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Some observations on a clopen version of the Rothberger property

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ABSTRACT

In this paper, we prove that a clopen version $S_1(\mathcal{C_O},\mathcal{C_O})$ of the Rothberger property and Borel strong measure zeroness are independent. For a zero-dimensional metric space (X,d), X satisfies $S_1(\mathcal{C_O},\mathcal{C_O})$ if, and only if, X has Borel strong measure zero with respect to each metric which has the same topology as d has. In a zero-dimensional space, the game $G_1(\mathcal{O},\mathcal{O})$ is equivalent to the game $G_1(\mathcal{C_O},\mathcal{C_O})$ and the point-open game is equivalent to the point-clopen game. Using reflections, we obtain that the game $G_1(\mathcal{C_O},\mathcal{C_O})$ and the point-clopen game are strategically and Markov dual. An example is given for a space on which the game $G_1(\mathcal{C_O},\mathcal{C_O})$ is undetermined.

RESUMEN

En este artículo, probamos que una versión clopen $S_1(\mathcal{C_O},\mathcal{C_O})$ de la propiedad de Rothberger y la nulidad de la medida fuerte de Borel son independientes. Para un espacio métrico (X,d) cero-dimensional, X satisface $S_1(\mathcal{C_O},\mathcal{C_O})$ si, y sólo si, X tiene una medida Borel fuerte cero con respecto a cada métrica que tenga la misma topología que d tiene. En un espacio cero-dimensional, el juego $G_1(\mathcal{O},\mathcal{O})$ es equivalente al juego $G_1(\mathcal{C_O},\mathcal{C_O})$ y el juego punto-abierto es equivalente al juego punto-cerrado. Usando reflexiones, obtenemos que el juego $G_1(\mathcal{C_O},\mathcal{C_O})$ y el juego punto-clopen son estratégicamente y Markov duales. Se entrega un ejemplo de un espacio para el cual el juego $G_1(\mathcal{C_O},\mathcal{C_O})$ es indeterminado

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1 Introduction

In 1938, Rothberger [12] (see also [9]) introduced covering property in topological spaces. A space X is said to have *Rothberger property* if for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle V_n : n \in \omega \rangle$ such that for each n, V_n is an element of $\mathcal{U}_n$ and each $x \in X$ belongs to V_n for some n. This property is stronger than Lindelöf and preserved under continuous images.

Usually, each selection principle $S_1(\mathcal{A}, \mathcal{B})$ can be associated with some topological game $G_1(\mathcal{A}, \mathcal{B})$. So the Rothberger property $S_1(\mathcal{O}, \mathcal{O})$ is associated with the Rothberger game $G_1(\mathcal{O}, \mathcal{O})$.

Let X be a topological space. The Rothberger game $G_1(\mathcal{O}, \mathcal{O})$ played on X is a game with two players Alice and Bob.

1st round: Alice chooses an open cover $\mathcal{U}_1$ of X. Bob chooses a set $U_1 \in \mathcal{U}_1$.

2nd round: Alice chooses an open cover $\mathcal{U}_2$ of X. Bob chooses a set $U_2 \in \mathcal{U}_2$.

etc.

If the family $\{U_n : n \in \omega\}$ is a cover of the space X then Bob wins the game $G_1(\mathcal{O}, \mathcal{O})$. Otherwise, Alice wins.

A topological space is Rothberger if, and only if, Alice has no winning strategy in the game $G_1(\mathcal{O}, \mathcal{O})$ [11].

In [8] Galvin proved that for a first-countable space X Bob has a winning strategy in $G_1(\mathcal{O}, \mathcal{O})$ if, and only if, X is countable.

In this paper, we continue to study the mildly Rothberger-type properties, started in papers [2, 3, 4], and, we define a new game - the mildly Rothberger game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$. In a zero-dimensional space, the Rothberger game is equivalent to the mildly Rothberger game. Using reflections, we obtained that $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ and the point-clopen game are strategically and Markov dual.

2 Preliminaries

Let (X, τ) or X be a topological space. If a set is open and closed in a topological space, then it is called *clopen*. Let ω be the first infinite cardinal and ω_1 the first uncountable cardinal. For the terms and symbols that we do not define, follow [7].

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a topological space X.

The symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle U_n : n \in \omega \rangle$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \omega\} \in \mathcal{B}$, [13].

In this paper $\mathcal{A}$ and $\mathcal{B}$ will be collections of the following open covers of a space X:

$\mathcal{O}$: the collection of all open covers of X.

$\mathcal{C}_{\mathcal{O}}$: the collection of all clopen covers of X.

Clearly, X has the Rothberger property if, and only if, X satisfies $S_1(\mathcal{O}, \mathcal{O})$.

A space X is said to have the *mildly Rothberger property* if it satisfies the selection principles $S_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$.

It can be noted that $S_1(\mathcal{O}, \mathcal{O}) \Rightarrow S_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$ and also every connected space must satisfy $S_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$. Then the set of real numbers with usual topology satisfies $S_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$ but it does not satisfy $S_1(\mathcal{O}, \mathcal{O})$.

Let (X, τ) be a topological space and $\mathcal{T}_X = \tau \setminus \{\emptyset\}$ be a topology without empty set.

- Let $\mathcal{T}_{X,x} = \{U \in \mathcal{T}_X : x \in U\}$ be the local point-base at $x \in X$.
- Let $\mathcal{P}_X = \{\mathcal{T}_{X,x} : x \in X\}$ be the collection of local point-bases of X.
- Let $\mathcal{C}_{\mathcal{T}_{X,x}} = \{U \in \mathcal{T}_X : U \text{ is a clopen set in } X,\ x \in U\}$.
- Let $\mathcal{C}_X = \{\mathcal{C}_{\mathcal{T}_{X,x}} : x \in X\}$.

3 Results on $S_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$

3.1 $S_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$ and Borel strong measure zeroness are independent

Recall that a set of reals X is *null* (or has measure zero) if for each positive ϵ there exists a cover $\{I_n\}_{n\in\omega}$ of X such that $\Sigma_n \operatorname{diam}(I_n) < \epsilon$.

To restrict the notion of measure zero or null set, in 1919, Borel [1] defined a notion stronger than measure zeroness. Now this notion is known as strong measure zeroness or strongly null set.

Borel strong measure zero: Y is *Borel strong measure zero* if there is for each sequence $\langle \epsilon_n : n \in \omega \rangle$ of positive real numbers a sequence $\langle J_n : n \in \omega \rangle$ of subsets of Y such that each J_n is of diameter $< \epsilon_n$, and Y is covered by $\{J_n : n \in \omega\}$.

But Borel was unable to construct a nontrivial (that is, an uncountable) example of a Borel strong measure zero set. He therefore conjectured that there exists no such examples.

In 1928, Sierpinski observed that every Luzin set is Borel strong measure zero, thus the Continuum Hypothesis implies that Borel's Conjecture is false.

Sierpinski asked whether the property of being Borel strong measure zero is preserved under taking homeomorphic (or even continuous) images.

In 1941, the answer given by Rothberger is negative under the Continuum Hypothesis. This lead Rothberger to introduce the following topological version of Borel strong measure zero (which is preserved under taking continuous images).

In 1988, Miller and Fremlin [10] proved that a space Y has the Rothberger property ($S_1(\mathcal{O},\mathcal{O})$) if, and only if, it has Borel strong measure zero with respect to each metric on Y which generates the topology of Y.

Recall that a space X is zero-dimensional if it has a base consisting clopen sets. Now we show that $S_1(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$ and Borel strong measure zeroness are independent to each other. Since the set of real numbers does not have measure zero, it does not have Borel strong measure zero but it satisfies $S_1(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$. Since every metric space with Borel strong measure zero must be zero-dimensional and separable, $S_1(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$ is equivalent to $S_1(\mathcal{O},\mathcal{O})$ (see below Theorem 3.1). So by Theorem 6(c) in [10], there is a subset of reals with Borel strong measure zero but it does not satisfy $S_1(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$.

The proof of the following result easily follows from replacing the open sets with sets of a clopen base of the topological space.

Theorem 3.1. *For a zero-dimensional space X, $S_1(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$ is equivalent to $S_1(\mathcal{O},\mathcal{O})$.*

From Theorem 1 in [10], we obtain the following corollary.

Corollary 3.2. *For a zero-dimensional metric space (X,d) the following statements are equivalent:*

(1) *X satisfies $S_1(\mathcal{O},\mathcal{O})$;*

(2) *X satisfies $S_1(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$;*

(3) *X has Borel strong measure zero with respect to every metric which generates the original topology;*

(4) *every continuous image of X in Baire space ω^ω with usual metric has Borel strong measure zero.*

3.2 Dual selection games

The *selection game* $G_1(\mathcal{A},\mathcal{B})$ is an ω-length game for two players, Alice and Bob. During round n, Alice choose $A_n \in \mathcal{A}$, followed by Bob choosing $B_n \in A_n$. Player Bob wins in the case that $\{B_n : n < \omega\} \in \mathcal{B}$, and Player Alice wins otherwise.

We consider the following strategies:

- A *strategy for player Alice* in $G_1(\mathcal{A}, \mathcal{B})$ is a function $\sigma : (\bigcup \mathcal{A})^{<\omega} \to \mathcal{A}$. A strategy σ for Alice is called *winning* if whenever $x_n \in \sigma\langle x_i : i < n\rangle$ for all $n < \omega$, $\{x_n : n \in \omega\} \notin \mathcal{B}$. If player Alice has a winning strategy, we write $Alice \uparrow G_1(\mathcal{A}, \mathcal{B})$.

- A strategy for player Bob in $G_1(\mathcal{A}, \mathcal{B})$ is a function $\tau : \mathcal{A}^{<\omega} \to \bigcup \mathcal{A}$. A strategy τ for Bob is *winning* if $A_n \in \mathcal{A}$ for all $n < \omega$, $\{\tau(A_0, \ldots, A_n) : n < \omega\} \in \mathcal{B}$.

- A *predetermined strategy* for Alice is a strategy which only considers the current turn number. Formally it is a function $\sigma : \omega \to \mathcal{A}$. If Alice has a winning predetermined strategy, we write $Alice \underset{pre}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$.

- A *Markov strategy* for Bob is a strategy which only considers the most recent move of player Alice and the current turn number. Formally it is a function $\tau : \mathcal{A} \times \omega \to \bigcup \mathcal{A}$. If Bob has a winning Markov strategy, we write $Bob \underset{mark}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$.

Note that, $Bob \underset{mark}{\uparrow} G_1(\mathcal{A}, \mathcal{B}) \Rightarrow Bob \uparrow G_1(\mathcal{A}, \mathcal{B}) \Rightarrow Alice \not\uparrow G_1(\mathcal{A}, \mathcal{B}) \Rightarrow Alice \underset{pre}{\not\uparrow} G_1(\mathcal{A}, \mathcal{B})$.

It is worth noting that $Alice \underset{pre}{\not\uparrow} G_1(\mathcal{A}, \mathcal{B})$ is equivalent to the selection principle $S_1(\mathcal{A}, \mathcal{B})$.

Two games G_1 and G_2 are said to be **strategically dual** provided that the following two hold:

- $Alice \uparrow G_1$ iff $Bob \uparrow G_2$
- $Alice \uparrow G_2$ iff $Bob \uparrow G_1$.

Two games G_1 and G_2 are said to be **Markov dual** provided that the following two hold:

- $Alice \underset{pre}{\uparrow} G_1$ iff $Bob \underset{mark}{\uparrow} G_2$
- $Alice \underset{pre}{\uparrow} G_2$ iff $Bob \underset{mark}{\uparrow} G_1$.

Two games G_1 and G_2 are said to be **dual** provided that they are both strategically dual and Markov dual.

For a set X, let $\mathcal{C}(X) = \{f \in (\bigcup X)^X : x \in X \Rightarrow f(x) \in x\}$ be the collection of all choice functions on X.

Write $X \preceq Y$ if X is coinitial in Y with respect to $\subseteq$; that is, $X \subseteq Y$, and for all $y \in Y$, there exists $x \in X$ such that $x \subseteq y$.

In the context of selection games, $\mathcal{A}'$ is a *selection basis* for $\mathcal{A}$ when $\mathcal{A}' \preceq \mathcal{A}$ [6].

Definition 3.3 ([6]). *The set $\mathcal{R}$ is said to be a* **reflection** *of the set $\mathcal{A}$ if $\{range(f) : f \in \mathcal{C}(\mathcal{R})\}$ is a selection basis for $\mathcal{A}$.*

Let $G_1(\mathcal{A}, \neg\mathcal{B}) := G_1(\mathcal{A}, \mathcal{P}(\bigcup \mathcal{A}) \setminus \mathcal{B})$.

Theorem 3.4 ([6], Corollary 26). *If $\mathcal{R}$ is a reflection of $\mathcal{A}$, then $G_1(\mathcal{A},\mathcal{B})$ and $G_1(\mathcal{R},\neg\mathcal{B})$ are dual.*

The point-open game $PO(X)$ is a game where Alice chooses points of X, Bob chooses an open neighborhood of each chosen point, and Alice wins if Bob's choices are a cover.

Theorem 3.5 ([8]). *The game $G_1(\mathcal{O},\mathcal{O})$ is strategically dual to the point-open game on each topological space.*

Theorem 3.6 ([5]). *The game $G_1(\mathcal{O},\mathcal{O})$ is Markov dual to the point-open game on each topological space.*

Corollary 3.7. *The game $G_1(\mathcal{O},\mathcal{O})$ is dual to the point-open game on each topological space.*

Recall that two games G and G' are *equivalent* (isomorphic) if

(1) $Alice \uparrow G$ iff $Alice \uparrow G'$.

(2) $Bob \uparrow G$ iff $Bob \uparrow G'$.

Since $\mathcal{P}_X$ is a reflection of $\mathcal{O}$ [6, Proposition 28], the Rothberger game $G_1(\mathcal{O},\mathcal{O})$ and $G_1(\mathcal{P}_X,\neg\mathcal{O})$ are dual [6, Corollary 29]. It is well known that the game $G_1(\mathcal{P}_X,\neg\mathcal{O})$ is equivalent to the point-open game.

3.3 The point-clopen and quasi-component-clopen games

The *point-clopen game* $PC(X)$ on a space X is played according to the following rules:

In each inning $n \in \omega$, Alice picks a point $x_n \in X$, and then Bob chooses a clopen set $U_n \subseteq X$ with $x_n \in U_n$. At the end of the play

$$x_0, U_0, x_1, U_1, x_2, U_2, \ldots, x_n, U_n, \ldots,$$

the winner is Alice if $X = \bigcup_{n\in\omega} U_n$, and Bob otherwise.

We denote the collection of all non-empty clopen subsets of a space X by τ_c and the collection of all finite subsets of τ_c by $\tau_c^{<\omega}$.

A strategy for Alice in the point-clopen game on a space X is a function $\varphi : \tau_c^{<\omega} \to X$.

A strategy for Bob in the point-clopen game on a space X is a function $\psi : X^{<\omega} \to \tau_c$ such that, for all $\langle x_0, x_1, \ldots, x_n\rangle \in X^{<\omega} \setminus \{\langle\rangle\}$, we have $x_n \in \psi(\langle x_0, \ldots, x_n\rangle) = U_n$.

A strategy $\varphi : \tau_c^{<\omega} \to X$ for Alice in the point-clopen game on a space X is a winning strategy for Alice if, for every sequence $\langle U_n : n \in \omega\rangle$ of clopen subsets of a space X such that $\forall n \in \omega$,

($x_n = \varphi(\langle U_0, U_1, \dots, U_{n-1}\rangle) \in U_n$), we have $X = \bigcup_{n\in\omega} U_n$. If Alice has a winning strategy in the point-clopen game on a space X, we write $Alice{\uparrow}PC(X)$.

A strategy $\psi : X^{<\omega} \to \tau_c$ for Bob in the point-clopen game on a space X is a winning strategy for Bob if, for every sequence $\langle x_n : n \in \omega\rangle$ of points of a space X, we have $X = \bigcup_{n\in\omega}\{U_n : U_n = \psi(\langle x_0, x_1, \dots, x_n\rangle)\}$. If Bob has a winning strategy in the point-clopen game on a space X, we write $Bob{\uparrow}PC(X)$.

The game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ is a game for two players, Alice and Bob, with an inning per each natural number n. In each inning, Alice picks a clopen cover of the space and Bob selects one member from this cover. Bob wins if the sets he selected throughout the game cover the space. If this is not the case, Alice wins.

The intersection of all clopen sets containing a component is called a *quasi-component* of the space [7].

The *quasi-component-clopen game* $QC(X)$ on a space X is played according to the following rules:

In each inning $n \in \omega$, Alice picks a quasi-component A_n of X, and then Bob chooses a clopen set $U_n \subseteq X$ with $A_n \subseteq U_n$. At the end of the play

$$A_0, U_0, A_1, U_1, A_2, U_2, \dots, A_n, U_n, \dots,$$

the winner is Alice if $X = \bigcup_{n\in\omega} U_n$, and Bob otherwise.

We denote the collection of all quasi-components of a space X by Q_X and the collection of all finite subsets of Q_X by $Q_X^{<\omega}$.

A strategy for Alice in the quasi-component-clopen game on a space X is a function $\varphi : \tau_c^{<\omega} \to Q_X$.

A strategy for Bob in the quasi-component-clopen game on a space X is a function $\psi : Q_X^{<\omega} \to \tau_c$ such that, for all $\langle A_0, A_1, \dots, A_n\rangle \in Q_X^{<\omega} \setminus \{\langle\rangle\}$, we have $A_n \subseteq \psi(\langle A_0, \dots, A_n\rangle) = U_n$.

A strategy $\varphi : \tau_c^{<\omega} \to Q_X$ for Alice in the quasi-component-clopen game on a space X is a winning strategy for Alice if, for every sequence $\langle U_n : n \in \omega\rangle$ of clopen subsets of a space X such that $\forall n \in \omega, (A_n = \varphi(\langle U_0, U_1, \dots, U_{n-1}\rangle) \subseteq U_n)$, we have $X = \bigcup_{n\in\omega} U_n$. If Alice has a winning strategy in the quasi-component-clopen game on a space X, we write $Alice{\uparrow}QC(X)$.

A strategy $\psi : Q_X^{<\omega} \to \tau_c$ for Bob in the quasi-component-clopen game on a space X is a winning strategy for Bob if, for every sequence $\langle A_n : n \in \omega\rangle$ of quasi-components of a space X, we have $X = \bigcup_{n\in\omega}\{U_n : U_n = \psi(\langle A_0, A_1, \dots, A_n\rangle)\}$. If Bob has a winning strategy in the quasi-component-clopen game on a space X, we write $Bob{\uparrow}QC(X)$.

Proposition 3.8. *The point-clopen game is equivalent to the quasi-component-clopen game.*

Proof. Let $\varphi : \tau_c^{<\omega} \to X$ be a winning strategy for Alice in the point-clopen game on a space

X. Then the function $\psi : \tau_c^{<\omega} \to Q_X$ such that $\psi(\langle U_0, U_1, \ldots, U_{n-1}\rangle) = Q[\varphi(\langle U_0, U_1, \ldots, U_{n-1}\rangle)]$ ($Q[x]$ is the quasi-component of x) for every sequence $\langle U_n : n \in \omega\rangle$ of clopen subsets of a space X and $n \in \omega$, is a winning strategy for Alice in the quasi-component-clopen game. This follows from the fact that $x_n = \varphi(\langle U_0, U_1, \ldots, U_{n-1}\rangle) \in Q[x_n] \subseteq U_n$.

Let $\varphi : \tau_c^{<\omega} \to Q_X$ be a winning strategy for Alice in the quasi-component-clopen game on a space X. Then the function $\psi : \tau_c^{<\omega} \to X$ such that $\psi(\langle U_0, U_1, \ldots, U_{n-1}\rangle) \in \varphi(\langle U_0, U_1, \ldots, U_{n-1}\rangle)$ for every sequence $\langle U_n : n \in \omega\rangle$ of clopen subsets of a space X and $n \in \omega$, is a winning strategy for Alice in the point-clopen game.

Let $\psi : X^{<\omega} \to \tau_c$ be a winning strategy for Bob in the point-clopen game on X. Then the function $\rho : Q_X^{<\omega} \to \tau_c$ such that $\rho(\langle A_0, A_1, \ldots, A_n\rangle) = \psi(\langle x_0, x_1, \ldots, x_n\rangle)$ for every sequence $\langle A_n : n \in \omega\rangle$ of quasi-components of a space X and some $x_0, \ldots, x_n$ that $A_i = Q[x_i]$ for each $i = 0, \ldots, n$, is a winning strategy for Bob in the quasi-component-clopen game.

Let $\psi : Q_X^{<\omega} \to \tau_c$ be a winning strategy for Bob in the quasi-component-clopen game on X. Then the function $\rho : X^{<\omega} \to \tau_c$ such that $\rho(\langle x_0, x_1, \ldots, x_n\rangle) = \psi(\langle A_0, A_1, \ldots, A_n\rangle)$ for every sequence $\langle x_n : n \in \omega\rangle$ of points of a space X where $A_i = Q[x_i]$ for each $i = 0, \ldots, n$, is a winning strategy for Bob in the point-clopen-clopen game. □

Proposition 3.9. *$\mathcal{C}_X$ is a reflection of $\mathcal{C}_\mathcal{O}$.*

Proof. For every clopen cover $\mathcal{U}$, the corresponding choice function $f \in \mathcal{C}(\mathcal{C}_X)$ is simply the witness that $x \in f(\mathcal{C}_{\mathcal{T}_{X,x}}) \in \mathcal{U}$. □

By Theorem 3.4, we get the following result.

Corollary 3.10. *$G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ and $G_1(\mathcal{C}_X, \neg\mathcal{C}_\mathcal{O})$ are dual.*

Note that $PC(X)$ and $G_1(\mathcal{C}_X, \neg\mathcal{C}_\mathcal{O})$ are the same game.

By Proposition 3.8, $PC(X)$ and $QC(X)$ are equivalent, hence, we get the following result.

Proposition 3.11. *The game $G_1(\mathcal{C}_X, \neg\mathcal{C}_\mathcal{O})$ is equivalent to the quasi-component-clopen game.*

Corollary 3.12. *If a space X is a union of countable number of quasi-components, then Bob $\uparrow G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$.*

The following chain of implications always holds:

$$\begin{gathered} X \text{ is a union of countable number of quasi-components} \\ \Downarrow \\ Bob \uparrow G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O}) \\ \Downarrow \\ Alice \not\uparrow G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O}) \\ \Updownarrow \\ X \text{ has mildly Rothberger property.} \end{gathered}$$

The proof of the following result easily follows from replacing the open sets with sets of a clopen base of the topological space.

Theorem 3.13. *For a zero-dimensional space, the following statements hold:*

(1) *The game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ is equivalent to the game $G_1(\mathcal{O}, \mathcal{O})$.*

(2) *The point-clopen game is equivalent to the point-open game.*

From [11] and [**?**], we have the following result.

Theorem 3.14. *For a space X, the following statements hold:*

(1) *[11] X satisfies $S_1(\mathcal{O}, \mathcal{O})$ iff $Alice \not\uparrow G_1(\mathcal{O}, \mathcal{O})$.*

(2) *[**?**] X satisfies $S_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ iff $Alice \not\uparrow G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$.*

Corollary 3.15. *For a space X, the following statements are equivalent:*

(1) *X satisfies $S_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$;*

(2) *$Alice \underset{pre}{\not\uparrow} G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$;*

(3) *$Alice \not\uparrow G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$;*

(4) *$Bob \not\uparrow G_1(\mathcal{C}_X, \neg\mathcal{C}_\mathcal{O})$;*

(5) *$Bob \underset{mark}{\not\uparrow} G_1(\mathcal{C}_X, \neg\mathcal{C}_\mathcal{O})$;*

(6) *$Bob \not\uparrow PC(X)$;*

(7) *$Bob \not\uparrow QC(X)$;*

(8) *$Bob \underset{mark}{\not\uparrow} PC(X)$;*

(9) *$Bob \underset{mark}{\not\uparrow} QC(X)$.*

Corollary 3.16. *For a zero-dimensional space X, the following statements are equivalent:*

(1) *X satisfies $S_1(\mathcal{O}, \mathcal{O})$;*

(2) *X satisfies $S_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$;*

(3) *$Alice \underset{pre}{\not\uparrow} G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$;*

(4) *$Alice \underset{pre}{\not\uparrow} G_1(\mathcal{O}, \mathcal{O})$;*

(5) *$Alice \not\uparrow G_1(\mathcal{O}, \mathcal{O})$;*

(6) *$Alice \not\uparrow G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$;*

(7) *Bob* $\not\uparrow G_1(\mathcal{P}_X, \neg\mathcal{O})$;

(8) *Bob* $\not\uparrow G_1(\mathcal{C}_X, \neg\mathcal{C}_{\mathcal{O}})$;

(9) *Bob* $\not\uparrow PO(X)$;

(10) *Bob* $\not\uparrow PC(X)$;

(11) *Bob* $\not\uparrow QC(X)$;

(12) *Bob* $\underset{mark}{\not\uparrow} PO(X)$;

(13) *Bob* $\underset{mark}{\not\uparrow} PC(X)$;

(14) *Bob* $\underset{mark}{\not\uparrow} QC(X)$.

In [8], Galvin and Telgársky in [14, Theorem 6.3] prove: If X is a Lindelöf space in which each element is G_δ, then Bob has a winning strategy in $G_1(\mathcal{O},\mathcal{O})$ if, and only if, X is countable.

Theorem 3.17. *Let X be a space in which each quasi-component is an intersection of countably many clopen sets, then Bob $\uparrow G_1(\mathcal{C}_{\mathcal{O}},\mathcal{C}_{\mathcal{O}})$ if, and only if, X is a union of countably many quasi-components.*

Proof. Let Bob have a winning strategy in the game $G_1(\mathcal{C}_{\mathcal{O}},\mathcal{C}_{\mathcal{O}})$ on X. Since the game $G_1(\mathcal{C}_{\mathcal{O}},\mathcal{C}_{\mathcal{O}})$ and the point-clopen game are dual and, by Proposition 3.8, the point-clopen game and the quasi-component-clopen game are equivalent.

Let Alice have a winning strategy in the quasi-component-clopen game. Let φ be a winning strategy of Alice in the quasi-component-clopen game on X. For every quasi-component Q, there is a sequence $\langle V_k : k\in\omega\rangle$ of clopen sets such that $Q=\bigcap_{k\in\omega} V_k$.

So we restrict the move of Bob from $\{V_k : k\in\omega\}$ for Q played by Alice.

Let Alice start the play of the point-clopen game by quasi-component $\varphi(\langle\rangle)=Q_{\langle\rangle}$. Then Bob replies with a clopen set of the form $V_{k_0,\langle\rangle}$ for some $k_0\in\omega$.

Alice's next move in the play is a quasi-component $\varphi(\langle V_{k_0,\langle\rangle}\rangle)=Q_{\langle k_0\rangle}$. Then Bob replies with a clopen set of the form $V_{k_1,\langle k_0\rangle}$ for some $k_1\in\omega$.

Now Alice's next move in the play is a quasi-component $\varphi(\langle V_{k_0,\langle\rangle}, V_{k_1,\langle k_0\rangle}\rangle)=Q_{\langle k_0,k_1\rangle}$. Then Bob replies with a clopen set of the form $V_{k_2,\langle k_0,k_1\rangle}$ for some $k_2\in\omega$ and so on.

Similarly we are defining $\langle Q_s : s\in\omega^{<\omega}\rangle$ by setting $Q_{\langle\rangle}=\varphi(\langle\rangle)$ and for each $s\in\omega^{<\omega}$ and for each $k\in\omega$, defining

$$Q_{s\frown\langle k\rangle}=\varphi(\langle V_{s(0),s\restriction 0}, V_{s(1),s\restriction 1},\ldots,V_{s(m-1),s\restriction(m-1)}, V_{k,s}\rangle),$$

where $m=dom(s)$. From this we construct a countable collection $\{Q_s : s\in\omega^{<\omega}\}$.

Now to show that $\bigcup\{Q_s : s\in\omega^{<\omega}\}=X$. If possible suppose that $\bigcup\{Q_s : s\in\omega^{<\omega}\}\neq X$, then there is $y\in X\setminus\{Q_s : s\in\omega^{<\omega}\}$. Then $y\notin Q_s$ for any $s\in\omega^{<\omega}$. For each $Q_n\in\{Q_s : s\in\omega^{<\omega}\}$, there is some k_n such that $y\notin V_{k_n,n}$. Then Alice loses the following play of the quasi-component-clopen game

$$\langle Q_0, V_{k_0,0}, Q_1, V_{k_1,1}, \ldots, Q_n, V_{k_n,n}, \ldots \rangle$$

in which Alice uses the strategy φ since $y \notin \bigcup_{n\in\omega} V_{k_n,n}$, a contradiction.

The converse follows from Corollary 3.12. □

3.4 Determinacy and $G_1(\mathcal{C_O}, \mathcal{C_O})$ game

A game G played between two players Alice and Bob is determined if either Alice has a winning strategy in game G or Bob has a winning strategy in game G. Otherwise G is undetermined.

It can be observed that the game $G_1(\mathcal{C_O}, \mathcal{C_O})$ is determined for every countable space. But in a mildly Rothberger space in which each quasi-component is an intersection of countably many clopen sets with uncountable many quasi-components, none of the players Alice and Bob have a winning strategy. So $G_1(\mathcal{C_O}, \mathcal{C_O})$ is undetermined for a mildly Rothberger space in which each quasi-component is an intersection of countably many clopen sets with uncountable many quasi-components. Thus every uncountable zero-dimensional mildly Rothberger metric space is undetermined.

Recall that an uncountable set L of reals is a *Luzin set* if for each meager set M, $L \cap M$ is countable. The Continuum Hypothesis implies the existence of a Luzin set. A Luzin set is an example of a space for which the game $G_1(\mathcal{C_O}, \mathcal{C_O})$ is undetermined.

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