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Maximum, anti-maximum principles and monotone methods for boundary value problems for Riemann-Liouville fractional differential equations in neighborhoods of simple eigenvalues

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ABSTRACT

It has been shown that, under suitable hypotheses, boundary value problems of the form, $Ly + \lambda y = f$, $BCy = 0$ where L is a linear ordinary or partial differential operator and BC denotes a linear boundary operator, then there exists $\Lambda > 0$ such that $f \geq 0$ implies $\lambda y \geq 0$ for $\lambda \in [-\Lambda, \Lambda] \setminus \{0\}$, where y is the unique solution of $Ly + \lambda y = f$, $BCy = 0$. So, the boundary value problem satisfies a maximum principle for $\lambda \in [-\Lambda, 0)$ and the boundary value problem satisfies an anti-maximum principle for $\lambda \in (0, \Lambda]$. In an abstract result, we shall provide suitable hypotheses such that boundary value problems of the form, $D_0^\alpha y + \beta D_0^{\alpha-1} y = f$, $BCy = 0$ where D_0^α is a Riemann-Liouville fractional differentiable operator of order α, $1 < \alpha \leq 2$, and BC denotes a linear boundary operator, then there exists $\mathcal{B} > 0$ such that $f \geq 0$ implies $\beta D_0^{\alpha-1} y \geq 0$ for $\beta \in [-\mathcal{B}, \mathcal{B}] \setminus \{0\}$, where y is the unique solution of $D_0^\alpha y + \beta D_0^{\alpha-1} y = f$, $BCy = 0$. Two examples are provided in which the hypotheses of the abstract theorem are satisfied to obtain the sign property of $\beta D_0^{\alpha-1} y$. The boundary conditions are chosen so that with further analysis a sign property of βy is also obtained. One application of monotone methods is developed to illustrate the utility of the abstract result.

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RESUMEN

Se ha demostrado que, bajo hipótesis apropiadas, problemas de valor en la frontera de la forma $Ly + \lambda y = f$, $BCy = 0$, donde L es un operador diferencial lineal ordinario o parcial y BC denota un operador lineal de frontera, entonces existe $\Lambda > 0$ tal que $f \geq 0$ implica $\lambda y \geq 0$ para $\lambda \in [-\Lambda, \Lambda] \setminus \{0\}$, donde y es la única solución de $Ly + \lambda y = f$, $BCy = 0$. Así, el problema de valor en la frontera satisface un principio del máximo para $\lambda \in [-\Lambda, 0)$ y el problema de valor en la frontera satisface un anti-principio del máximo para $\lambda \in (0, \Lambda]$. En un resultado abstracto, entregaremos hipótesis apropiadas tales que los problemas de valor en la frontera de la forma $D_0^\alpha y + \beta D_0^{\alpha-1} y = f$, $BCy = 0$ donde D_0^α es un operador diferencial fraccionario de Riemann-Liouville de orden α, $1 < \alpha \leq 2$, y BC denota un operador lineal de frontera, entonces existe $\mathcal{B} > 0$ tal que $f \geq 0$ implica $\beta D_0^{\alpha-1} y \geq 0$ para $\beta \in [-\mathcal{B}, \mathcal{B}] \setminus \{0\}$, donde y es la única solución de $D_0^\alpha y + \beta D_0^{\alpha-1} y = f$, $BCy = 0$. Se entregan dos ejemplos en los cuales las hipótesis del teorema abstracto se satisfacen para obtener la propiedad de signo de $\beta D_0^{\alpha-1} y$. Las condiciones de frontera se eligen de tal forma de obtener también una propiedad de signo para βy con un análisis adicional. Se desarrolla una aplicación de métodos monótonos para ilustrar la utilidad del resultado abstracto.

Keywords and Phrases: Maximum principle, anti-maximum principle, Riemann-Liouville fractional differential equation, boundary value problem, monotone methods, upper and lower solution.

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1 Introduction

For $\gamma > 0$, $y \in \mathcal{L}[0,1]$, the space of Lebesgue integrable functions, the expression

$$I_0^\gamma y(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} y(s)\, ds, \quad 0 \le t \le 1,$$

denotes a Riemann-Liouville fractional integral of y of order γ, where Γ denotes the special gamma function. For $\gamma = 0$, I_0^0 is defined to be the identity operator.

Let n denote a positive integer and assume $n-1 < \alpha \le n$. Then $D_0^\alpha y(t) = D^n I_0^{n-\alpha} y(t)$, where $D^n = \dfrac{d^n}{dt^n}$ and if this expression exists, denotes a Riemann-Liouville fractional derivative of y of order α. So, if $1 < \alpha < 2$, $D_0^\alpha y(t) = \dfrac{d^2}{dt^2} \displaystyle\int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} y(s)\, ds$ if the right hand side exists. In the case α is a positive integer, we may write $D_0^\alpha y(t) = D^\alpha y(t)$ or $I_0^\alpha y(t) = I^\alpha y(t)$ since the Riemann-Liouville derivative or integral agrees with the classical derivative or integral if α is a positive integer.

For authoritative accounts on the development of fractional calculus, we refer to the monographs [11, 16, 20]. For the sake of self-containment, we state properties that we shall employ in this study. It is well-know that the Riemann-Liouville fractional integrals commute; that is if $\gamma_1, \gamma_2 > 0$, and $y \in \mathcal{L}[0,1]$, then

$$I_0^{\gamma_1} I_0^{\gamma_2} y(t) = I_0^{\gamma_1+\gamma_2} y(t) = I_0^{\gamma_2} I_0^{\gamma_1} y(t).$$

A power rule is valid for the Riemann-Liouville fractional integral; if $\delta > -1$ and $\gamma \ge 0$, then

$$I_0^\gamma t^\delta = I_0^\gamma (t-0)^\delta = \frac{\Gamma(\delta+1)}{\Gamma(\delta+1+\gamma)} t^{\delta+\gamma}.$$

A power rule is valid for the Riemann-Liouville fractional derivative; if $\delta > -1$ and $\gamma \ge 0$ then

$$D_0^\gamma t^\delta = \frac{\Gamma(\delta+1)}{\Gamma(\delta+1-\gamma)} t^{\delta-\gamma}.$$

Since the gamma function is unbounded at 0, it is the convention that if $\delta + 1 - \gamma = 0$, then $D_0^\gamma t^\delta = 0$. Note that if $1 < \alpha \le 2$, and if $D_0^\alpha y(t)$ exists, then $D_0^{\alpha-1} y(t)$ exists and

$$D_0^\alpha y(t) = D^2 I_0^{2-\alpha} y(t) = DDI_0^{1-(\alpha-1)} y(t) = DD_0^{\alpha-1} y(t).$$

In [12], a boundary value problem,

$$D_0^\alpha y(t) = f(t, y(t)), \quad 0 < t \le 1, \tag{1.1}$$

$$y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1), \tag{1.2}$$

where $1 < \alpha \leq 2$, was studied. This is an example of a boundary value problem at resonance since $< t^{\alpha-1} >$, the linear span of $t^{\alpha-1}$, denotes the solution space of the homogeneous problem, $D_0^\alpha y = 0$, with homogeneous boundary conditions, (1.2). In [12], the purpose of that article was to consider an equivalent shifted equation, $D_0^\alpha y(t) - K^2 y(t) = f(t, y(t)) - K^2 y(t)$, $0 < t \leq 1$, and apply the method of quasilinearization to the shifted boundary value problem. The method of quasilinearization is ideally suited when the boundary value problem, in this case the shifted boundary value problem, satisfies a maximum principle [19]. In particular, in [12], a nonpositive Green's function for the shifted boundary value problem was explicitly constructed. Not surprisingly, Mittag-Leffler functions were key to the construction and signing of the Green's function. The case, $D_0^\alpha y(t) + K^2 y(t) = f(t, y(t)) + K^2 y(t)$, $0 < t \leq 1$, was not addressed in [12].

The maximum principle is well-known and is an important tool in the qualitative study of differential equations; we refer the reader to the well-known monograph [19] for many applications. In recent years, the maximum principle has become an important tool in the study of boundary value problems for fractional differential equations. Early applications appear in [24] and [3] where explicit Green's functions, expressed in terms of power functions, where constructed and sign properties were analyzed so that fixed point theorems could be applied. Many authors have continued the strategy to construct and analyze explicit Green's functions and apply fixed point theory to nonlinear boundary value problems for fractional differential equations.

In the example, $y'' + \lambda y = f$, with Neumann boundary conditions, $y'(0) = 0$, $y'(1) = 0$, if $\lambda < 0$, then this boundary value problem satisfies a maximum principle. In particular, for $f \in \mathcal{L}[0, 1]$, the boundary value problem is uniquely solvable and f nonnegative implies y is nonpositive where y is the unique solution associated with f. Clément and Peletier [9] were the first to discover an anti-maximum principle. They were primarily interested in partial differential equations, but they illustrated the anti-maximum principle with the the same boundary value problem, $y'' + \lambda y = f$, $y'(0) = 0$, $y'(1) = 0$, but now, $0 < \lambda < \frac{\pi^2}{4}$. For this particular boundary value problem, if $0 < \lambda < \frac{\pi^2}{4}$ and if $f \in \mathcal{L}[0, 1]$, then the boundary value problem is uniquely solvable and f nonnegative implies y is nonnegative where y is the unique solution associated with f. At $\lambda = 0$, the boundary value problem is at resonance, and more precisely, $\lambda = 0$ denotes a simple eigenvalue of the linear problem. So there has been a change in the sign property, maximum principle or anti-maximum principle, through the simple eigenvalue $\lambda = 0$. Since the publication of [9] there have been many studies of boundary value problems with parameter and the change of behavior from maximum to anti-maximum principles as a function of the parameter. In the case of partial differential equations, we refer to [1, 2, 8, 10, 14, 17, 18, 21]. In the case of ordinary differential equations we refer to [4, 5, 6, 7, 13, 22].

In this article, we intend to study this change in behavior for a boundary value problem for a Riemann-Liouville fractional differential equation. We shall modify the methods developed in [8], where in [8], those authors began with an ordinary differential equation

$$y''(t) + \lambda y(t) = f(t), \quad t \in [0,1], \tag{1.3}$$

and considered either periodic boundary conditions or Neumann boundary conditions. Key to their argument is that for $f = 0$, at $\lambda = 0$, the boundary value problem, (1.3) with periodic or Neumann boundary conditions, is at resonance since constant functions are nontrivial solutions. That is, $\lambda = 0$ is a simple eigenvalue for the problem, (1.3) with periodic or Neumann boundary conditions, and the eigenspace is $< 1 >$, where $< 1 >$ denotes the linear span of the 1 function. Rewriting the boundary value problem as an abstract equation and employing the resolvent, the inverse of $(D^2 + \lambda \mathcal{I})$ for $\lambda \neq 0$, under the imposed boundary conditions, if it exists, and the partial resolvent for $\lambda = 0$, then under the assumption that $f \geq 0$ (with $f \in \mathcal{L}[0,1]$), the authors in [8] exhibited sufficient conditions for the existence of $\Lambda > 0$, and a constant $K > 0$, independent of f, such that

$$\lambda y(t) \geq K|f|_1, \quad \lambda \in [-\Lambda, \Lambda] \setminus \{0\}, \quad 0 \leq t \leq 1,$$

where y is the unique solution of the boundary value problem associated with (1.3) and $|f|_1 = \int_0^1 |f(s)|\,ds$. With this one inequality the authors showed that for $-\Lambda \leq \lambda < 0$ the boundary value problem, (1.3) with periodic or Neumann boundary conditions, satisfies a maximum principle and for $0 < \lambda \leq \Lambda$, the boundary value problem (1.3) with periodic or Neumann boundary conditions, satisfies an anti-maximum principle. They referred to this principle as a maximum principle (we shall take the liberty to refer to it as a signed maximum principle in y) and then proceeded to produce many nice examples.

Recently, [13], the arguments developed in [8] were adapted to study boundary value problems for the ordinary differential equation

$$y''(t) + \beta y'(t) = f(t), \quad t \in [0,1]; \tag{1.4}$$

sufficient conditions for a signed maximum principle in Dy, where $Dy = y'$, were obtained. That is, under the assumption that $f \geq 0$ (with $f \in \mathcal{L}[0,1]$), sufficient conditions were exhibited to imply the existence of $\mathcal{B} > 0$, and a constant $K > 0$, independent of f such that

$$\beta Dy(t) \geq K|f|_1, \quad \lambda \in [-\mathcal{B}, \mathcal{B}] \setminus \{0\}, \quad 0 \leq t \leq 1.$$

Two examples of boundary value problems were presented in which if a solution y of the boundary value problem is such that $Dy = y'$ has constant sign on $[0,1]$, then y has constant sign on $[0,1]$. For one of the examples, an appropriate partial order in $C^1[0,1]$, depending on the sign of β, was defined and the method of upper and lower solutions, coupled with monotone methods, was employed to obtain sufficient conditions for the existence of solutions of the boundary value problem for a nonlinear differential equation,

$$y''(t) = f(t, y(t), y'(t)), \quad t \in [0, 1].$$

Motivated by the work in [13], we shall adapt the methods developed in [8] and exhibit sufficient conditions to obtain a signed maximum principle in $D_0^{\alpha-1}y$, for the boundary value problem $D_0^{\alpha}y(t) + \beta D_0^{\alpha-1}y(t) = f(t)$, with boundary conditions

$$BCy = 0, \quad D_0^{\alpha-1}y(0) = D_0^{\alpha-1}y(1), \tag{1.5}$$

where BC denotes a linear boundary operator mapping a function y to the reals. In particular, we shall exhibit sufficient conditions that imply the existence of $\mathcal{B} > 0$, and a constant $K > 0$, independent of f, such that

$$\beta D_0^{\alpha-1}y(t) \geq K|f|_1, \quad \beta \in [-\mathcal{B}, \mathcal{B}] \setminus \{0\}, \quad 0 \leq t \leq 1. \tag{1.6}$$

In two examples, the boundary condition BC will be such that if y satisfies the boundary conditions (1.5) and $\beta D_0^{\alpha-1}y(t) > 0$ on $[0, 1]$, then $\beta y(t) \geq 0$ on $(0, 1]$. In one of the examples, an appropriate partial order in a Banach space is defined and the method of upper and lower solutions, coupled with monotone methods, is applied to obtain sufficient conditions for the existence of solutions of the nonlinear differential equation

$$D_0^{\alpha}y(t) = f(t, y(t), \beta D_0^{\alpha-1}y(t)), \quad t \in (0, 1],$$

satisfying the boundary conditions, (1.5).

In Section 2, following the lead of [8], we shall first define the concept of a signed maximum principle in $D_0^{\alpha-1}y$. Then analogous to Lemma 1, Lemma 2 and Lemma 3 in [8], we shall prove the main theorem and obtain sufficient conditions for (1.6) and hence, obtain sufficient conditions for adherence to a signed maximum principle in $D_0^{\alpha-1}y$. In Section 3, we shall exhibit two examples that adhere to a strong signed maximum principle in $D_0^{\alpha-1}y$ and furthermore (1.6) implies $\beta y(t) \geq 0$ on $(0, 1]$. We shall close in Section 4 with an application of a monotone method applied to a nonlinear problem related to one of the examples produced in Section 4. At $\beta = 0$, the problem is at resonance. The problem is shifted [15] by $\beta D_0^{\alpha-1}y$ and $\beta > 0$ or $\beta < 0$ is chosen as a function of the monotonicity properties of the nonlinear term $f(t, y(t), \beta D_0^{\alpha-1}y(t))$.

2 The main theorem

As is standard, let $C[0,1]$ denote the Banach space of continuous functions defined on $[0,1]$ with the supremum norm, $|\cdot|_0$, and let $\mathcal{L}[0,1]$ denote the space of Lebesgue integrable functions with the usual $\mathcal{L}_1$ norm. Employing notation introduced in [23], assume $1<\alpha\leq 2$ and define

$$C_{\alpha-2}[0,1]=\left\{y:(0,1]\to\mathbb{R}: y(t) \text{ is continuous for } t\in(0,1], \text{ and } \lim_{t\to 0^+} t^{2-\alpha}y(t) \text{ exists}\right\}.$$

It is clear that $y\in C_{\alpha-2}[0,1]$ if, and only if, there exists $z\in C[0,1]$ such that $y(t)=t^{\alpha-2}z(t)$ for $t\in(0,1]$. Define $|y|_{\alpha-2}=|z|_0$ and $C_{\alpha-2}[0,1]$ with norm $|\cdot|_{\alpha-2}$ is a Banach space.

Let $\mathcal{X}_{\alpha-2}$ denote the Banach space

$$\mathcal{X}_{\alpha-2}=\{y:(0,1]\to\mathbb{R}: y\in C_{\alpha-2}[0,1], D_0^{\alpha-1}y\in C[0,1]\},$$

with

$$||y||=\max\{|y|_{\alpha-2},|D_0^{\alpha-1}y|_0\}.$$

The following definition is motivated by Definition 1 found in [8].

Definition 2.1. *Assume $\mathcal{A}$ is a linear operator with $Dom(\mathcal{A})\subset\mathcal{X}_{\alpha-2}$ and $Im(\mathcal{A})\subset\mathcal{L}[0,1]$. For $\beta\in\mathbb{R}\setminus\{0\}$, the operator $\mathcal{A}+\beta D_0^{\alpha-1}$ satisfies a* ***signed maximum principle in*** *$D_0^{\alpha-1}y$ if for each $f\in\mathcal{L}[0,1]$, the equation*

$$(\mathcal{A}+\beta D_0^{\alpha-1})y=f,\quad y\in Dom(\mathcal{A}),$$

has unique solution, y, and $f(t)\geq 0$, $0\leq t\leq 1$, implies $\beta D_0^{\alpha-1}y(t)\geq 0$, $0\leq t\leq 1$. The operator $\mathcal{A}+\beta D_0^{\alpha-1}$ satisfies a ***strong signed maximum principle in*** *$D_0^{\alpha-1}y$ if $f(t)\geq 0$, $0\leq t\leq 1$, and $f(t)\not\equiv 0$ a.e., implies $\beta D_0^{\alpha-1}y(t)>0$, $0\leq t\leq 1$.*

Remark 2.2. *Throughout this article, the phrases "maximum principle" or "anti-maximum principle" may be used loosely. If so, we mean the following. If $f\geq 0$ implies $D_0^{\alpha-1}y\leq 0$ the phrase maximum principle may be used. This is precisely the case for the classical second order ordinary differential equation with Dirichlet boundary conditions. If $f\geq 0$ implies $D_0^{\alpha-1}y\geq 0$ the phrase anti-maximum principle may be used. This is the case observed in [9] for $\alpha=2$, where the phrase anti-maximum principle was coined.*

For $f\in\mathcal{L}[0,1]$ (or $f\in C[0,1]$), let $|f|_1=\int_0^1|f(s)|\,ds$ and define $\overline{f}=\int_0^1 f(t)\,dt$. Define

$$\tilde{\mathcal{C}}\subset C[0,1]=\{f\in C[0,1]:\overline{f}=0\},\quad \tilde{\mathcal{L}}\subset\mathcal{L}[0,1]=\{f\in\mathcal{L}[0,1]:\overline{f}=0\}.$$

Assume $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \to \mathcal{L}[0,1]$ denotes a linear operator satisfying

$$\mathrm{Dom}(\mathcal{A}) \subset \mathcal{X}_{\alpha-2}, \quad \ker(\mathcal{A}) = < t^{\alpha-1} + ct^{\alpha-2} >, \quad \mathrm{Im}(\mathcal{A}) = \tilde{\mathcal{L}}, \tag{2.1}$$

for some real constant c, where $< t^{\alpha-1} + ct^{\alpha-2} >$ denotes the linear span of $t^{\alpha-1} + ct^{\alpha-2}$. Assume further that for $\tilde{f} \in \tilde{\mathcal{L}}$, the problem $\mathcal{A}y = \tilde{f}$ is uniquely solvable with solution $y \in \mathrm{Dom}(\mathcal{A})$ and such that $\int_0^1 D_0^{\alpha-1} y(t)\, dt = \overline{(D_0^{\alpha-1} y)} = 0$. In particular, define

$$\mathrm{Dom}(\tilde{\mathcal{A}}) = \left\{ y \in \mathrm{Dom}(\mathcal{A}) : \overline{(D_0^{\alpha-1} y)} = 0 \right\},$$

and then

$$\mathcal{A}|_{\mathrm{Dom}(\tilde{\mathcal{A}})} : \mathrm{Dom}(\tilde{\mathcal{A}}) \to \tilde{\mathcal{L}}$$

is one to one and onto. Moreover, if $\mathcal{A}\tilde{y} = \tilde{f}$ for $\tilde{f} \in \tilde{\mathcal{L}}$, $\tilde{y} \in \mathrm{Dom}(\tilde{\mathcal{A}})$, assume there exists a constant $M > 0$ depending only on $\mathcal{A}$ such that

$$|D_0^{\alpha-1} \tilde{y}|_0 \leq M |\tilde{f}|_1. \tag{2.2}$$

For $f \in \mathcal{L}$, define

$$\tilde{f} = f - \overline{f},$$

which implies $\tilde{f} \in \tilde{\mathcal{L}}$, and for $y \in \mathrm{Dom}(\mathcal{A})$ define

$$\tilde{y} = y - \overline{(D_0^{\alpha-1} y)} \frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

which implies $\tilde{y} \in \mathrm{Dom}(\tilde{\mathcal{A}})$ since

$$D_0^{\alpha-1}(\tilde{y}) = D_0^{\alpha-1} y - \overline{(D_0^{\alpha-1} y)}.$$

Finally assume there exists $\mathcal{A}' : \mathrm{Dom}(\mathcal{A}') \to \mathcal{L}$ such that $\mathcal{A} = \mathcal{A}' D_0^{\alpha-1}$. In this context, we rewrite

$$\mathcal{A}y + \beta D_0^{\alpha-1} y = f, \quad y \in \mathrm{Dom}(\mathcal{A}), \tag{2.3}$$

as

$$(\mathcal{A}' + \beta \mathcal{I}) D_0^{\alpha-1} y = f, \quad D_0^{\alpha-1} y \in \mathrm{Dom}(\mathcal{A}'). \tag{2.4}$$

Define $\mathrm{Dom}(\tilde{\mathcal{A}}') = \{ v \in \mathrm{Dom}(\mathcal{A}') : \overline{v} = 0 \} \subset C[0,1]$ and it follows that

$$\mathcal{A}'|_{\mathrm{Dom}(\tilde{\mathcal{A}}')} : \mathrm{Dom}(\tilde{\mathcal{A}}') \to \tilde{\mathcal{L}}$$

is one to one and onto. With the decompositions $\tilde{f} = f - \overline{f}$ and $\tilde{y} = y - \overline{D_0^{\alpha-1}y}\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, it follows that $\tilde{f} \in \tilde{\mathcal{L}}$ and $\tilde{y} \in \mathrm{Dom}(\tilde{\mathcal{A}})$, or more appropriately, $D_0^{\alpha-1}\tilde{y} \in \mathrm{Dom}(\tilde{\mathcal{A}}')$. So, equation (2.3) or equation (2.4) decouples as follows:

$$\mathcal{A}'D_0^{\alpha-1}\tilde{y} + \beta D_0^{\alpha-1}\tilde{y} = (\mathcal{A}' + \beta\mathcal{I})D_0^{\alpha-1}\tilde{y} = \tilde{f}, \tag{2.5}$$

$$\beta D_0^{\alpha-1}\left(\overline{D_0^{\alpha-1}y}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right) = \beta\overline{D_0^{\alpha-1}y} = \overline{f}. \tag{2.6}$$

Denote the inverse of $(\mathcal{A}' + \beta\mathcal{I})$, if it exists, by $\mathcal{R}_\beta$ and denote the inverse of $\mathcal{A}'|_{\mathrm{Dom}(\tilde{\mathcal{A}}')}$ by $\mathcal{R}_0$. So, $\mathcal{R}_0 : \tilde{\mathcal{L}} \to C[0,1]$ and

$$D_0^{\alpha-1}\tilde{y} = \mathcal{R}_0\tilde{f} \quad \text{if, and only if,} \quad \mathcal{A}'(D_0^{\alpha-1}\tilde{y}) = \tilde{f}. \tag{2.7}$$

Note that (2.7) implies that since $D_0^{\alpha-1}\tilde{y} \in \mathrm{Dom}(\tilde{\mathcal{A}}')$, then

$$D_0^{\alpha-1}\tilde{y} = \mathcal{R}_0\mathcal{A}'D_0^{\alpha-1}\tilde{y}. \tag{2.8}$$

Since $\tilde{\mathcal{C}} \subset \tilde{\mathcal{L}}$, we can also consider $\mathcal{R}_0 : \tilde{\mathcal{C}} \to C[0,1]$. Let

$$||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}} = \sup_{|v|_0=1} |\mathcal{R}_0 v|_0, \quad v, \mathcal{R}_0 v \in C[0,1],$$

and

$$||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}} = \sup_{|v|_1=1} |\mathcal{R}_0 v|_0, \quad v \in \mathcal{L}[0,1], \quad \mathcal{R}_0 v \in C[0,1].$$

Since $D_0^{\alpha-1}\tilde{y} \in \tilde{\mathcal{C}}$ then $|\mathcal{R}_0 D_0^{\alpha-1}\tilde{y}|_0 \le ||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}|D_0^{\alpha-1}\tilde{y}|_0$. Similarly, $\tilde{f} \in \tilde{\mathcal{L}}$ implies $|\mathcal{R}_0\tilde{f}|_0 \le ||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}|\tilde{f}|_1$.

The following theorem is proved in [13] for the case $\alpha = 2$ and closely models the motivating lemmas and proofs found in [8]. We supply the proof again for $1 < \alpha \le 2$, for the sake of self-containment.

Theorem 2.3. *Assume $\mathcal{A} : Dom(\mathcal{A}) \to \mathcal{L}[0,1]$ denotes a linear operator satisfying* (2.1) *and* (2.2), *and assume that for $\tilde{f} \in \tilde{\mathcal{L}}$, the problem $\mathcal{A}y = \tilde{f}$ is uniquely solvable with solution $y \in Dom(\mathcal{A})$ such that $\overline{D_0^{\alpha-1}y} = 0$. Further, assume there exists $\mathcal{A}' : Dom(\mathcal{A}') \to \mathcal{L}[0,1]$ such that $\mathcal{A} = \mathcal{A}'D_0^{\alpha-1}$. Assume $\tilde{\mathcal{A}}'|_{Dom(\tilde{\mathcal{A}}')} : Dom(\tilde{\mathcal{A}}') \to \tilde{\mathcal{L}}$ is one to one and onto. Then there exists $B_1 > 0$ such if $0 < |\beta| \le B_1$, then $\mathcal{R}_\beta$, the inverse of $(\mathcal{A}' + \beta\mathcal{I})$, exists. Moreover, if $\tilde{f} \in \tilde{L}$, if $B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}} < 1$, where $\mathcal{R}_0$ denotes the inverse of $\mathcal{A}'|_{Dom(\tilde{\mathcal{A}}')}$, and if $0 < |\beta| \le B_1$, then*

$$|\mathcal{R}_\beta\tilde{f}|_0 \le \frac{||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}|\tilde{f}|_1. \tag{2.9}$$

Further, there exists $\mathcal{B} \in (0, B_1)$ such that if $0 < |\beta| \le \mathcal{B}$, then the operator $(\mathcal{A} + \beta D_0^{\alpha-1})$ satisfies *a strong signed maximum principle in* $D_0^{\alpha-1}y$.

Proof. Employ (2.8) and apply $\mathcal{R}_0$ to (2.5) to obtain

$$D_0^{\alpha-1}\tilde{y} + \beta\mathcal{R}_0 D_0^{\alpha-1}\tilde{y} = \mathcal{R}_0\tilde{f}.$$

Note that (2.2) implies that $\mathcal{R}_0 : \tilde{\mathcal{L}} \to \tilde{\mathcal{C}}$ is continuous and hence, bounded. Assume $|\beta|||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}} < 1$. Then $(\mathcal{I} + \beta\mathcal{R}_0) : \tilde{C} \to \tilde{C}$ is invertible and

$$D_0^{\alpha-1}\tilde{y} = (\mathcal{I} + \beta\mathcal{R}_0)^{-1}\mathcal{R}_0\tilde{f}.$$

So, assume $0 < B_1 < \dfrac{1}{||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}$ and assume $|\beta| \le B_1$. Then $\mathcal{R}_\beta = (\mathcal{I} + \beta\mathcal{R}_0)^{-1}\mathcal{R}_0$ exists. Moreover,

$$\begin{aligned}|D_0^{\alpha-1}\tilde{y}|_0 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}|D_0^{\alpha-1}\tilde{y}|_0 &\le |D_0^{\alpha-1}\tilde{y}|_0 - |\beta|||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}|D_0^{\alpha-1}\tilde{y}|_0\\ &\le |(\mathcal{I} + \beta\mathcal{R}_0)D\tilde{y}|_0 = |\mathcal{R}_0\tilde{f}|_0 \le ||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}|\tilde{f}|_1\end{aligned}$$

and (2.9) is proved since $\mathcal{R}_\beta\tilde{f} = D_0^{\alpha-1}\tilde{y} \subset C[0,1]$.

Now assume $f \in \mathcal{L}$ and assume $f \ge 0$ a.e. Then $\bar{f} = |f|_1$. Let $0 < |\beta| \le B_1 < \dfrac{1}{||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}$, write $f = \bar{f} + \tilde{f}$ and consider

$$\beta D_0^{\alpha-1}y = \beta\mathcal{R}_\beta f = \beta\mathcal{R}_\beta(\bar{f} + \tilde{f}).$$

Note that $\beta\mathcal{R}_\beta\bar{f} = \bar{f}$ since $(\mathcal{A}' + \beta\mathcal{I})\bar{f} = \beta\bar{f}$. So,

$$\begin{aligned}\beta D_0^{\alpha-1}y = \beta\mathcal{R}_\beta f &= \beta\mathcal{R}_\beta(\bar{f} + \tilde{f})\\ &= \bar{f} + \beta\mathcal{R}_\beta\tilde{f} \ge |f|_1 - |\beta||\mathcal{R}_\beta\tilde{f}|_0.\end{aligned}$$

Continuing to assume that $0 < |\beta| \le B_1$, it now follows from (2.9) that

$$\beta D_0^{\alpha-1}y \ge |f|_1 - |\beta|\Big(\frac{||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}\Big)|\tilde{f}|_1.$$

Since $\tilde{f} = f - \bar{f}$, and $|\tilde{f}|_1 \le |f|_1 + \bar{f} = 2|f|_1$, the theorem is proved with

$$\mathcal{B} < \min\Big\{B_1, \Big(\frac{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}{2||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}\Big)\Big\}.$$

In particular, if $0 < |\beta| \le \mathcal{B}$, then

$$\beta D_0^{\alpha-1}y(t) \ge K|f|_1 = \Big(1 - \mathcal{B}\Big(\frac{2||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}\Big)\Big)|f|_1. \quad \square$$

3 Two examples

This article is modeled after [13] and in [13] the conclusion of the theorem analogous to Theorem 2.3 is that the operator $(\mathcal{A} + \beta D)$ satisfies a strong signed maximum principle in Dy. So the elementary observation that $\beta Dy > 0$ on an interval implies that βy is monotone increasing on that interval is employed to consider boundary value problems for which $\beta Dy > 0$ on $[0, 1]$ implies that βy has constant sign on $(0, 1)$. In the following lemma, we state and prove a modest extension of this principle to the fractional Riemann-Liouville derivative of order $\gamma = \alpha - 1$, $0 < \gamma \leq 1$.

Lemma 3.1. *Assume* $0 < \gamma \leq 1$. *Assume* $\beta \neq 0$. *Assume* $y \in C_{\gamma-1}[0, 1]$ *and assume* $D_0^\gamma y(t) \in C[0, 1]$. *Assume* $\beta D_0^\gamma y(t) > 0$, $0 \leq t \leq 1$, *and assume* $\beta \lim_{t\to 0^+} t^{1-\gamma} y(t) \geq 0$. *Then* $\beta y(t) > 0$, $0 < t \leq 1$.

Proof. If $\gamma = 1$, then y can be extended continuously to $[0, 1]$ and $\beta y(0) \geq 0$. Then βy is increasing on $[0, 1]$ and the result is true.

So, assume $0 < \gamma < 1$ and define $a = \lim\limits_{t\to 0^+} t^{1-\gamma} y(t)$. Thus, $\beta a \geq 0$. Then [11, Theorem 2.23] or [23, Proposition 6.1],

$$y(t) = at^{\gamma-1} + I_0^\gamma D_0^\gamma y(t), \quad 0 < t \leq 1,$$

and

$$\beta y(t) = \beta at^{\gamma-1} + I_0^\gamma \beta D_0^\gamma y(t), \quad 0 < t \leq 1.$$

If $a = 0$, then $I_0^\gamma \beta D_0^\gamma y(t) > 0$ if $0 < t \leq 1$ and the statement is proved. If $\beta a > 0$, then both terms $\beta at^{\gamma-1}$ and $I_0^\gamma \beta D_0^\gamma y(t)$ are positive for $t \in (0, 1]$, and the statement is proved. □

Example 3.2. Let $1 < \alpha \leq 2$, and consider the linear boundary value problem

$$D_0^\alpha y(t) + \beta D_0^{\alpha-1} y(t) = f(t), \quad 0 \leq t \leq 1, \tag{3.1}$$

$$y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1). \tag{3.2}$$

For the boundary value problem (3.1), (3.2), $\mathcal{A} = D_0^\alpha$, $\mathcal{A}' = D = \frac{d}{dt}$, $\ker(\mathcal{A}) =< t^{\alpha-1} >$. We show that the operators $\mathcal{A}$ and $\mathcal{A}'$ satisfy the hypotheses of Theorem 2.3.

One can show directly that $\mathrm{Im}(\mathcal{A}) = \tilde{\mathcal{L}}$. If $f \in \mathrm{Im}(\mathcal{A})$ then there exists a solution y of

$$D_0^\alpha y(t) = f(t), \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1),$$

which implies

$$0 = D_0^{\alpha-1} y(1) - D_0^{\alpha-1} y(0) = \int_0^1 D_0^\alpha y(t)\, dt = \int_0^1 f(t)\, dt,$$

and $f \in \tilde{\mathcal{L}}$. Likewise, if $f \in \tilde{\mathcal{L}}$, then

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds = I_0^\alpha f(t) \in \mathrm{Dom}(\tilde{A}) \tag{3.3}$$

is a solution of

$$D_0^\alpha y(t) = f(t), \quad 0 \le t \le 1, \quad y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1).$$

To see that $y \in \mathrm{Dom}(\tilde{A})$, note that

$$D_0^{\alpha-1} y(t) = D_0^{\alpha-1} I_0^\alpha f(t) = D_0^{\alpha-1} I_0^{\alpha-1} I^1 f(t) = \int_0^t f(s)\, ds.$$

So, $\overline{D_0^{\alpha-1} y} = \overline{f} = 0$. To see that the boundary conditions are satisfied, $y(0) = I_0^\alpha y|_{t=0}$, and the condition $y(0) = 0$ is clear. Moreover, $D_0^{\alpha-1} I_0^\alpha f(t) = \int_0^t f(s)\, ds$, which implies $D_0^{\alpha-1} I_0^\alpha f|_{t=0} = D_0^{\alpha-1} I_0^\alpha f|_{t=1} = 0$ since $f \in \tilde{\mathcal{L}}$.

To argue that $\mathcal{A} y = \tilde{f}$ is uniquely solvable with solution $y \in \mathrm{Dom}(\tilde{A})$, (3.3) implies the solvability. For uniqueness, if y_1 and y_2 are two such solutions, then $(y_1 - y_2)(t) = c t^{\alpha-1}$ and $y_1 - y_2 \in \mathrm{Dom}(\tilde{\mathcal{A}})$ implies $c = 0$.

Finally, (3.3) implies (2.2) is satisfied with $M = 1$ since

$$|D_0^{\alpha-1} y(t)| = \left| \int_0^t f(s)\, ds \right| \le |f|_1.$$

Theorem 2.3 applies and there exists $\mathcal{B} > 0$ such that if $0 < |\beta| \le \mathcal{B}$ then $(\mathcal{A} + \beta D_0^{\alpha-1} y)$ has the strong maximum principle in $D_0^{\alpha-1} y$. Thus, $f \ge 0$ implies $\beta D_0^{\alpha-1} y \ge 0$. To apply Lemma 3.1, recall [11, Theorem 2.23] or [23, Theorem 6.8], that

$$\begin{aligned}
y(t) &= a t^{\alpha-2} + \frac{D_0^{\alpha-1} y|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + I_0^\alpha D_0^\alpha y(t) \\
&= a t^{(\alpha-1)-1} + \frac{D_0^{\alpha-1} y|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + I_0^{\alpha-1} I D D_0^{\alpha-1} y(t) \\
&= a t^{(\alpha-1)-1} + \frac{D_0^{\alpha-1} y|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + I_0^{\alpha-1} (D_0^{\alpha-1} y(t) - D_0^{\alpha-1} y|_{t=0}) \\
&= a t^{(\alpha-1)-1} + I_0^{\alpha-1} D_0^{\alpha-1} y(t).
\end{aligned}$$

where $a = \lim_{t \to 0^+} t^{2-\alpha} y(t) = \lim_{t \to 0^+} t^{1-(\alpha-1)} y(t)$. Since $y(0) = 0$ implies $a = 0$, Lemma 3.1 applies with $\gamma = \alpha - 1$ and $\beta a = 0$. Thus, $\beta y(t) \ge 0$, for $0 < t \le 1$, and if $|f|_1 > 0$, then $\beta y(t) > 0$, for $0 < t \le 1$.

Hence, a natural partial order on $\mathcal{X}_{\alpha-2}$ in which to apply the method of upper and lower solutions and monotone methods to a nonlinear boundary value problem is

$$y \in \mathcal{X}_{\alpha-2} \succeq 0 \Longleftrightarrow \beta y(t) \geq 0, \quad 0 < t \leq 1, \quad \text{and} \quad \beta D_0^{\alpha-1} y(t) \geq 0, \quad 0 \leq t \leq 1, \tag{3.4}$$

and

$$y \in \mathcal{X}_{\alpha-2} \succ 0 \Longleftrightarrow \beta y(t) > 0, \quad 0 < t \leq 1, \quad \text{and} \quad \beta D_0^{\alpha-1} y(t) > 0, \quad 0 \leq t \leq 1.$$

In Section 4, we shall employ monotone methods with respect to this partial order and obtain sufficient conditions for existence of maximal and minimal solutions of a nonlinear boundary value problem

$$D_0^{\alpha} y(t) = f(t, y(t), D_0^{\alpha-1} y(t)), \quad t \in (0, 1],$$

associated with the boundary conditions (3.2).

Example 3.3. For the second example, let $0 < h < 1$, and consider a family of boundary conditions

$$\lim_{t \to 0^+} t^{2-\alpha} y(t) = h y(1), \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1). \tag{3.5}$$

Remark 3.4. *Note that the boundary condition $y(1)$ can be expressed as $t^{2-\alpha} y(t)|_{t=1}$ and so, if $h = 1$ in (3.5), we intend that these boundary conditions represent a Riemann-Liouville fractional analogue of periodic boundary conditions. In this example however, we require that $0 < h < 1$.*

For the boundary value problem (3.1), (3.5), $\mathcal{A} = D_0^{\alpha}$, $\mathcal{A}' = D = \frac{d}{dt}$ and

$$\ker(\mathcal{A}) = \left\langle t^{\alpha-1} + \frac{h}{1-h} t^{\alpha-2} \right\rangle.$$

Precisely as in Example (3.2), $\operatorname{Im}(\mathcal{A}) = \tilde{\mathcal{L}}$. Again, $f \in \tilde{\mathcal{L}}$ implies

$$\operatorname{Dom}(\tilde{\mathcal{A}}) = \left\{ y \in \mathcal{X}_{\alpha-2} : \overline{D_0^{\alpha-1} y} = 0 \right\}.$$

Again, M in (2.2) can be computed since if $f \in \tilde{\mathcal{L}}$, then

$$\tilde{y}(t) = I_0^{\alpha} f(t) + \frac{h}{1-h} I_0^{\alpha} f(1) t^{\alpha-2}$$

denotes the unique solution $y \in \operatorname{Dom}(\tilde{\mathcal{A}})$ of the boundary value problem $D_0^{\alpha} y = f$, (3.5). Thus, Theorem 2.3 applies and there exists $\mathcal{B} > 0$ such that if $0 < |\beta| \leq \mathcal{B}$ then $(\mathcal{A} + \beta D_0^{\alpha-1})$ satisfies a strong maximum principle in $D_0^{\alpha-1} y$.

To determine a sign condition on βy we appeal to Lemma 3.1. Let $a = \lim_{t \to 0^+} t^{2-\alpha} y(t)$. We first rule out the case $a = 0$. Assume $0 < |\beta| \leq \mathcal{B}$, and $0 < h < 1$. If $a = 0$, then $y(t) = I_0^{\alpha-1} D_0^{\alpha-1} y(t)$ and

$\beta y(1) > 0$. In particular, $y(1) \neq 0$. Since $\lim_{t \to 0^+} t^{2-\alpha} y(t) = a = 0$, y does not satisfy the boundary condition, $\lim_{t \to 0^+} t^{2-\alpha} y(t) = hy(1)$. Thus, $a \neq 0$.

Now, continue to assume $0 < |\beta| \leq \mathcal{B}$, and assume $0 < h < 1$. If $\beta D_0^{\alpha-1} y(t) > 0$, $0 \leq t \leq 1$, we rule out the case $\beta a < 0$. The condition $0 < h < 1$, the boundary condition, $\lim_{t \to 0^+} t^{2-\alpha} y(t) = hy(1)$ and the identity $y(t) = at^{\alpha-2} + I_0^{\alpha-1} D_0^{\alpha-1} y(t)$ imply that with $a = \lim_{t \to 0^+} t^{2-\alpha} y(t) = \lim_{t \to 0^+} t^{1-(\alpha-1)} y(t)$, then

$$0 < \frac{a}{a + I_0^{\alpha-1} D_0^{\alpha-1} y|_{t=1}} < 1,$$

or

$$0 < \frac{\beta a}{\beta a + I_0^{\alpha-1} \beta D_0^{\alpha-1} y|_{t=1}} < 1.$$

If $\beta a < 0$, then $\beta a < \beta a + I_0^{\alpha-1} \beta D_0^{\alpha-1} y|_{t=1} < 0$ and $|\beta a| > \left|\beta a + I_0^{\alpha-1} \beta D_0^{\alpha-1} y|_{t=1}\right|$, which implies

$$\frac{\beta a}{\beta a + I_0^{\alpha-1} \beta D_0^{\alpha-1} y|_{t=1}} > 1,$$

and so the condition $0 < h < 1$ is contradicted. So, $\beta a > 0$ and Lemma 3.1 applies with $\gamma = \alpha - 1$. Thus, if $0 < |\beta| \leq \mathcal{B}$ and $0 < h < 1$, then a natural partial order in which to apply the method of upper and lower solutions and monotone methods to a nonlinear problem is

$$y \in \mathcal{X}_{\alpha-2} \succeq 0 \iff \beta y(t) \geq 0, \quad 0 < t \leq 1, \quad \text{and} \quad \beta D_0^{\alpha-1} y(t) \geq 0, \quad 0 \leq t \leq 1,$$

and

$$y \in \mathcal{X}_{\alpha-2} \succ 0 \iff \beta y(t) > 0, \quad 0 < t \leq 1, \quad \text{and} \quad \beta D_0^{\alpha-1} y(t) > 0, \quad 0 \leq t \leq 1.$$

In particular, there is a transition from a maximum principle to an anti-maximum principle at $\beta = 0$.

Remark 3.5. *The work in this article extends the work produced in [13], where* $\alpha = 2$. *In [13], it is shown if* $1 < h$, *then* $\beta D_0^{\alpha-1} y(t) = \beta D^1 y(t) \geq 0$, $0 \leq t \leq 1$, *implies* $\beta y(t) \leq 0$, $0 \leq t \leq 1$. *In [13], the sign of the derivative implies monotonicity of the function. For the fractional case,* $1 < \alpha < 2$, *the case* $1 < h$ *remains open.*

4 A Monotone Method

Assume $1 < \alpha \leq 2$. Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Consider the boundary value problem

$$D_0^\alpha y(t) = f(t, y(t), D_0^{\alpha-1} y(t)), \quad 0 \leq t \leq 1, \tag{4.1}$$

$$y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1). \tag{4.2}$$

Assume that f satisfies the following monotonicity properties.

$$f(t,y,z_1) < f(t,y,z_2) \quad \text{for} \quad (t,y) \in [0,1]\times\mathbb{R}, \quad z_1 > z_2, \tag{4.3}$$
$$f(t,y_1,z) < f(t,y_2,z) \quad \text{for} \quad (t,z) \in [0,1]\times\mathbb{R}, \quad y_1 > y_2;$$

that is, f is monotone decreasing in each of the third component and second component.

Note for $y \in C_{\alpha-2}$, one should initially consider the differential equation $D_0^\alpha y(t) = f(t, y(t), D_0^{\alpha-1}y(t))$ on $(0,1]$. The boundary condition $y(0) = 0$ implies the functions produced in the following iterative schemes exist on $[0,1]$ and so, we assume (4.1) on $[0,1]$.

Apply a shift [15] to (4.1) and consider the equivalent boundary value problem,

$$D_0^\alpha y(t) + \beta D_0^{\alpha-1}y(t) = f(t, y(t), D_0^{\alpha-1}y(t)) + \beta D_0^{\alpha-1}y(t), \quad 0 \le t \le 1,$$

with boundary conditions (4.2), where $-\mathcal{B} \le \beta < 0$ and $\mathcal{B} > 0$ is shown to exist in Theorem 2.3. Note that if $g(t,y,z) = f(t,y,z) + \beta z$ and f satisfies (4.3), then g satisfies (4.3) if $\beta < 0$.

Assume the existence of solutions, $w_1, v_1 \in \mathcal{X}_{\alpha-2}$, of the following boundary value problems for fractional differential inequalities

$$D_0^\alpha w_1(t) \ge f(t, w_1(t), D_0^{\alpha-1}w_1(t)), \quad D_0^\alpha v_1(t) \le f(t, v_1(t), D_0^{\alpha-1}v_1(t)), \quad 0 \le t \le 1, \tag{4.4}$$
$$w_1(0) = 0, \quad D_0^{\alpha-1}w_1(0) = D_0^{\alpha-1}w_1(1), \qquad v_1(0) = 0, \quad D_0^{\alpha-1}v_1(0) = D_0^{\alpha-1}v_1(1).$$

Assume further that

$$(v_1(t) - w_1(t)) \ge 0, \quad 0 \le t \le 1, \quad (D_0^{\alpha-1}v_1(t) - D_0^{\alpha-1}w_1(t)) \ge 0, \quad 0 \le t \le 1. \tag{4.5}$$

Motivated by (3.4) and noting that $\beta < 0$, define a partial order $\succeq_{\beta<0}$ on $\mathcal{X}_{\alpha-2}$ by

$$u \in \mathcal{X}_{\alpha-2} \succeq_{\beta<0} 0 \iff u(t) < 0, \quad 0 < t \le 1, \quad \text{and} \quad D_0^{\alpha-1}u(t) \le 0, \quad 0 \le t \le 1.$$

Then the assumption (4.5) implies $w_1 \succeq_{\beta<0} v_1$.

Define iteratively, the sequences $\{v_k\}_{k=1}^\infty$, $\{w_k\}_{k=1}^\infty$, where

$$D_0^\alpha v_{k+1}(t) + \beta D_0^{\alpha-1}v_{k+1}(t) = f(t, v_k(t), D_0^{\alpha-1}v_k(t)) + \beta D_0^{\alpha-1}v_k(t), \quad 0 \le t \le 1, \tag{4.6}$$
$$v_{k+1}(0) = 0, \quad D_0^{\alpha-1}v_{k+1}(0) = D_0^{\alpha-1}v_{k+1}(1),$$

and

$$D_0^\alpha w_{k+1}(t)+\beta D_0^{\alpha-1}w_{k+1}(t)=f(t,w_k(t),D_0^{\alpha-1}w_k(t))+\beta D_0^{\alpha-1}w_k(t),\quad 0\le t\le 1,\qquad (4.7)$$
$$w_{k+1}(0)=0,\quad D_0^{\alpha-1}w_{k+1}(0)=D_0^{\alpha-1}w_{k+1}(1).$$

Theorem 2.3 implies the existence of each v_{k+1}, w_{k+1} since if $|\beta|\le\mathcal{B}$, the inverse of $(\mathcal{A}+\beta D)$ exists.

Theorem 4.1. *Assume $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is continuous and assume f satisfies the monotonicity properties* (4.3)*. Assume the existence of functions $v_1,w_1\in\mathcal{X}_{\alpha-2}$ satisfying* (4.4) *and* (4.5)*. Define the sequences of iterates $\{v_k\}_{k=1}^\infty$, $\{w_k\}_{k=1}^\infty$ by* (4.6) *and* (4.7) *respectively. Then, for each $k\in\mathbb{N}_1$,*

$$w_k\succeq_{\beta<0}w_{k+1}\succeq_{\beta<0}v_{k+1}\succeq_{\beta<0}v_k.\qquad (4.8)$$

Moreover, $\{v_k\}_{k=1}^\infty$ converges in $\mathcal{X}_{\alpha-2}$ to a solution v of the boundary value problem (4.1)*,* (4.2) *and $\{w_k\}_{k=1}^\infty$ converges in $\mathcal{X}_{\alpha-2}$ to a solution w of the boundary value problem* (4.1)*,* (4.2) *satisfying*

$$w_k\succeq_{\beta<0}w_{k+1}\succeq_{\beta<0}w\succeq_{\beta<0}v\succeq_{\beta<0}v_{k+1}\succeq_{\beta<0}v_k.\qquad (4.9)$$

Proof. Since v_1 satisfies a differential inequality given in (4.5), then for $0\le t\le 1$,

$$D_0^\alpha v_2(t)+\beta D_0^{\alpha-1}v_2(t)=f(t,v_1(t),D_0^{\alpha-1}v_1(t))+\beta D_0^{\alpha-1}v_1(t)\ge D_0^\alpha v_1(t)+\beta D_0^{\alpha-1}v_1(t).$$

Set $u=v_2-v_1$ and u satisfies a boundary value problem for a differential inequality,

$$D_0^\alpha u(t)+\beta D_0^{\alpha-1}u(t)\ge 0,\quad 0\le t\le 1,\quad u(0)=0,\quad D_0^{\alpha-1}u(0)=D_0^{\alpha-1}u(1).$$

The signed maximum principle applies and $u\succeq_{\beta<0}0$; in particular, $v_2\succeq_{\beta<0}v_1$. Similarly, $w_1\succeq_{\beta<0}w_2$. Now set $u=w_2-v_2$ and

$$\begin{aligned}D_0^\alpha u(t)+\beta D_0^{\alpha-1}u(t)&=\Big(f(t,w_1(t),D_0^{\alpha-1}w_1(t))-f(t,v_1(t),D_0^{\alpha-1}v_1(t))\Big)\\&\quad+\beta(D_0^{\alpha-1}w_1(t)-D_0^{\alpha-1}v_1(t)),\quad 0\le t\le 1,\\ u(0)=0,&\quad D_0^{\alpha-1}u(0)=D_0^{\alpha-1}u(1).\end{aligned}$$

Since f satisfies (4.3) and $w_1\succeq_{\beta<0}v_1$, then

$$D_0^\alpha u(t)+\beta D_0^{\alpha-1}u(t)\ge 0,\quad 0\le t\le 1,$$

and again the signed maximum principle applies and $u\succeq_{\beta<0}0$. In particular, $w_2\succeq_{\beta<0}v_2$. Thus, (4.8) is proved for $k = 1$. A straightforward induction implies that (4.8) is valid using the arguments presented in this paragraph.

To obtain the existence of limiting solutions v and w satisfying (4.9), note that the sequence $\{D_0^{\alpha-1}v_k\}$ is monotone decreasing and bounded below by $\{D_0^{\alpha-1}w_1\}$. So the sequence $\{D_0^{\alpha-1}v_k\}$ is converging pointwise on $[0,1]$ to some function g defined on $[0,1]$. Moreover, $D_0^{\alpha}v_k = DD_0^{\alpha-1}v_k$ is uniformly bounded on

$$\Omega = \{(t,y,z) : w_1(t) \le y \le v_1(t),\ D_0^{\alpha-1}w_1(t) \le z \le D_0^{\alpha-1}v_1(t),\ 0 \le t \le 1\},$$

and so the pointwise limit g is continuous on $[0,1]$. Dini's theorem applies and $\{D_0^{\alpha-1}v_k\}$ is converging uniformly to g on $[0,1]$. Note $a = 0$, and so, we can define $v_k(0) = 0$ and extend v_k to a continuous function on $[0,1]$. The sequence $\{v_k\}$ is monotone decreasing and bounded below, and so there exists v such that $\{v_k\}$ is converging pointwise to v on $[0,1]$. Note that since $v_k(0) = 0$, then $v_k = I_0^{\alpha-1}D_0^{\alpha-1}v_k$ which converges uniformly $I_0^{\alpha-1}g$. So $v = I_0^{\alpha-1}g$ which implies $D_0^{\alpha-1}v = g$. To summarize, v_k is converging to v in $C_{\alpha-2}$ and $\{D_0^{\alpha-1}v_k\}$ is converging to $\{D_0^{\alpha-1}v\}$ in $C[0,1]$.

Finally, using $D_0^{\alpha}v_{k+1}(t) = f(t, v_k(t), D_0^{\alpha-1}v_k(t)) + \beta(D_0^{\alpha-1}v_k(t) - D_0^{\alpha-1}v_{k+1}(t))$, it now follows that the sequence $\{D_0^{\alpha}v_k\}$ converges uniformly on $[0,1]$ to $f(t, v(t), D_0^{\alpha-1}v(t))$. Since $D_0^{\alpha}v_k = D^1D_0^{\alpha-1}v_k$, we conclude that $\lim_{k\to\infty} D_0^{\alpha}v_k = D_0^{\alpha}v$.

Similar details apply to $\{w_k\}$ and the theorem is proved. □

Suppose now f satisfies the "anti"-inequalities to (4.3); that is suppose f satisfies

$$\begin{aligned} f(t,y,z_1) > f(t,y,z_2) \quad &\text{for} \quad (t,y) \in [0,1]\times\mathbb{R}, \quad z_1 > z_2, \\ f(t,y_1,z) > f(t,y_2,z) \quad &\text{for} \quad (t,z) \in [0,1]\times\mathbb{R}, \quad y_1 > y_2. \end{aligned} \tag{4.10}$$

One can appeal to the signed maximum principle and apply a shift to (4.1) and consider the equivalent boundary value problem, $D_0^{\alpha}y(t) + \beta D_0^{\alpha-1}y(t) = f(t, y(t), D_0^{\alpha-1}y(t)) + \beta D_0^{\alpha-1}y(t), 0 \le t \le 1$, where $0 < \beta < \mathcal{B}$. Note, if f satisfies (4.10) and $\beta > 0$, then $g(t,y,z) = f(t,y,z) + \beta z$ satisfies (4.10).

Now, assume the existence of solutions, $w_1, v_1 \in \mathcal{X}_{\alpha-2}$, of the following differential inequalities

$$\begin{gathered} D_0^{\alpha}w_1(t) \le f(t, w_1(t), D_0^{\alpha-1}w_1(t)), \quad D_0^{\alpha}v_1(t) \ge f(t, v_1(t), D_0^{\alpha-1}v_1(t)), \quad 0 \le t \le 1, \\ w_1(0) = 0, \quad D_0^{\alpha-1}w_1(0) = D_0^{\alpha-1}w_1(1), \quad v_1(0) = 0, \quad D_0^{\alpha-1}v_1(0) = D_0^{\alpha-1}v_1(1). \end{gathered} \tag{4.11}$$

Assume further that

$$(v_1(t) - w_1(t)) \ge 0, \quad 0 < t \le 1, \quad (D_0^{\alpha-1}v_1(t) - D_0^{\alpha-1}w_1(t)) \ge 0, \quad 0 \le t \le 1. \tag{4.12}$$

Noting that $\beta > 0$ define a partial order $\succeq_{\beta>0}$ on $\mathcal{X}_{\alpha-2}$ by

$$u \in \mathcal{X}_{\alpha-2} \succeq_{\beta>0} 0 \Longleftrightarrow u(t) \geq 0, \quad 0 < t \leq 1, \quad \text{and} \quad D_0^{\alpha-1}u(t) \geq 0, \quad 0 \leq t \leq 1.$$

In particular, in (4.12), assume $v_1 \succeq_{\beta>0} w_1$.

Theorem 4.2. *Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and assume f satisfies the monotonicity properties, (4.10). Assume the existence of $w_1, v_1 \in \mathcal{X}_{\alpha-2}$ satisfying (4.11) and (4.12). Define the sequences of iterates $\{v_k\}_{k=1}^{\infty}$, $\{w_k\}_{k=1}^{\infty}$ by (4.6) and (4.7) respectively. Then, for each $k \in \mathbb{N}_1$,*

$$v_k \succeq_{\beta>0} v_{k+1} \succeq_{\beta>0} w_{k+1} \succeq_{\beta>0} w_k.$$

Moreover, $\{v_k\}_{k=1}^{\infty}$ converges in $\mathcal{X}_{\alpha-2}$ to a solution v of (4.1) and $\{w_k\}_{k=1}^{\infty}$ converges in $\mathcal{X}_{\alpha-2}$ to a solution w of (4.1) satisfying

$$v_k \succeq_{\beta>0} v_{k+1} \succeq_{\beta>0} v \succeq_{\beta>0} w \succeq_{\beta>0} w_{k+1} \succeq_{\beta>0} w_k.$$

We close the article with two corollaries of Theorem 4.2 in which upper and lower solutions, v_1 and w_1 are explicitly produced.

Corollary 4.3. *Let $\mathcal{B}$ be given by Theorem 2.3. Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, assume there exists $\beta \in (0, \mathcal{B}]$ such that $f(t,y,z) + \beta z$ is bounded on $[0,1] \times \mathbb{R}^2$, and assume $g(t,y,z) = f(t,y,z) + \beta z$ satisfies the monotonicity conditions (4.10). Then $v_1(t) = \dfrac{M}{\beta\Gamma(\alpha)} t^{\alpha-1} \in \mathcal{X}_{\alpha-2}$ and $w_1(t) = -v_1(t) \in \mathcal{X}_{\alpha-2}$ satisfy (4.11) and (4.12) where $M = \sup\limits_{[0,1]\times\mathbb{R}^2} |f(t,y,z) + \beta z|$; in particular, there exists a solution $y \in \mathcal{X}_{\alpha-2}$ of the boundary value problem (4.1), (4.2) satisfying*

$$v_1 \succeq_{\beta>0} y \succeq_{\beta>0} w_1.$$

Remark 4.4. *Remove the hypothesis that g satisfies (4.10), and the Schauder fixed point theorem implies the existence of a solution of the boundary value problem (4.1), (4.2) in the case g is bounded.*

Corollary 4.5. *Let $\mathcal{B}$ be given by Theorem 2.3. Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, assume there exists $\beta \in (0, \mathcal{B}]$ such that $g(t,y,z) = f(t,y,z) + \beta z$ satisfies the monotonicity conditions (4.10). Assume there exist $\sigma \in C[0,1]$ and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|g(t,y,z)| \leq \sigma(t)\psi(|y|), \quad (t,y,z) \in [0,1] \times \mathbb{R}^2.$$

Moreover, assume there exists $M > 0$ such that

$$\frac{\beta M}{|\sigma|_0 \psi\left(\frac{M}{\Gamma(\alpha)}\right)} > 1.$$

Then there exists a solution of the boundary value problem (4.1), (4.2).

Proof. Set $v_1(t) = \dfrac{M}{\Gamma(\alpha)} t^{\alpha-1} \in \mathcal{X}_{\alpha-2}$. Then

$$D_0^\alpha v_1(t) + \beta D_0^{\alpha-1} v_1(t) = \beta M > |\sigma|_0 \psi\left(\frac{M}{\Gamma(\alpha)}\right) \geq g(t, v_1, D_0^{\alpha-1} v_1(t)).$$

Set $w_1(t) = -v_1(t)$ and $v_1(t), w_1(t) \in \mathcal{X}_{\alpha-2}$ satisfy (4.11) and (4.12). □

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