This result is proved, using an integration technique, in [3, problem 3.29]. Here we give another proof. We apply Abel's summation formula with $a_n = 1$ and $b_n = x_n^2$, where $x_n = \dfrac{1}{n} - \dfrac{1}{n+1} + \dfrac{1}{n+2} - \cdots$. Observe that $x_n + x_{n+1} = \dfrac{1}{n}$.

We have

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2 &= \lim_{n\to\infty} nx_{n+1}^2+\sum_{n=1}^{\infty} n\left(x_n^2-x_{n+1}^2\right)\\ &= \sum_{n=1}^{\infty} n\left(x_n-x_{n+1}\right)\left(x_n+x_{n+1}\right)\\ &= \sum_{n=1}^{\infty}\left(x_n-x_{n+1}\right)=x_1=\ln 2.\end{aligned}$$

We used that $\lim_{n\to\infty} nx_{n+1}^2=0$, which follows based on (1.2).

(*h*) We need the following power series formula $\sum_{n=1}^{\infty}\frac{H_n}{n}x^n=\operatorname{Li}_2(x)+\frac{1}{2}\ln^2(1-x)$, $x\in[-1,1)$. For a proof of this result see [5, p. 403].

We calculate the series by Abel's summation formula with $a_n=\frac{1}{n}$ and $b_n=x_n^2$, where $x_n=\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots$. Observe that $x_n+x_{n+1}=\frac{1}{n}$.

We have

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2=\lim_{n\to\infty}H_nx_{n+1}^2+\sum_{n=1}^{\infty}H_n\left(x_n-x_{n+1}\right)\left(x_n+x_{n+1}\right)\\ &=\sum_{n=1}^{\infty}\frac{H_n}{n}\left(2x_n-\frac{1}{n}\right)=2\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)-\sum_{n=1}^{\infty}\frac{H_n}{n^2}\\ &\overset{(1.1)}{=}2\sum_{n=1}^{\infty}\frac{H_n}{n}\int_0^1\frac{x^{n-1}}{1+x}\,\mathrm{d}x-2\zeta(3)=2\int_0^1\frac{1}{x(1+x)}\left(\sum_{n=1}^{\infty}\frac{H_n}{n}x^n\right)\mathrm{d}x-2\zeta(3)\\ &=2\int_0^1\frac{1}{x(1+x)}\left(\operatorname{Li}_2(x)+\frac{1}{2}\ln^2(1-x)\right)\mathrm{d}x-2\zeta(3) \qquad (2.7)\\ &=2\int_0^1\left(\frac{1}{x}-\frac{1}{1+x}\right)\left(\operatorname{Li}_2(x)+\frac{1}{2}\ln^2(1-x)\right)\mathrm{d}x-2\zeta(3)\\ &=2\int_0^1\frac{\operatorname{Li}_2(x)}{x}\,\mathrm{d}x+\int_0^1\frac{\ln^2(1-x)}{x}\,\mathrm{d}x-2\int_0^1\frac{\operatorname{Li}_2(x)}{1+x}\,\mathrm{d}x-\int_0^1\frac{\ln^2(1-x)}{1+x}\,\mathrm{d}x-2\zeta(3).\end{aligned}$$

We calculate

$$\begin{aligned}\int_0^1\frac{\ln^2(1-x)}{x}\,\mathrm{d}x&=\int_0^1\frac{\ln^2 y}{1-y}\,\mathrm{d}y=\int_0^1\ln^2 y\left(\sum_{n=0}^{\infty}y^n\right)\mathrm{d}y\\ &=\sum_{n=0}^{\infty}\int_0^1 y^n\ln^2 y\,\mathrm{d}y=\sum_{n=0}^{\infty}\frac{2}{(n+1)^3}=2\zeta(3).\end{aligned} \tag{2.8}$$

We also have, see [5, p. 110], that

$$\int_0^1\frac{\ln^2(1-x)}{1+x}\,\mathrm{d}x=\frac{7}{4}\zeta(3)-\zeta(2)\ln 2+\frac{\ln^3 2}{3}. \tag{2.9}$$

It follows, based on (2.5), (2.6), (2.7), (2.8) and (2.9), that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right)^2 = \frac{3}{2}\zeta(3) - \zeta(2)\ln 2 - \frac{\ln^3 2}{3}.$$

We mention that the series in part (g) of Lemma 1.3 was calculated by a different method by Boyadzhiev in [2].

(i) This formula was proved by Boyadzhiev in [2, entry (19)]. □

Now we are ready to prove Theorem 1.1.

3 Proof of Theorem 1.1

Proof. (a) We calculate the series by shifting the index of summation. We have

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right)^3 \\
&= \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots \right)^3 + \sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right)^3 \\
&\overset{n-1=i}{=} \ln^3 2 + \sum_{i=1}^{\infty} \left(\frac{1}{i+1} - \frac{1}{i+2} + \frac{1}{i+3} - \cdots \right)^3 \\
&= \ln^3 2 + \sum_{i=1}^{\infty} \left[\frac{1}{i} - \left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \cdots \right) \right]^3 \\
&= \ln^3 2 + \sum_{i=1}^{\infty} \left[\frac{1}{i^3} - \frac{3}{i^2} \left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \cdots \right) \right. \\
&\quad \left. + \frac{3}{i} \left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \cdots \right)^2 - \left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \cdots \right)^3 \right] \\
&= \ln^3 2 + \zeta(3) - 3 \sum_{i=1}^{\infty} \frac{1}{i^2} \left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \cdots \right) \\
&\quad + 3 \sum_{i=1}^{\infty} \frac{1}{i} \left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \cdots \right)^2 - S.
\end{aligned}$$

It follows, based on parts (d) and (h) of Lemma 1.3, that

$$\begin{aligned}
2S &= \ln^3 2 + \zeta(3) - 3 \sum_{i=1}^{\infty} \frac{1}{i^2} \left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \cdots \right) + 3 \sum_{i=1}^{\infty} \frac{1}{i} \left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \cdots \right)^2 \\
&= \ln^3 2 + \zeta(3) - 3 \left(\frac{13}{8}\zeta(3) - \zeta(2)\ln 2 \right) + 3 \left(\frac{3}{2}\zeta(3) - \zeta(2)\ln 2 - \frac{\ln^3 2}{3} \right) = \frac{5}{8}\zeta(3)
\end{aligned}$$

and part (a) of Theorem 1.1 is proved.

(b) We calculate the series using Abel's summation formula. We apply formula (1.3) with $a_n = 1$ and $b_n = (-1)^n x_n^3$, where $x_n = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots$. We have

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3 &= \lim_{n\to\infty}(-1)^{n+1}nx_{n+1}^3+\sum_{n=1}^{\infty}n(-1)^n\left(x_n^3+x_{n+1}^3\right)\\
&= \sum_{n=1}^{\infty}(-1)^n n\left(x_n+x_{n+1}\right)\left(x_n^2-x_nx_{n+1}+x_{n+1}^2\right)\\
&\overset{x_n+x_{n+1}=\frac{1}{n}}{=} \sum_{n=1}^{\infty}(-1)^n\left(x_n^2-x_nx_{n+1}+x_{n+1}^2\right)\\
&\overset{x_n+x_{n+1}=\frac{1}{n}}{=} \sum_{n=1}^{\infty}(-1)^n\left(3x_n^2-\frac{3}{n}x_n+\frac{1}{n^2}\right)\\
&= 3\sum_{n=1}^{\infty}(-1)^n x_n^2-3\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x_n+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\\
&\overset{\text{Lemma 1.3 }(f),(a)}{=} 3\left(-\frac{\zeta(2)}{4}\right)-3\left(\frac{\ln^2 2}{2}-\frac{\zeta(2)}{2}\right)-\frac{\zeta(2)}{2}\\
&= \frac{\zeta(2)}{4}-\frac{3}{2}\ln^2 2.
\end{aligned}$$

We used in the preceding calculations that $\lim_{n\to\infty} nx_{n+1}^3 = 0$, which follows from (1.2).

(c) Let $x_n = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots$. We calculate the series by shifting the index of summation. We have

$$\begin{aligned}
S &= \sum_{n=1}^{\infty}n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3=\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\cdots\right)^3+\sum_{n=2}^{\infty}nx_n^3\\
&\overset{n-1=i}{=} \ln^3 2+\sum_{i=1}^{\infty}(i+1)\left(\frac{1}{i+1}-\frac{1}{i+2}+\frac{1}{i+3}-\cdots\right)^3\\
&= \ln^3 2+\sum_{i=1}^{\infty}(i+1)\left[\frac{1}{i}-\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)\right]^3\\
&= \ln^3 2+\sum_{i=1}^{\infty}(i+1)\left(\frac{1}{i^3}-\frac{3}{i^2}x_i+\frac{3}{i}x_i^2-x_i^3\right)\\
&= \ln^3 2+\sum_{i=1}^{\infty}\left(\frac{1}{i^2}+\frac{1}{i^3}-\frac{3}{i}x_i-\frac{3}{i^2}x_i+3x_i^2+\frac{3}{i}x_i^2-ix_i^3-x_i^3\right)\\
&= \ln^3 2+\zeta(2)+\zeta(3)-3\sum_{i=1}^{\infty}\frac{x_i}{i}-3\sum_{i=1}^{\infty}\frac{x_i}{i^2}+3\sum_{i=1}^{\infty}x_i^2+3\sum_{i=1}^{\infty}\frac{x_i^2}{i}-S-\sum_{i=1}^{\infty}x_i^3.
\end{aligned}$$

It follows, based on part (a) of Theorem 1.1 and parts (b), (d), (g) and (h) of Lemma 1.3, that

$$\begin{aligned}2S &= \ln^3 2+\zeta(2)+\zeta(3)-3\left(\frac{\ln^2 2}{2}+\frac{\zeta(2)}{2}\right)-3\left(\frac{13}{8}\zeta(3)-\zeta(2)\ln 2\right)\\ &+3\ln 2+3\left(\frac{3}{2}\zeta(3)-\zeta(2)\ln 2-\frac{\ln^3 2}{3}\right)-\frac{5}{16}\zeta(3)\\ &=-\frac{\zeta(2)}{2}-\frac{3}{2}\ln^2 2+3\ln 2+\frac{5}{16}\zeta(3),\end{aligned}$$

and Theorem 1.1 is proved. □

Now we give the proof of Theorem 1.2.

4 Proof of Theorem 1.2

Proof. (a) We apply Abel's summation formula with $a_n = 1$ and $b_n = x_n^4$, where $x_n = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots$. We have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^4 = \lim_{n\to\infty} n x_{n+1}^4 + \sum_{n=1}^{\infty} n\left(x_n^4 - x_{n+1}^4\right)\\ &= \sum_{n=1}^{\infty} n\left(x_n - x_{n+1}\right)\left(x_n + x_{n+1}\right)\left(x_n^2 + x_{n+1}^2\right) \overset{x_n+x_{n+1}=\frac{1}{n}}{=} \sum_{n=1}^{\infty}\left(x_n - x_{n+1}\right)\left(x_n^2+x_{n+1}^2\right)\\ &= \sum_{n=1}^{\infty}\left(x_n^3 - x_{n+1}^3 + \frac{x_n}{n^2} - \frac{3}{n}x_n^2 + 2x_n^3\right) = x_1^3 + \sum_{n=1}^{\infty}\frac{x_n}{n^2} - 3\sum_{n=1}^{\infty}\frac{x_n^2}{n} + 2\sum_{n=1}^{\infty} x_n^3\\ &\overset{(*)}{=} \ln^3 2 + \frac{13}{8}\zeta(3) - \zeta(2)\ln 2 - 3\left(\frac{3}{2}\zeta(3) - \zeta(2)\ln 2 - \frac{\ln^3 2}{3}\right) + \frac{5}{8}\zeta(3)\\ &= 2\ln^3 2 + 2\zeta(2)\ln 2 - \frac{9}{4}\zeta(3).\end{aligned}$$

We have applied at step $(*)$ parts (d) and (h) of Lemma 1.3 and part (a) of Theorem 1.1. We also used that $\lim_{n\to\infty} n x_{n+1}^4 = 0$, which follows from (1.2).

(b) We calculate the series by applying Abel's summation formula with $a_n = n$ and $b_n = x_n^4$, where $x_n = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots$.

We have

$$\begin{aligned}&\sum_{n=1}^{\infty} n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^4 = \frac{1}{2}\lim_{n\to\infty} n(n+1)x_{n+1}^4 + \frac{1}{2}\sum_{n=1}^{\infty} n(n+1)\left(x_n^4 - x_{n+1}^4\right)\\ &\overset{x_n+x_{n+1}=\frac{1}{n}}{=} \frac{1}{2}\sum_{n=1}^{\infty}(n+1)\left(x_n - x_{n+1}\right)\left(x_n^2 + x_{n+1}^2\right)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}(n+1)\left(x_n^3 + x_n x_{n+1}^2 - x_{n+1}x_n^2 - x_{n+1}^3\right)\end{aligned}$$

$$
\begin{aligned}
&\overset{x_n+x_{n+1}=\frac{1}{n}}{=} \frac{1}{2}\sum_{n=1}^{\infty}\left[nx_n^3-(n+1)x_{n+1}^3+2nx_n^3+3x_n^3-3x_n^2-\frac{3}{n}x_n^2+\frac{x_n}{n}+\frac{x_n}{n^2}\right]\\
&=\frac{1}{2}\left[x_1^3+2\sum_{n=1}^{\infty}nx_n^3+3\sum_{n=1}^{\infty}x_n^3-3\sum_{n=1}^{\infty}x_n^2-3\sum_{n=1}^{\infty}\frac{x_n^2}{n}+\sum_{n=1}^{\infty}\frac{x_n}{n}+\sum_{n=1}^{\infty}\frac{x_n}{n^2}\right]\\
&\overset{(*)}{=}\frac{\ln^3 2}{2}+\left(-\frac{\zeta(2)}{4}-\frac{3}{4}\ln^2 2+\frac{3}{2}\ln 2+\frac{5}{32}\zeta(3)\right)+\frac{15}{32}\zeta(3)-\frac{3}{2}\ln 2\\
&-\frac{3}{2}\left(\frac{3}{2}\zeta(3)-\zeta(2)\ln 2-\frac{\ln^3 2}{3}\right)+\frac{\ln^2 2}{4}+\frac{\zeta(2)}{4}+\frac{13}{16}\zeta(3)-\frac{\zeta(2)\ln 2}{2}\\
&=\ln^3 2-\frac{1}{2}\ln^2 2+\zeta(2)\ln 2-\frac{13}{16}\zeta(3).
\end{aligned}
$$

We used at step $(*)$ parts (c) and (a) of Theorem 1.1 and parts (g), (h), (b) and (d) of Lemma 1.3. We also used that $\lim_{n\to\infty} n(n+1)x_{n+1}^4=0$, which follows from (1.2). □

The next corollary answers an open problem posed in [5, Open problem p. 107].

Corollary 4.1. *The following identities hold:*

(a) $$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\cdots\right)^3=4\ln^3 2+6\zeta(2)\ln 2-\frac{27}{4}\zeta(3);$$

(b) $$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\cdots\right)^3=4\ln^3 2+2\zeta(2)-12\ln^2 2-3\zeta(2)\ln 2+\frac{15}{4}\zeta(3).$$

Proof. (a) Let $x_n=\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots$ and observe that $\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\cdots=x_n-x_{n+1}$ and $x_n+x_{n+1}=\frac{1}{n}$. It follows that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\cdots\right)^3&=\sum_{n=1}^{\infty}(x_n-x_{n+1})^3\\
&=\sum_{n=1}^{\infty}\left[x_n^3-x_{n+1}^3+3x_nx_{n+1}(x_{n+1}-x_n)\right]\\
&=\sum_{n=1}^{\infty}\left[x_n^3-x_{n+1}^3+3x_n\left(\frac{1}{n}-x_n\right)\left(\frac{1}{n}-2x_n\right)\right]\\
&=\sum_{n=1}^{\infty}\left[x_n^3-x_{n+1}^3+3\frac{x_n}{n^2}-\frac{9x_n^2}{n}+6x_n^3\right]\\
&=x_1^3+3\sum_{n=1}^{\infty}\frac{x_n}{n^2}-9\sum_{n=1}^{\infty}\frac{x_n^2}{n}+6\sum_{n=1}^{\infty}x_n^3,
\end{aligned}
$$

and the result follows based on part (a) of Theorem 1.1 and parts (d) and (g) of Lemma 1.3.

(b) We have, exactly as in the proof of part (a), that

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\cdots\right)^3\\
&=\sum_{n=1}^{\infty}(-1)^n\left(x_n-x_{n+1}\right)^3=\sum_{n=1}^{\infty}(-1)^n\left[x_n^3-x_{n+1}^3+3x_nx_{n+1}(x_{n+1}-x_n)\right]\\
&=\sum_{n=1}^{\infty}(-1)^nx_n^3-\sum_{n=1}^{\infty}(-1)^nx_{n+1}^3+3\sum_{n=1}^{\infty}(-1)^n\frac{x_n}{n^2}-9\sum_{n=1}^{\infty}(-1)^n\frac{x_n^2}{n}+6\sum_{n=1}^{\infty}(-1)^nx_n^3\\
&=8\sum_{n=1}^{\infty}(-1)^nx_n^3+x_1^3+3\sum_{n=1}^{\infty}(-1)^n\frac{x_n}{n^2}-9\sum_{n=1}^{\infty}(-1)^n\frac{x_n^2}{n},
\end{aligned}$$

and the result follows based on part (b) of Theorem 1.1 and parts (e) and (i) of Lemma 1.3.

□

Remark 4.2. *The calculation of the quintic series* $\sum_{n=1}^{\infty}x_n^5$, *where* $x_n=\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots$, *which we believe it can be expressed in terms of well known constants, can be approached by reducing the series to the calculation of quadratic, cubic and quartic sums* $\sum_{n=1}^{\infty}\frac{x_n^2}{n^3}$, $\sum_{n=1}^{\infty}\frac{x_n^3}{n^2}$ *and* $\sum_{n=1}^{\infty}\frac{x_n^4}{n}$. *These series and other higher power sums involving the tail of* $\ln 2$ *are the topics of a research project that will be investigated by the authors.*

We mention that other challenging quadratic and cubic series involving the tail of various special functions, as well as open problems can be found in [5].

References

[1] B. C. Berndt, *Ramanujan's notebooks. Part I*, New York: Springer-Verlag, 1985.

[2] K. N. Boyadzhiev, "Power series with skew-harmonic numbers, dilogarithms, and double integrals", Tatra Mt. Math. Publ., vol. 56, no. 1, pp. 93–108, 2013.

[3] O. Furdui, *Limits, Series and Fractional Part Integrals*, Problems in Mathematical Analysis, New York: Springer, 2013.

[4] O. Furdui and A. Sîntămărian, "Alternating quadratic and cubic series with the tail of ln 2", Creat. Math. Inform., to be published.

[5] A. Sîntămărian and O. Furdui, *Sharpening mathematical analysis skills*, Problem Books in Mathematics, Cham: Springer, 2021.