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Several inequalities for an integral transform of positive operators in Hilbert spaces with applications

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ABSTRACT

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a postive operator on a complex Hilbert space H.
We show among others that, if $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right] \\
&\leq \frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right](B-A)^{-1} \\
&\leq \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\
&\leq \frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right].
\end{aligned}$$

Some examples for operator monotone and operator convex functions as well as for integral transforms $\mathcal{D}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

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RESUMEN

Para una función contínua y positiva $w(\lambda)$, $\lambda > 0$ y μ una medida positiva sobre $(0, \infty)$ consideramos la siguiente *transformada integral*

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

donde se asume que la integral existe para un operador positivo T, sobre el espacio complejo de Hilbert H.
Mostramos, entre otras cosas, que si $\beta \geq A \geq \alpha > 0$, $B > 0$ con $M \geq B - A \geq m > 0$ para algunas constantes α, β, m, M, entonces

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\mathcal{D}(w, \mu)(\beta) - \mathcal{D}(w, \mu)(M + \beta)\right] \\
&\leq \frac{m^2}{M}\left[\mathcal{D}(w, \mu)(\beta) - \mathcal{D}(w, \mu)(M + \beta)\right](B - A)^{-1} \\
&\leq \mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B) \\
&\leq \frac{M^2}{m}\left[\mathcal{D}(w, \mu)(\alpha) - \mathcal{D}(w, \mu)(m + \alpha)\right](B - A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\mathcal{D}(w, \mu)(\alpha) - \mathcal{D}(w, \mu)(m + \alpha)\right].
\end{aligned}$$

También se proporcionan algunos ejemplos para las funciones operador monótono y operador convexo, así como de transformadas integrales $\mathcal{D}(\cdot, \cdot)$ relacionadas con las funciones exponencial y logarítmica.

Keywords and Phrases: Operator monotone functions, Operator convex functions, Operator inequalities, Löwner-Heinz inequality, Logarithmic operator inequalities.

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1 Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [6], see for instance [1, p. 144–145]:

Theorem 1.1. *A function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} \, d\mu(\lambda), \tag{1.1}$$

where $b \geq 0$ and a positive measure μ on $[0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} \, d\mu(\lambda) < \infty. \tag{1.2}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex. We have the following representation of operator convex functions [1, p. 147]:

Theorem 1.2. *A function $f : [0, \infty) \to \mathbb{R}$ is operator convex in $[0, \infty)$ with $f'_+(0) \in \mathbb{R}$ if and only if it has the representation*

$$f(t) = f(0) + f'_+(0)\, t + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} \, d\mu(\lambda), \tag{1.3}$$

where $c \geq 0$ and a positive measure μ on $[0, \infty)$ such that (1.2) holds.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} \, d\lambda. \tag{1.4}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln\left(\frac{u+t}{u+1}\right), \quad \text{for all } u > 0$$

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)}, \quad \text{for all } t > 0. \tag{1.5}$$

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda), \quad t > 0, \tag{1.6}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.6) exists for all $t > 0$. For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda, \quad t > 0. \tag{1.7}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \quad t > 0. \tag{1.8}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1)\mathcal{D}(w_{\ln})(t), \quad t > 0. \tag{1.9}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda), \tag{1.10}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \quad \text{for } T > 0. \tag{1.11}$$

From (1.8) we have the representation

$$T^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(T) \tag{1.12}$$

where $T > 0$ and from (1.9)

$$(T-1)^{-1} \ln T = \mathcal{D}(w_{\ln})(T) \tag{1.13}$$

provided $T > 0$ and $T-1$ is invertible.

In what follows, if A is an operator and a is a real number, then by $A \geq a$ we understand $A \geq aI$, where I is the identity operator.

In this paper we show among others that, if $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right] \\
&\leq \frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right](B-A)^{-1} \\
&\leq \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\
&\leq \frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right].
\end{aligned}$$

Some examples for operator monotone and operator convex functions as well as for integral transforms $\mathcal{D}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2 Main results

In the following, whenever we write $\mathcal{D}(w,\mu)$ we mean that the integral from (1.6) exists and is finite for all $t > 0$.

Theorem 2.1. *For all A, $B > 0$ with $B - A \geq 0$ we have the representation*

$$\begin{aligned}
0 &\leq (B-A)^{1/2}\left[\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B)\right](B-A)^{1/2} \qquad (2.1)\\
&= \int_0^\infty \left(\int_0^1 \left[(B-A)^{1/2}(\lambda + sB + (1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right) \times w(\lambda)\, d\mu(\lambda).
\end{aligned}$$

Proof. Observe that, for all A, $B > 0$

$$\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) = \int_0^\infty w(\lambda)\left[(\lambda+B)^{-1} - (\lambda+A)^{-1}\right] d\mu(\lambda). \qquad (2.2)$$

Let $T, S > 0$. The function $f(t) = -t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t\to 0}\left[\frac{f(T+tS) - f(T)}{t}\right] = T^{-1}ST^{-1}, \quad \text{for } T, S > 0 \qquad (2.3)$$

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D] : \{(1-t)C + tD,\ t \in [0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t) := f((1-t)C + tD),\ t \in [0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t))\, dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C)\, dt. \qquad (2.4)$$

If we write this equality for the function $f(t) = -t^{-1}$ and $C, D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1} (D - C) ((1-t)C + tD)^{-1} dt. \tag{2.5}$$

Now, if we take in (2.5) $C = \lambda + B$, $D = \lambda + A$, then

$$\begin{aligned} &(\lambda + B)^{-1} - (\lambda + A)^{-1} \\ &= \int_0^1 ((1-t)(\lambda + B) + t(\lambda + A))^{-1} (A - B) \times ((1-t)(\lambda + B) + t(\lambda + A))^{-1} dt \\ &= \int_0^1 (\lambda + (1-t)B + tA)^{-1} (A - B) (\lambda + (1-t)B + tA)^{-1} dt \end{aligned} \tag{2.6}$$

and by (2.2) we derive

$$\begin{aligned} &\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\ &= \int_0^\infty w(\lambda) \left(\int_0^1 (\lambda + (1-t)B + tA)^{-1} (B - A) \times (\lambda + (1-t)B + tA)^{-1} dt \right) d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \left(\int_0^1 (\lambda + sB + (1-s)A)^{-1} (B - A) \times (\lambda + sB + (1-s)A)^{-1} ds \right) d\mu(\lambda) \end{aligned} \tag{2.7}$$

for all $A, B > 0$, where for the last equality we used the change of variable $s = 1 - t$, $t \in [0, 1]$. Now, since $B - A \geq 0$, hence by multiplying both sides with $(B - A)^{1/2}$ we get

$$\begin{aligned} &(B - A)^{1/2} [\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B)] (B - A)^{1/2} \\ &= \int_0^\infty w(\lambda) \left(\int_0^1 (B - A)^{1/2} (\lambda + sB + (1-s)A)^{-1} (B - A) \right. \\ &\left. \times (\lambda + sB + (1-s)A)^{-1} (B - A)^{1/2} ds \right) d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \left(\int_0^1 (B - A)^{1/2} (\lambda + sB + (1-s)A)^{-1} (B - A)^{1/2} \right. \\ &\left. \times (B - A)^{1/2} (\lambda + sB + (1-s)A)^{-1} (B - A)^{1/2} ds \right) d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \times \left(\int_0^1 \left[(B - A)^{1/2} (\lambda + sB + (1-s)A)^{-1} (B - A)^{1/2} \right]^2 ds \right) d\mu(\lambda), \end{aligned} \tag{2.8}$$

which proves the identity in (2.1). Since

$$\left[(B - A)^{1/2} (\lambda + sB + (1-s)A)^{-1} (B - A)^{1/2} \right]^2 \geq 0$$

then by integrating over s on $[0, 1]$, multiplying by $w(\lambda) \geq 0$ and integrating over $d\mu(\lambda)$, we deduce the inequality in (2.1). $\square$

The case of operator monotone functions is as follows:

Corollary 2.2. *Assume that f is operator monotone on $[0,\infty)$, then all A, $B>0$ with $B-A\geq 0$ we have the equality*

$$\begin{aligned}0 &\leq (B-A)^{1/2}\left[f(A)A^{-1}-f(B)B^{-1}\right](B-A)^{1/2}\\ &- f(0)(B-A)^{1/2}\left(A^{-1}-B^{-1}\right)(B-A)^{1/2}\\ &=\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)\lambda\, d\mu(\lambda)\end{aligned} \tag{2.9}$$

for some positive measure $\mu(\lambda)$. If $f(0)=0$, then

$$\begin{aligned}0 &\leq (B-A)^{1/2}\left[f(A)A^{-1}-f(B)B^{-1}\right](B-A)^{1/2}\\ &=\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)\times\lambda\, d\mu(\lambda).\end{aligned} \tag{2.10}$$

Proof. From (1.1) we have the representation

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t), \tag{2.11}$$

with $\ell(\lambda)=\lambda$, for some positive measure $\mu(\lambda)$ and nonnegative number b. Since

$$\begin{aligned}\mathcal{D}(\ell,\mu)(A)-\mathcal{D}(\ell,\mu)(B) &= [f(A)-f(0)]A^{-1}-[f(B)-f(0)]B^{-1}\\ &= f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right),\end{aligned}$$

hence by (2.1) we get (2.9). □

The case of operator convex functions is as follows:

Corollary 2.3. *Assume that f is operator convex on $[0,\infty)$, then all A, $B>0$ with $B-A\geq 0$ we have that*

$$\begin{aligned}0 &\leq (B-A)^{1/2}\left[f(A)A^{-2}-f(B)B^{-2}\right](B-A)^{1/2}\\ &- f'_+(0)(B-A)^{1/2}\left(A^{-1}-B^{-1}\right)(B-A)^{1/2}\\ &- f(0)(B-A)^{1/2}\left(A^{-2}-B^{-2}\right)(B-A)^{1/2}\\ &=\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)\times\lambda\, d\mu(\lambda),\end{aligned} \tag{2.12}$$

for some positive measure $\mu(\lambda)$. If $f(0)=0$, then

$$\begin{aligned}0 &\leq (B-A)^{1/2}\left[f(A)A^{-2}-f(B)B^{-2}\right](B-A)^{1/2}\\ &- f'_+(0)(B-A)^{1/2}\left(A^{-1}-B^{-1}\right)(B-A)^{1/2}\\ &=\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)\times\lambda\, d\mu(\lambda).\end{aligned} \tag{2.13}$$

Proof. From (1.3) we have that

$$\frac{f(t)-f(0)-f'_+(0)\,t}{t^2}-c=\mathcal{D}(\ell,\mu)(t),$$

for $t>0$. Then for A, $B>0$,

$$\begin{aligned}\mathcal{D}(\ell,\mu)(A)-\mathcal{D}(\ell,\mu)(B)&=f(A)A^{-2}-f'_+(0)A^{-1}-f(0)A^{-2}-f(A)B^{-2}+f'_+(0)B^{-1}+f(0)B^{-2}\\&=f(A)A^{-2}-f(B)B^{-2}-f'_+(0)\left(A^{-1}-B^{-1}\right)-f(0)\left(A^{-2}-B^{-2}\right)\end{aligned}$$

and by (2.1) we derive (2.13). □

When more conditions are imposed on the operators A and B we have the following refinements and reverses of the inequality

$$0\le\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)$$

that hold for $B-A\ge0$.

Theorem 2.4. *If $\beta\ge A\ge\alpha>0$, $B>0$ with $M\ge B-A\ge m>0$ for some constants α, β, m, M, then*

$$\begin{aligned}0&\le\frac{m^2}{M^2}\left[\mathcal{D}(w,\mu)(\beta)-\mathcal{D}(w,\mu)(M+\beta)\right]\\&\le\frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta)-\mathcal{D}(w,\mu)(M+\beta)\right](B-A)^{-1}\\&\le\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\\&\le\frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}(w,\mu)(m+\alpha)\right](B-A)^{-1}\\&\le\frac{M^2}{m^2}\left[\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}(w,\mu)(m+\alpha)\right].\end{aligned}\tag{2.14}$$

Proof. For $s\in[0,1]$ we have

$$\lambda+sB+(1-s)A=\lambda+s(B-A)+A.$$

We have

$$\lambda+s(B-A)+A\ge\lambda+sm+A\ge\lambda+sm+\alpha=\lambda+(1-s)\alpha+s(m+\alpha),$$

$s\in[0,1]$ and $\lambda\ge0$, which implies that

$$(\lambda+sB+(1-s)A)^{-1}\le[\lambda+(1-s)\alpha+s(m+\alpha)]^{-1}$$

and, by multiplying both sides by $(B-A)^{1/2}\ge0$,

$$\begin{aligned}(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}&\le[\lambda+(1-s)\alpha+s(m+\alpha)]^{-1}(B-A)\\&\le M[\lambda+(1-s)\alpha+s(m+\alpha)]^{-1}.\end{aligned}$$

Furthermore,

$$\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 \leq M^2\left[\lambda+(1-s)\alpha+s(m+\alpha)\right]^{-2},$$

for $s\in[0,1]$ and $\lambda\geq 0$, which implies by integration that

$$\begin{aligned}
&\int_0^\infty w(\lambda)\left(\int_0^1\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)d\mu(\lambda)\\
&\leq M^2\int_0^\infty w(\lambda)\left(\int_0^1\left[\lambda+(1-s)\alpha+s(m+\alpha)\right]^{-2}ds\right)d\mu(\lambda)\\
&=\frac{M^2}{m}\int_0^\infty w(\lambda)\left(\int_0^1\left[\lambda+(1-s)\alpha+s(m+\alpha)\right]^{-1}(m+\alpha-\alpha)\right.\\
&\left.\times\left[\lambda+(1-s)\alpha+s(m+\alpha)\right]^{-1}ds\right)d\mu(\lambda)\quad\text{(and by (2.7))}\\
&=\frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}(w,\mu)(m+\alpha)\right].
\end{aligned}$$

Using (2.8) we get

$$(B-A)^{1/2}\left[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\right](B-A)^{1/2}\leq\frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}(w,\mu)(m+\alpha)\right].$$

Multiplying both sides with $(B-A)^{-1/2}$ we deduce the fourth inequality in (2.14). We also have

$$\lambda+s(B-A)+A\leq\lambda+sM+A\leq\lambda+sM+\beta=\lambda+(1-s)\beta+s(M+\beta),$$

which implies that

$$(\lambda+sB+(1-s)A)^{-1}\geq\left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-1}$$

and, by multiplying both sides by $(B-A)^{1/2}\geq 0$,

$$\begin{aligned}
(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}&\geq\left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-1}(B-A)\\
&\geq m\left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-1},
\end{aligned}$$

for $s\in[0,1]$ and $\lambda\geq 0$. By taking the square, we get

$$\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2\geq m^2\left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-2},$$

for $s\in[0,1]$ and $\lambda\geq 0$. By taking the integrals in this inequality we obtain

$$\begin{aligned}
&\int_0^\infty w(\lambda)\left(\int_0^1\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)d\mu(\lambda)\\
&\geq m^2\int_0^\infty w(\lambda)\left(\int_0^1\left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-2}ds\right)d\mu(\lambda)\\
&=\frac{m^2}{M}\int_0^\infty w(\lambda)\left(\int_0^1\left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-1}(M+\beta-\beta)\right.\\
&\left.\times\left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-1}ds\right)d\mu(\lambda)\quad\text{(and by (2.7))}\\
&=\frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta)-\mathcal{D}(w,\mu)(M+\beta)\right].
\end{aligned}$$

Using (2.8) we get

$$(B-A)^{1/2}\left[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\right](B-A)^{1/2}\geq\frac{m^{2}}{M}\left[\mathcal{D}(w,\mu)(\beta)-\mathcal{D}(w,\mu)(M+\beta)\right].$$

Multiplying both sides with $(B-A)^{-1/2}$ we deduce the second inequality in (2.14). The rest of the inequalities are obvious. □

It is well known that, if $P\geq 0$, then

$$|\langle Px,y\rangle|^{2}\leq\langle Px,x\rangle\langle Py,y\rangle,$$

for all x, $y\in H$. Therefore, if $T>0$, then

$$0\leq\langle x,x\rangle^{2}=\left\langle T^{-1}Tx,x\right\rangle^{2}=\left\langle Tx,T^{-1}x\right\rangle^{2}\leq\langle Tx,x\rangle\left\langle TT^{-1}x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle,$$

for all $x\in H$. If $x\in H$, $\|x\|=1$, then

$$1\leq\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle\leq\langle Tx,x\rangle\sup_{\|x\|=1}\left\langle x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1}\leq T. \tag{2.15}$$

Remark 2.5. *If $A>0$ and $B-A>0$, then obviously $\|A\|\geq A\geq\left\|A^{-1}\right\|^{-1}$ and $\|B-A\|\geq B-A\geq\left\|(B-A)^{-1}\right\|^{-1}$. So, if we take $\beta=\|A\|$, $\alpha=\left\|A^{-1}\right\|^{-1}$, $M=\|B-A\|$ and $m=\left\|(B-A)^{-1}\right\|^{-1}$ in (2.14), then we get*

$$\begin{aligned}
0&\leq\frac{\mathcal{D}(w,\mu)(\|A\|)-\mathcal{D}(w,\mu)(\|B-A\|+\|A\|)}{\|B-A\|^{2}\left\|(B-A)^{-1}\right\|^{2}}\\
&\leq\frac{\mathcal{D}(w,\mu)(\|A\|)-\mathcal{D}(w,\mu)(\|B-A\|+\|A\|)}{\|B-A\|\left\|(B-A)^{-1}\right\|^{2}}(B-A)^{-1}\\
&\leq\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\\
&\leq\|B-A\|^{2}\left\|(B-A)^{-1}\right\|\\
&\times\left[\mathcal{D}(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right)-\mathcal{D}(w,\mu)\left(\left\|(B-A)^{-1}\right\|^{-1}+\left\|A^{-1}\right\|^{-1}\right)\right]\times(B-A)^{-1}\\
&\leq\|B-A\|^{2}\left\|(B-A)^{-1}\right\|^{2}\\
&\times\left[\mathcal{D}(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right)-\mathcal{D}(w,\mu)\left(\left\|(B-A)^{-1}\right\|^{-1}+\left\|A^{-1}\right\|^{-1}\right)\right].
\end{aligned} \tag{2.16}$$

Corollary 2.6. *Assume that f is operator monotone on $[0,\infty)$. If $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then*

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\frac{f(\beta)}{\beta} - \frac{f(M+\beta)}{M+\beta} - \frac{M}{\beta(M+\beta)} f(0)\right] \\
&\leq \frac{m^2}{M}\left[\frac{f(\beta)}{\beta} - \frac{f(M+\beta)}{M+\beta} - \frac{M}{\beta(M+\beta)} f(0)\right](B-A)^{-1} \\
&\leq f(A) A^{-1} - f(B) B^{-1} - f(0)\left(A^{-1} - B^{-1}\right) \\
&\leq \frac{M^2}{m}\left[\frac{f(\alpha)}{\alpha} - \frac{f(m+\alpha)}{m+\alpha} - \frac{m}{\alpha(m+\alpha)} f(0)\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\frac{f(\alpha)}{\alpha} - \frac{f(m+\alpha)}{m+\alpha} - \frac{m}{\alpha(m+\alpha)} f(0)\right].
\end{aligned} \tag{2.17}$$

If $f(0) = 0$, then

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\frac{f(\beta)}{\beta} - \frac{f(M+\beta)}{M+\beta}\right] \leq \frac{m^2}{M}\left[\frac{f(\beta)}{\beta} - \frac{f(M+\beta)}{M+\beta}\right](B-A)^{-1} \\
&\leq f(A) A^{-1} - f(B) B^{-1} \leq \frac{M^2}{m}\left[\frac{f(\alpha)}{\alpha} - \frac{f(m+\alpha)}{m+\alpha}\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\frac{f(\alpha)}{\alpha} - \frac{f(m+\alpha)}{m+\alpha}\right].
\end{aligned} \tag{2.18}$$

The proof follows by (2.14) and the representation (2.11).

Remark 2.7. *If $A > 0$ and $B - A > 0$, then for f an operator monotone function on $[0,\infty)$ with $f(0) = 0$, we obtain from (2.18) some similar inequalities to the ones in Remark 2.5. We omit the details.*

The case of operator convex functions is as follows:

Corollary 2.8. *Assume that f is operator convex on $[0,\infty)$. If $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then*

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\frac{f(\beta)}{\beta^2} - \frac{f(M+\beta)}{(M+\beta)^2} - f'_+(0)\frac{M}{\beta(M+\beta)} - f(0)\frac{M(M+2\beta)}{\beta^2(M+\beta)^2}\right] \\
&\leq \frac{m^2}{M}\left[\frac{f(\beta)}{\beta^2} - \frac{f(M+\beta)}{(M+\beta)^2} - f'_+(0)\frac{M}{\beta(M+\beta)} - f(0)\frac{M(M+2\beta)}{\beta^2(M+\beta)^2}\right] \times (B-A)^{-1} \\
&\leq f(A) A^{-2} - f(B) B^{-2} - f'_+(0)\left(A^{-1} - B^{-1}\right) - f(0)\left(A^{-2} - B^{-2}\right) \\
&\leq \frac{M^2}{m}\left[\frac{f(\alpha)}{\alpha^2} - \frac{f(m+\alpha)}{(m+\alpha)^2} - f'_+(0)\frac{m}{\alpha(m+\alpha)} - f(0)\frac{m(m+2\alpha)}{\alpha^2(m+\alpha)^2}\right] \times (B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\frac{f(\alpha)}{\alpha^2} - \frac{f(m+\alpha)}{(m+\alpha)^2} - f'_+(0)\frac{m}{\alpha(m+\alpha)} - f(0)\frac{m(m+2\alpha)}{\alpha^2(m+\alpha)^2}\right].
\end{aligned} \tag{2.19}$$

If $f(0)=0$, *then*

$$
\begin{aligned}
0 &\le \frac{m^2}{M^2}\left[\frac{f(\beta)}{\beta^2}-\frac{f(M+\beta)}{(M+\beta)^2}-f'_+(0)\frac{M}{\beta(M+\beta)}\right] \\
&\le \frac{m^2}{M}\left[\frac{f(\beta)}{\beta^2}-\frac{f(M+\beta)}{(M+\beta)^2}-f'_+(0)\frac{M}{\beta(M+\beta)}\right](B-A)^{-1} \\
&\le f(A)A^{-2}-f(B)B^{-2}-f'_+(0)\left(A^{-1}-B^{-1}\right) \qquad (2.20)\\
&\le \frac{M^2}{m}\left[\frac{f(\alpha)}{\alpha^2}-\frac{f(m+\alpha)}{(m+\alpha)^2}-f'_+(0)\frac{m}{\alpha(m+\alpha)}\right](B-A)^{-1} \\
&\le \frac{M^2}{m^2}\left[\frac{f(\alpha)}{\alpha^2}-\frac{f(m+\alpha)}{(m+\alpha)^2}-f'_+(0)\frac{m}{\alpha(m+\alpha)}\right].
\end{aligned}
$$

Remark 2.9. *If $A>0$ and $B-A>0$, then for f an operator convex function on $[0,\infty)$ with $f(0)=0$, we obtain from (2.20) some similar inequalities to the ones in Remark 2.5. We omit the details.*

3 Some examples

The function $f(t)=t^r$, $r\in(0,1]$ is operator monotone on $[0,\infty)$ and by (2.18) we obtain the power inequalities

$$
\begin{aligned}
0 &\le \frac{m^2}{M^2}\left[\beta^{r-1}-(M+\beta)^{r-1}\right] \le \frac{m^2}{M}\left[\beta^{r-1}-(M+\beta)^{r-1}\right](B-A)^{-1} \\
&\le A^{r-1}-B^{r-1} \le \frac{M^2}{m}\left[\alpha^{r-1}-(m+\alpha)^{r-1}\right](B-A)^{-1} \qquad (3.1)\\
&\le \frac{M^2}{m^2}\left[\alpha^{r-1}-(m+\alpha)^{r-1}\right],
\end{aligned}
$$

provided that $\beta\ge A\ge\alpha>0$, $B>0$ with $M\ge B-A\ge m>0$ for some constants α, β, m, M.

The function $f(t)=\ln(t+1)$ is operator monotone on $[0,\infty)$ and by (2.18) we get

$$
\begin{aligned}
0 &\le \frac{m^2}{M^2}\left[\frac{\ln(\beta+1)}{\beta}-\frac{\ln(M+\beta+1)}{M+\beta}\right] \le \frac{m^2}{M}\left[\frac{\ln(\beta+1)}{\beta}-\frac{\ln(M+\beta+1)}{M+\beta}\right](B-A)^{-1} \\
&\le A^{-1}\ln(A+1)-B^{-1}\ln(B+1) \le \frac{M^2}{m}\left[\frac{\ln(\alpha+1)}{\alpha}-\frac{\ln(m+\alpha+1)}{m+\alpha}\right](B-A)^{-1} \qquad (3.2)\\
&\le \frac{M^2}{m^2}\left[\frac{\ln(\alpha+1)}{\alpha}-\frac{\ln(m+\alpha+1)}{m+\alpha}\right],
\end{aligned}
$$

provided that $\beta\ge A\ge\alpha>0$, $B>0$ with $M\ge B-A\ge m>0$ for some constants α, β, m, M.

The function $f(t) = -\ln(t+1)$ is operator convex, and by (2.20) we obtain

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\frac{\ln(M+\beta+1)}{(M+\beta)^2} - \frac{\ln(\beta+1)}{\beta^2} + \frac{M}{\beta(M+\beta)}\right] \\
&\leq \frac{m^2}{M}\left[\frac{\ln(M+\beta+1)}{(M+\beta)^2} - \frac{\ln(\beta+1)}{\beta^2} + \frac{M}{\beta(M+\beta)}\right](B-A)^{-1} \\
&\leq B^{-2}\ln(B+1) - A^{-2}\ln(A+1) + A^{-1} - B^{-1} \\
&\leq \frac{M^2}{m}\left[\frac{\ln(m+\alpha+1)}{(m+\alpha)^2} - \frac{\ln(\alpha+1)}{\alpha^2} + \frac{m}{\alpha(m+\alpha)}\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\frac{\ln(m+\alpha+1)}{(m+\alpha)^2} - \frac{\ln(\alpha+1)}{\alpha^2} + \frac{m}{\alpha(m+\alpha)}\right],
\end{aligned} \tag{3.3}$$

provided that $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M.

Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a > 0$. Then

$$D(e_{-a})(t) := \int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda}\,d\lambda = E_1(at)\exp(at), \quad t \geq 0,$$

where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}\,du, \quad t \geq 0. \tag{3.4}$$

For $a = 1$ we have

$$D(e_{-1})(t) := \int_0^\infty \frac{\exp(-\lambda)}{t+\lambda}\,d\lambda = E_1(t)\exp(t), \quad t \geq 0.$$

Let $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M. Then by (2.14) we have

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[E_1(a\beta)\exp(a\beta) - E_1(a(M+\beta))\exp(a(M+\beta))\right] \\
&\leq \frac{m^2}{M}\left[E_1(a\beta)\exp(a\beta) - E_1(a(M+\beta))\exp(a(M+\beta))\right](B-A)^{-1} \\
&\leq E_1(aA)\exp(aA) - E_1(aB)\exp(aB) \\
&\leq \frac{M^2}{m}\left[E_1(a\alpha)\exp(a\alpha) - E_1(a(m+\alpha))\exp(a(m+\alpha))\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[E_1(a\alpha)\exp(a\alpha) - E_1(a(m+\alpha))\exp(a(m+\alpha))\right],
\end{aligned} \tag{3.5}$$

for $a > 0$. For $a = 1$ we have

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[E_1(\beta)\exp(\beta) - E_1(M+\beta)\exp(M+\beta)\right] \\
&\leq \frac{m^2}{M}\left[E_1(\beta)\exp(\beta) - E_1(M+\beta)\exp(M+\beta)\right](B-A)^{-1} \\
&\leq E_1(A)\exp(A) - E_1(B)\exp(B) \\
&\leq \frac{M^2}{m}\left[E_1(\alpha)\exp(\alpha) - E_1(m+\alpha)\exp(m+\alpha)\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[E_1(\alpha)\exp(\alpha) - E_1(m+\alpha)\exp(m+\alpha)\right].
\end{aligned} \tag{3.6}$$

More examples of such transforms are

$$D(w_{1/(\ell^2+a^2)})\,(t) := \int_0^\infty \frac{1}{(t+\lambda)\,(\lambda^2+a^2)}\,d\lambda = \frac{\pi t - 2a\ln(t/a)}{2a\,(t^2+a^2)}, \quad t \geq 0$$

and

$$D(w_{\ell/(\ell^2+a^2)})\,(t) := \int_0^\infty \frac{\lambda}{(t+\lambda)\,(\lambda^2+a^2)}\,d\lambda = \frac{\pi a + 2t\ln(t/a)}{2a\,(t^2+a^2)}, \quad t \geq 0,$$

for $a > 0$. The interested reader may state other similar results by employing the examples of monotone operator functions provided in [2, 3, 4, 7] and [8].

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