CUBO, A Mathematical Journal
Vol. 25, no. 2, pp. 273–288, August 2023
DOI: 10.56754/0719-0646.2502.273

Stability of ternary antiderivation in ternary Banach algebras via fixed point theorem

Mehdi Dehghanian[1]
Choonkil Park[2], ✉
Yamin Sayyari[1]

[1] *Department of Mathematics, Sirjan University of Technology, Sirjan, Iran.*
mdehghanian@sirjantech.ac.ir
y.sayyari@sirjantech.ac.ir

[2] *Research Institute for Convergence of Basic Science, Hanyang University, Seoul 04763, South Korea.*
baak@hanyang.ac.kr

ABSTRACT

In this paper, we introduce the concept of ternary antiderivation on ternary Banach algebras and investigate the stability of ternary antiderivation in ternary Banach algebras, associated to the (α, β)-functional inequality:

$$\begin{aligned}&\|\mathcal{F}(x+y+z)-\mathcal{F}(x+z)-\mathcal{F}(y-x+z)-\mathcal{F}(x-z)\|\\&\leq\|\alpha(\mathcal{F}(x+y-z)+\mathcal{F}(x-z)-\mathcal{F}(y))\|+\|\beta(\mathcal{F}(x-z)\\&+\mathcal{F}(x)-\mathcal{F}(z))\|\end{aligned}$$

where α and β are fixed nonzero complex numbers with $|\alpha|+|\beta|<2$ by using the fixed point method.

RESUMEN

En este arículo, introducimos el concepto de antiderivación ternaria en álgebras de Banach ternarias e investigamos la estabilidad de las antiderivaciones ternaria en álgebras de Banach ternarias, asociadas a la (α, β)- desigualdad funcional:

$$\begin{aligned}&\|\mathcal{F}(x+y+z)-\mathcal{F}(x+z)-\mathcal{F}(y-x+z)-\mathcal{F}(x-z)\|\\&\leq\|\alpha(\mathcal{F}(x+y-z)+\mathcal{F}(x-z)-\mathcal{F}(y))\|+\|\beta(\mathcal{F}(x-z)\\&+\mathcal{F}(x)-\mathcal{F}(z))\|\end{aligned}$$

donde α y β son números complejos no cero fijos, con $|\alpha|+|\beta|<2$ usando el método de punto fijo.

Keywords and Phrases: Hyers-Ulam stability; stability; fixed point method; ternary antiderivation; ternary Banach algebra; additive functional inequality.

2020 AMS Mathematics Subject Classification: 47B47, 11E20, 17B40, 39B72, 47H10.

Accepted: 11 July, 2023
Received: 01 September, 2022

1 Introduction

A ternary Banach algebra is a complex Banach space $\mathcal{A}$, endowed with a ternary product $(x, y, z) \to [x, y, z]$ of $\mathcal{A}^3$ into $\mathcal{A}$, which is $\mathbb{C}-$linear in each variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$ for all $x, y, z, w, u \in \mathcal{A}$.

If a ternary Banach algebra $(\mathcal{A}, [\cdot, \cdot, \cdot])$ has an unit, *i.e.*, an element $e \in \mathcal{A}$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in \mathcal{A}$, then it is routine to verify that $\mathcal{A}$, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital algebra. Conversely, if $(\mathcal{A}, \circ)$ is a unital algebra, then $[x, y, z] := x \circ y^* \circ z$ makes $\mathcal{A}$ into a ternary Banach algebra.

A $\mathbb{C}-$linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a ternary homomorphism if $H([x, y, z]) = [H(x), H(y), H(z)]$ for all $x, y, z \in \mathcal{A}$. A $\mathbb{C}-$linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a ternary derivation if

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in \mathcal{A}$.

We say that an equation is stable if any function satisfying the equation approximately is near to an exact solution of the equation.

The stability problem of functional equations started from a question of Ulam, in 1940, on the stability of group homomorphisms. In 1941, Hyers [17] gave an answer to the question of Ulam in the context of Banach spaces in the case of additive mappings, that was an major step toward further solutions in this field.

During the last two decades, a number of articles and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, k-additive mappings, multiplicative mappings, bounded nth differences, Euler-Lagrange functional equations, differential equations, and Navier-Stokes equations (see [1, 2, 4, 5, 19, 22, 25, 26, 27, 28, 29]).

Also, approximate generalized Lie derivations have been already established in [6, 7].

Ternary algebraic structures appear in various domains of theoretical and mathematical physics, such as the quark model and Nambu mechanics [18, 21]. Today, many physical systems can be modeled as a linear system. The principle of additivity has various applications in physics especially in calculating the internal energy in thermodynamic and also the meaning of the superposition principle.

In recent years, the Hyers-Ulam stability of various (among others functional, differential and integral) equations and other objects (for example in groups, Banach algebra, ternary Banach algebras and C^*-ternary algebras) has been intensively studied (see [8, 9, 10, 11, 15, 16, 30]).

Fixed-point theory has been studied by various methods. The study on fixed point theory provides essential tools for solving problems arising in various fields of functional analysis, such as dynamical systems, equilibrium problems and differential equations (see for instance [3, 14, 24]).

We recall a fundamental result in fixed point theory.

Definition 1.1 ([12])**.** *Let $\mathcal{X}$ be a non-empty set and $d : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ a mapping such that*

(1) $d(x, y) = 0$ *if and only if* $x = y$,

(2) $d(x, y) = d(y, x)$ *for all* $x, y \in \mathcal{X}$,

(3) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in \mathcal{X}$.

Then d is called a generalized metric and $(\mathcal{X}, d)$ is a generalized metric space.

Theorem 1.2 ([12])**.** *Let $(\mathcal{X}, d)$ be a complete generalized metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping, that is,*

$$d(\mathcal{T}x, \mathcal{T}y) \leq Ld(x, y)$$

for some $L < 1$ and all $x, y \in \mathcal{X}$. Then for each given element $x \in \mathcal{X}$, either

$$d(\mathcal{T}^n x, \mathcal{T}^{n+1} x) = +\infty$$

for all $n \geq 0$ or

$$d(\mathcal{T}^n x, \mathcal{T}^{n+1} x) < +\infty, \qquad \forall n \geq n_0,$$

for some positive integer n_0. Moreover, if the second alternative holds, then

(i) the sequence $\{\mathcal{T}^n x\}$ is convergent to a fixed point y^* of $\mathcal{T}$;

(ii) y^* is the unique fixed point of $\mathcal{T}$ in the set $Y := \{y \in \mathcal{X}, d(\mathcal{T}^{n_0} x, y) < +\infty\}$ and $d(y, y^*) \leq \frac{1}{1-L} d(y, \mathcal{T}y)$ for all $y \in Y$.

In this paper, we consider the following functional inequality

$$\begin{aligned} &\|\mathcal{F}(x+y+z) - \mathcal{F}(x+z) - \mathcal{F}(y-x+z) - \mathcal{F}(x-z)\| \\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| \end{aligned} \tag{1.1}$$

for all $x, y, z \in \mathcal{A}$, where α and β are fixed nonzero complex numbers with $|\alpha| + |\beta| < 2$.

Throughout this paper, assume that $\mathcal{A}$ is a ternary Banach algebra and α and β are fixed nonzero complex numbers with $|\alpha| + |\beta| < 2$.

The aim of the present paper is to establish the stability problem of ternary antiderivations in complex ternary Banach algebras by using the fixed point method.

2 Stability of (α, β)-functional inequality (1.1)

In this section, we prove the Hyers-Ulam stability of the additive (α, β)-functional inequality (1.1) by using the fixed point method.

Lemma 2.1. *Let $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying*

$$\begin{aligned} &\|\mathcal{F}(x+y+z) - \mathcal{F}(x+z) - \mathcal{F}(y-x+z) - \mathcal{F}(x-z)\| \\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) - \mathcal{F}(x) + \mathcal{F}(z))\| \end{aligned} \tag{2.1}$$

for all $x, y, z \in \mathcal{A}$. Then the mapping $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is additive.

Proof. Assume that $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ satisfies (2.1).

Putting $x = y = z = 0$ in (2.1), we have

$$2\|\mathcal{F}(0)\| \leq (|\alpha| + |\beta|)\|\mathcal{F}(0)\|$$

and thus $\mathcal{F}(0) = 0$, since $|\alpha| + |\beta| < 2$.

Letting $z = x$ in (2.1), we obtain

$$\|\mathcal{F}(2x + y) - \mathcal{F}(2x) - \mathcal{F}(y)\| \leq 0$$

and so $\mathcal{F}(2x + y) = \mathcal{F}(2x) + \mathcal{F}(y)$ for all $x, y \in \mathcal{A}$. Therefore $\mathcal{F}$ is additive. □

Theorem 2.2. *Suppose that $\Lambda : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ is a function such that there exists an $L < 1$ with*

$$\Lambda(x, y, z) \leq \frac{L}{2}\Lambda(2x, 2y, 2z) \tag{2.2}$$

for all $x, y, z \in \mathcal{A}$. Let $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying

$$\begin{aligned} &\|\mathcal{F}(x+y+z) - \mathcal{F}(x+z) - \mathcal{F}(y-x+z) - \mathcal{F}(x-z)\| \\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \Lambda(x, y, z) \end{aligned} \tag{2.3}$$

for all $x, y, z \in \mathcal{A}$. Then there exists a unique additive mapping $\Delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|\mathcal{F}(x) - \Delta(x)\| \leq \frac{L}{2(1-L)}\Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right) \tag{2.4}$$

for all $x \in \mathcal{A}$.

Proof. Setting $x = y = z = 0$ in (2.3), we have

$$2\|\mathcal{F}(0)\| \leq (|\alpha| + |\beta|)\|\mathcal{F}(0)\| + \Lambda(0, 0, 0)$$

and thus $\mathcal{F}(0) = 0$, since $|\alpha| + |\beta| < 2$ and by (2.2) $\Lambda(0, 0, 0) = 0$.

Letting $x = z = \frac{t}{2}$ and $y = t$ in (2.3), we get

$$\|\mathcal{F}(2t) - 2\mathcal{F}(t)\| \leq \Lambda\left(\frac{t}{2}, t, \frac{t}{2}\right) \tag{2.5}$$

for all $t \in \mathcal{A}$.

Now, consider the set $\Omega = \{\omega : \mathcal{A} \to \mathcal{A} : \omega(0) = 0\}$ and the mapping d defined on $\Omega \times \Omega$ by

$$d(\delta, \omega) = \inf\left\{k \in \mathbb{R}_+ : \|\delta(x) - \omega(x)\| \leq k\Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right), \forall x \in \mathcal{A}\right\},$$

where as usual, $\inf \emptyset = +\infty$. d is a complete generalized metric on Ω (see [20]).

Now, let us consider the linear mapping $\mathcal{T} : \Omega \to \Omega$ such that

$$\mathcal{T}\delta(x) := 2\delta\left(\frac{x}{2}\right)$$

for all $x \in \mathcal{A}$. Thus $d(\delta, \omega) = \varepsilon$ implies that

$$\|\delta(x) - \omega(x)\| \leq \epsilon\Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right)$$

for all $x \in \mathcal{A}$. Hence

$$\|\mathcal{T}\delta(x) - \mathcal{T}\omega(x)\| = \left\|2\delta\left(\frac{x}{2}\right) - 2\omega\left(\frac{x}{2}\right)\right\| \leq 2\varepsilon\Lambda\left(\frac{x}{4}, \frac{x}{2}, \frac{x}{4}\right) \leq L\varepsilon\Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right)$$

for all $x \in \mathcal{A}$, that is $d(\delta, \omega) = \varepsilon$ implies that $d(\mathcal{T}\delta(x), \mathcal{T}\omega(x)) \leq L\varepsilon$. This means that

$$d(\mathcal{T}\delta(x), \mathcal{T}\omega(x)) \leq Ld(\delta, \omega)$$

for all $\delta, \omega \in \Omega$.

Next, from (2.5), we get

$$\left\|\mathcal{F}(x)-2\mathcal{F}\left(\frac{x}{2}\right)\right\| \leq \Lambda\left(\frac{x}{4},\frac{x}{2},\frac{x}{4}\right) \leq \frac{L}{2}\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$, it follows that $d(\mathcal{F},\mathcal{T}\mathcal{F})\leq\frac{L}{2}$.

Using the fixed point alternative we deduce the existence of a unique fixed point of $\mathcal{T}$, that is, the existence of a mapping $\Delta:\mathcal{A}\to\mathcal{A}$ such that

$$\Delta(x)=2\Delta\left(\frac{x}{2}\right)$$

with the following property: there exists a $k\in(0,\infty)$ satisfying

$$\|\mathcal{F}(x)-\Delta(x)\|\leq k\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$.

Since $\lim_{n\to\infty} d(\mathcal{T}^n\mathcal{F},\Delta)=0$,

$$\lim_{n\to\infty}2^n\mathcal{F}\left(\frac{x}{2^n}\right)=\Delta(x)$$

for all $x\in\mathcal{A}$.

Also, $d(\mathcal{F},\Delta)\leq\dfrac{1}{1-L}d(\mathcal{F},\mathcal{T}\mathcal{F})$ which implies

$$\|\mathcal{F}(x)-\Delta(x)\|\leq\frac{L}{2(1-L)}\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$. It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\|\Delta(x+y+z)-\Delta(x+z)-\Delta(y-x+z)-\Delta(x-z)\|\\
&=\lim_{n\to\infty}2^n\left\|\mathcal{F}\left(\frac{x+y+z}{2^n}\right)-\mathcal{F}\left(\frac{x+z}{2^n}\right)-\mathcal{F}\left(\frac{y-x+z}{2^n}\right)-\mathcal{F}\left(\frac{x-z}{2^n}\right)\right\|\\
&\leq\lim_{n\to\infty}2^n\left\|\alpha\left(\mathcal{F}\left(\frac{x+y-z}{2^n}\right)+\mathcal{F}\left(\frac{x-z}{2^n}\right)-\mathcal{F}\left(\frac{y}{2^n}\right)\right)\right\|\\
&\quad+\lim_{n\to\infty}2^n\left\|\beta\left(\mathcal{F}\left(\frac{x-z}{2^n}\right)+\mathcal{F}\left(\frac{x}{2^n}\right)-\mathcal{F}\left(\frac{z}{2^n}\right)\right)\right\|+\lim_{n\to\infty}2^n\Lambda\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)\\
&=\|\alpha(\Delta(x+y-z)+\Delta(x-z)-\Delta(y))\|+\|\beta(\Delta(x-z)+\Delta(x)-\Delta(z))\|
\end{aligned}$$

for all $x,y,z\in\mathcal{A}$. Therefore, by Lemma 2.1, the mapping Δ is additive. □

Corollary 2.3. *Let $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying*

$$\begin{aligned}&\|\mathcal{F}(x+y+z) - \mathcal{F}(x+z) - \mathcal{F}(y-x+z) - \mathcal{F}(x-z)\| \\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \|[x,y,z]\|\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Then there exists a unique additive mapping $\Delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|\mathcal{F}(x) - \Delta(x)\| \leq \|[x,x,x]\|$$

for all $x \in \mathcal{A}$.

Proof. The proof follows from Theorem 2.2 by taking $L = \frac{8}{9}$ and $\Lambda(x,y,z) = \|[x,y,z]\|$ for all $x, y, z \in \mathcal{A}$. □

3 Stability of ternary antiderivations in ternary algebras

In this section we introduce the concept of ternary antiderivation in ternary Banach algebras and prove the stability of ternary antiderivations associated to (1.1) in ternary Banach algebras.

Definition 3.1. *Let $\mathcal{A}$ be a ternary Banach algebra. A $\mathbb{C}-$linear mapping $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ is called a ternary antiderivation if it satisfies*

$$[\mathcal{I}(x), \mathcal{I}(y), \mathcal{I}(z)] = \mathcal{I}[\mathcal{I}(x), y, z] + \mathcal{I}[x, \mathcal{I}(y), z] + \mathcal{I}[x, y, \mathcal{I}(z)]$$

for all $x, y, z \in \mathcal{A}$.

Example 3.2. *The complex number set $\mathbb{C}$ with a ternary product $[x,y,z] = xyz$ for all $x, y, z \in \mathbb{C}$, is a ternary Banach algebra.*

Define $\mathcal{I} : \mathbb{C} \to \mathbb{C}$ by

$$\mathcal{I}(x) = 3x$$

for all $x \in \mathbb{C}$. Then $\mathcal{I}$ is a ternary antiderivation.

Definition 3.3 ([13])**.** *Let $\mathcal{A}$ be a ternary Banach algebra. A double sequence $\{a_{n,m}\}$ in $\mathcal{A}$ converges to $L \in \mathcal{A}$ and we write $\lim_{n,m\to\infty} a_{n,m} = L$ if for every $\epsilon > 0$ there is an integer N such that for all $n, m \geq N$,*

$$|a_{n,m} - L| < \epsilon.$$

If no such number L exists, we say that $\{a_{n,m}\}$ diverges.

Lemma 3.4 ([23]). *Let $\mathcal{A}$ be complex Banach algebra and let $f : \mathcal{A} \to \mathcal{A}$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x \in \mathcal{A}$, then f is $\mathbb{C}-$linear.*

Theorem 3.5. *Let $\Lambda : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a function, and let there exists an $L < 1$ with satisfying*

$$\Lambda(x, y, z) \leq \frac{L}{8}\Lambda(2x, 2y, 2z) \tag{3.1}$$

for all $x, y, z \in \mathcal{A}$. Assume that $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is a mapping such that

$$\begin{aligned} &\|\mathcal{F}(\mu(x + y + z)) - \mu\mathcal{F}(x + z) - \mu\mathcal{F}(y - x + z) - \mu\mathcal{F}(x - z)\| \\ &\leq \|\alpha(\mathcal{F}(x + y - z) + \mathcal{F}(x - z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x - z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \Lambda(x, y, z) \end{aligned} \tag{3.2}$$

and

$$\|[\mathcal{F}(x), \mathcal{F}(y), \mathcal{F}(z)] - \mathcal{F}[\mathcal{F}(x), y, z] - \mathcal{F}[x, \mathcal{F}(y), z] - \mathcal{F}[x, y, \mathcal{F}(z)]\| \leq \Lambda(x, y, z) \tag{3.3}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{A}$. If $\mathcal{F}$ is continuous and in addition, $\mathcal{F}_n(x) := 2^n\mathcal{F}\left(\frac{x}{2^n}\right)$ converges uniformly for all $x \in \mathcal{A}$, double sequences $\{2^{n+m}\mathcal{F}\left(\mathcal{F}\left(\frac{x}{2^n}\right)\frac{y}{2^m}\right)\}$ and $\{2^{n+m}\mathcal{F}\left(\frac{x}{2^n}\mathcal{F}\left(\frac{y}{2^m}\right)\right)\}$ are convergent for all $x, y \in \mathcal{A}$, then there exists a unique continuous ternary antiderivation $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ such that

$$\|\mathcal{F}(x) - \mathcal{I}(x)\| \leq \frac{L}{2(4 - L)}\Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right)$$

for all $x \in \mathcal{A}$.

Proof. Assume that $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ satisfies (3.2).

Putting $\mu = 1$ and $x = y = z = 0$ in (3.2), we obtain

$$2\|\mathcal{F}(0)\| \leq (|\alpha| + |\beta|)\|\mathcal{F}(0)\| + \Lambda(0, 0, 0)$$

and thus $\mathcal{F}(0) = 0$, since $|\alpha| + |\beta| < 2$ and by (3.1), $\Lambda(0, 0, 0) = 0$.

Letting $x = z = \frac{t}{2}$ and $y = t$ in (3.2), we have

$$\|\mathcal{F}(2\mu t) - 2\mu\mathcal{F}(t)\| \leq \Lambda\left(\frac{t}{2}, t, \frac{t}{2}\right) \tag{3.4}$$

for all $\mu \in \mathbb{T}^1$ and all $t \in \mathcal{A}$.

Next, consider the set

$$\Omega := \{\omega : \mathcal{A} \to \mathcal{A} : \omega(0) = 0\}$$

and define the generalized metric on Ω

$$d(\theta,\omega)=\inf\left\{k\in\mathbb{R}_{\geq 0}:\|\theta(x)-\omega(x)\|\leq k\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right),\forall x\in A\right\},$$

where as usual, $\inf\emptyset=+\infty$. By [20, Lemma 1.2], (Ω,d) is a complete generalized metric space.

Now we define the linear mapping $\mathcal{T}:\Omega\to\Omega$ such that

$$\mathcal{T}\theta(x)=2\theta\left(\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$.

Let $\theta,\omega\in\Omega$ be given such that $d(\theta,\omega)=\varepsilon$. Then

$$\|\theta(x)-\omega(x)\|\leq\varepsilon\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$. Hence

$$\|\mathcal{T}\theta(x)-\mathcal{T}\omega(x)\|=\left\|2\theta\left(\frac{x}{2}\right)-2\omega\left(\frac{x}{2}\right)\right\|\leq 2\varepsilon\Lambda\left(\frac{x}{4},\frac{x}{2},\frac{x}{4}\right)\leq\frac{L}{4}\varepsilon\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$. So $d(\theta,\omega)=\varepsilon$ implies that $d(\mathcal{T}\theta(x),\mathcal{T}\omega(x))\leq\frac{L}{4}\varepsilon$. Hence

$$d(\mathcal{T}\theta(x),\mathcal{T}\omega(x))\leq\frac{L}{4}d(\theta,\omega)$$

for all $\theta,\omega\in\Omega$. It follows from (3.4) that

$$\left\|\mathcal{F}(x)-2\mathcal{F}\left(\frac{x}{2}\right)\right\|\leq\Lambda\left(\frac{x}{4},\frac{x}{2},\frac{x}{4}\right)\leq\frac{L}{8}\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$ and so $d(\mathcal{F},\mathcal{T}\mathcal{F})\leq\frac{L}{8}$.

Using the fixed point alternative we deduce the existence of a unique fixed point of $\mathcal{T}$, that is, the existence of a mapping $\mathcal{I}:\mathcal{A}\to\mathcal{A}$ such that

$$\mathcal{I}(x)=2\mathcal{I}\left(\frac{x}{2}\right)$$

with the following property: there exists a $k\in(0,\infty)$ satisfying

$$\|\mathcal{F}(x)-\mathcal{I}(x)\|\leq k\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$.

Since $\lim_{n\to\infty}d(\mathcal{T}^n\mathcal{F},\mathcal{I})=0$,

$$\lim_{n\to\infty}2^n\mathcal{F}\left(\frac{x}{2^n}\right)=\mathcal{I}(x)$$

for all $x \in \mathcal{A}$. Also, $d(\mathcal{F},\mathcal{I}) \le \frac{1}{1-\frac{L}{4}} d(\mathcal{F},\mathcal{T}\mathcal{F})$ which implies

$$\|\mathcal{F}(x) - \mathcal{I}(x)\| \le \frac{L}{2(4-L)} \Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right)$$

for all $x \in \mathcal{A}$. It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\|\mathcal{I}(x+y+z) - \mathcal{I}(x+z) - \mathcal{I}(y-x+z) - \mathcal{I}(x-z)\| \\
&= \lim_{n\to\infty} \left\| 2^n \left(\mathcal{F}\left(\frac{x+y+z}{2^n}\right) - \mathcal{F}\left(\frac{x+z}{2^n}\right) - \mathcal{F}\left(\frac{y-x+z}{2^n}\right) - \mathcal{F}\left(\frac{x-z}{2^n}\right)\right)\right\| \\
&\le \lim_{n\to\infty} 2^n \left\| \alpha \left(\mathcal{F}\left(\frac{x+y-z}{2^n}\right) + \mathcal{F}\left(\frac{x-z}{2^n}\right) - \mathcal{F}\left(\frac{y}{2^n}\right)\right)\right\| \\
&\quad + \lim_{n\to\infty} 2^n \left\| \beta \left(\mathcal{F}\left(\frac{x-z}{2^n}\right) + \mathcal{F}\left(\frac{x}{2^n}\right) - \mathcal{F}\left(\frac{z}{2^n}\right)\right)\right\| + \lim_{n\to\infty} 2^n \Lambda\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&= \|\alpha(\mathcal{I}(x+y-z) + \mathcal{I}(x-z) - \mathcal{I}(y))\| + \|\beta(\mathcal{I}(x-z) + \mathcal{I}(x) - \mathcal{I}(z))\|
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. By Lemma 2.1, the mapping $\mathcal{I}$ is additive.

Letting $x = z = \frac{t}{2}$ and $y = 0$ in (3.2), we get

$$\|\mathcal{F}(\mu t) - \mu\mathcal{F}(t)\| \le \Lambda\left(\frac{t}{2}, 0, \frac{t}{2}\right)$$

for all $\mu \in \mathbb{T}^1$ and all $t \in \mathcal{A}$. Thus

$$\begin{aligned}
\|\mathcal{I}(\mu x) - \mu\mathcal{I}(x)\| &= \lim_{n\to\infty} 2^n \left\| \mathcal{F}\left(\mu\frac{x}{2^n}\right) - \mu\mathcal{F}\left(\frac{x}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty} 2^n \Lambda\left(\frac{x}{2^{n+1}}, 0, \frac{x}{2^{n+1}}\right) \le \lim_{n\to\infty} \left(\frac{L}{4}\right)^n \Lambda\left(\frac{x}{2}, 0, \frac{x}{2}\right),
\end{aligned}$$

which tends to zero as $n \to \infty$ and so $\mathcal{I}(\mu x) = \mu\mathcal{I}(x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Therefore, by Lemma 3.4, the mapping $\mathcal{I}$ is $\mathbb{C}-$linear.

Since $\mathcal{F}$ is continuous and $\mathcal{F}_n$ converges uniformly, $\mathcal{I}$ is continuous. It follows from (3.1) and (3.3) that

$$\begin{aligned}
&\|[\mathcal{I}(x), \mathcal{I}(y), \mathcal{I}(z)] - \mathcal{I}[\mathcal{I}(x), y, z] - \mathcal{I}[x, \mathcal{I}(y), z] - \mathcal{I}[x, y, \mathcal{I}(z)]\| \\
&= \lim_{n\to\infty} \left\| 2^{3n} \left[\mathcal{F}\left(\frac{x}{2^n}\right), \mathcal{F}\left(\frac{y}{2^n}\right), \mathcal{F}\left(\frac{z}{2^n}\right)\right] - 2^n \mathcal{I}\left[\mathcal{F}\left(\frac{x}{2^n}\right), y, z\right] \right. \\
&\qquad \left. - 2^n \mathcal{I}\left[x, \mathcal{F}\left(\frac{y}{2^n}\right), z\right] - 2^n \mathcal{I}\left[x, y, \mathcal{F}\left(\frac{z}{2^n}\right)\right]\right\| \\
&= \lim_{n\to\infty} 2^{3n} \left\| \left[\mathcal{F}\left(\frac{x}{2^n}\right), \mathcal{F}\left(\frac{y}{2^n}\right), \mathcal{F}\left(\frac{z}{2^n}\right)\right] - \mathcal{F}\left[\mathcal{F}\left(\frac{x}{2^n}\right), \frac{y}{2^n}, \frac{z}{2^n}\right] \right. \\
&\qquad \left. - \mathcal{F}\left[\frac{x}{2^n}, \mathcal{F}\left(\frac{y}{2^n}\right), \frac{z}{2^n}\right] - \mathcal{F}\left[\frac{x}{2^n}, \frac{y}{2^n}, \mathcal{F}\left(\frac{z}{2^n}\right)\right]\right\| \\
&\le \lim_{n\to\infty} 2^{3n} \Lambda\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \le \lim_{n\to\infty} L^n \Lambda(x, y, z)
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{A}$. Since $L < 1$, the $\mathbb{C}-$linear mapping $\mathcal{I}$ is a ternary antiderivation. □

Corollary 3.6. *Let $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying*

$$\begin{aligned}&\|\mathcal{F}(\mu(x+y+z)) - \mu\mathcal{F}(x+z) - \mu\mathcal{F}(y-x+z) - \mu\mathcal{F}(x-z)\|\\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \|[[x,x,y],y,z]\|,\end{aligned}$$

$$\|[\mathcal{F}(x), \mathcal{F}(y), \mathcal{F}(z)] - \mathcal{F}[\mathcal{F}(x), y, z] + \mathcal{F}[x, \mathcal{F}(y), z] + \mathcal{F}[x, y, \mathcal{F}(z)]\| \leq \|[[x,x,y],y,z]\|$$

for all $x, y, z \in \mathcal{A}$. If $\mathcal{F}$ is continuous and in addition, $\mathcal{F}_n(x) := 2^n\mathcal{F}\left(\frac{x}{2^n}\right)$ converges uniformly for all $x \in \mathcal{A}$, double sequences $\{2^{n+m}\mathcal{F}\left(\mathcal{F}\left(\frac{x}{2^n}\right)\frac{y}{2^m}\right)\}$ and $\{2^{n+m}\mathcal{F}\left(\frac{x}{2^n}\mathcal{F}\left(\frac{y}{2^m}\right)\right)\}$ are convergent for all $x, y \in \mathcal{A}$, then there exists a unique continuous ternary antiderivation $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ such that

$$\|\mathcal{F}(x) - \mathcal{I}(x)\| \leq \frac{1}{50}\|x\|^5$$

for all $x \in \mathcal{A}$.

Proof. The proof follows from Theorem 3.5 by taking $L = \frac{32}{33}$ and $\Lambda(x, y, z) = \|[[x,x,y],y,z]\|$ for all $x, y, z \in \mathcal{A}$. □

4 Stability of continuous ternary antiderivations in ternary Banach algebras

In this section, we prove the stability of continuous ternary antiderivations in ternary Banach algebras.

Theorem 4.1. *Let $\Lambda : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a function. If there exists an $L < 1$ with satisfying*

$$\Lambda(x, y, z) \leq \frac{L}{8}\Lambda(2x, 2y, 2z) \tag{4.1}$$

for all $x, y, z \in \mathcal{A}$. Assume that $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying

$$\begin{aligned}&\|\mathcal{F}(\mu(x+y+z)) - \mu\mathcal{F}(x+z) - \mu\mathcal{F}(y-x+z) - \mu\mathcal{F}(x-z)\|\\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \Lambda(x,y,z)\end{aligned} \tag{4.2}$$

and (3.3) for all μ with $|\mu| < 1$ (resp. $|\mu| > 1$). If $\mathcal{F}$ is continuous and in addition, $\mathcal{F}_n(x) := 2^n\mathcal{F}\left(\frac{x}{2^n}\right)$ converges uniformly for all $x \in \mathcal{A}$, double sequences $\{2^{n+m}\mathcal{F}\left(\mathcal{F}\left(\frac{x}{2^n}\right)\frac{y}{2^m}\right)\}$ and

$\{2^{n+m}\mathcal{F}\left(\frac{x}{2^n}\mathcal{F}\left(\frac{y}{2^m}\right)\right)\}$ *are convergent for all* $x, y \in \mathcal{A}$*, then there exists a unique continuous ternary antiderivation* $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ *such that*

$$\|\mathcal{F}(x) - \mathcal{I}(x)\| \leq \frac{L}{2(4-L)}\Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right) \tag{4.3}$$

for all $x \in \mathcal{A}$*.*

Proof. Let $\mu \in \mathbb{T}^1$. Then there exists a sequence $\{\mu_n\}_{n=1}^{\infty}$ with $|\mu_n| < 1$ (resp. $|\mu_n| > 1$) such that

$$\lim_{n\to\infty} \mu_n = \mu.$$

By (4.2) we get

$$\begin{aligned}&\|\mathcal{F}(\mu_n(x+y+z)) - \mu_n\mathcal{F}(x+z) - \mu_n\mathcal{F}(y-x+z) - \mu_n\mathcal{F}(x-z)\| \\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \Lambda(x,y,z)\end{aligned}$$

for all positive integers n, all μ_n with $|\mu_n| < 1$ (resp. $|\mu_n| > 1$) and all $x, y, z \in \mathcal{A}$.

Passing to the limit as $n \to \infty$, and using the continuity of $\mathcal{F}$ and $\|\cdot\|$, we obtain

$$\begin{aligned}&\|\mathcal{F}(\mu(x+y+z)) - \mu\mathcal{F}(x+z) - \mu\mathcal{F}(y-x+z) - \mu\mathcal{F}(x-z)\| \\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \Lambda(x,y,z)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{A}$

Therefore, by the same reasoning as in the proof of Theorem 3.5, there exists a unique ternary antiderivation $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ satisfying (4.3). □

Declarations

Availablity of data and materials

Not applicable.

Human and animal rights

We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest

The authors declare that they have no competing interests.

Fundings

The authors declare that there is no funding available for this paper.

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Acknowledgements

We would like to express our sincere gratitude to the anonymous referee for his/her helpful comments that will help to improve the quality of the manuscript.

References

[1] M. R. Abdollahpoura, R. Aghayaria and M. Th. Rassias, "Hyers-Ulam stability of associated Laguerre differential equations in a subclass of analytic functions", *J. Math. Anal. Appl.*, vol. 437, no. 1, pp. 605–612, 2016. DOI: https://doi.org/10.1016/j.jmaa.2016.01.024.

[2] J. Brzdęk, "On a method of proving the Hyers-Ulam stability of functional equations on restricted domains", *Aust. J. Math. Anal. Appl.*, vol. 6, no. 1, Art. ID 4, 2009.

[3] J. Brzdęk, L. Cădariu and K. Ciepliński, "Fixed point theory and the Ulam stability", *J. Funct. Spaces*, vol. 2014, Article ID 829419, 2014. DOI: https://doi.org/10.1155/2014/829419.

[4] J. Brzdęk, L. Cădariu, K. Ciepliński, A. Fošner and Z. Leśniak, "Survey on recent Ulam stability results concerning derivations", *J. Funct. Spaces*, vol. 2016, Article ID 1235103, 2016. DOI: https://doi.org/10.1155/2016/1235103.

[5] J. Brzdęk and K. Ciepliński, "Hyperstability and superstability", *Abstr. Appl. Anal.*, vol. 2013, Art. ID 401756, 2013. DOI: https://doi.org/10.1155/2013/401756.

[6] J. Brzdęk and A. Fošner, "On approximate generalized Lie derivation", *Glas. Mat. Ser. III*, vol. 50, no. 1, pp. 77–99, 2015. DOI: https://doi.org/10.3336/gm.50.1.07.

[7] J. Brzdęk, A. Fošner and Z. Leśniak, "A note on asymptotically approximate generalized Lie derivations", *J. Fixed Point Theory Appl.*, vol. 22, no. 2, Art. ID 40, 2020. DOI: https://doi.org/10.1007/s11784-020-00775-8.

[8] M. Dehghanian and S. M. S. Modarres, "Ternary γ-homomorphisms and ternary γ-derivations on ternary semigroups", *J. Inequal. Appl.*, vol. 2012, Art. ID 34, 2012. DOI: https://doi.org/10.1186/1029-242X-2012-34.

[9] M. Dehghanian, S. M. S. Modarres, C. Park and D. Y. Shin, "C^*-Ternary 3-derivations on C^*-ternary algebras", *J. Comput. Anal. Appl.*, vol. 2013, Art. ID 124, 2013. DOI: https://doi.org/10.1186/1029-242X-2013-124.

[10] M. Dehghanian and C. Park, "C^*-Ternary 3-homomorphisms on C^*-ternary algebras", *Results Math.*, vol. 66, no. 1-2, pp. 87–98, 2014. DOI: https://doi.org/10.1007/s00025-014-0365-7.

[11] M. Dehghanian, Y. Sayyari and C. Park, "Hadamard homomorphisms and Hadamard derivations on Banach algebras", *Miskolc Math. Notes*, vol. 24, no. 1, pp. 129–137, 2023. DOI: https://doi.org/10.18514/MMN.2023.3928.

[12] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative for contractions on a generalized complete metric space", *Bull. Amer. Math. Soc.*, vol. 74, pp. 305–309, 1968. DOI: https://doi.org/10.1090/S0002-9904-1968-11933-0.

[13] E. D. Habil, "Double sequences and double series", *Islamic Univ. J. Ser. Nat. Stud. Eng.*, vol. 14, no. 1, pp. 1–32, 2006.

[14] M. R. Haddadi and M. Dehghanian, "Existence and convergence theorems for equilibrium problems and fixed point problem", *Southeast Asian Bull. Math.*, vol. 42, no. 4, pp. 559–568, 2018.

[15] S. A. Hosseinioun, R. Farrokhzad and M. Eshaghi Gordj, "Nearly higher ternary derivations in Banach ternary algebras: An alternative fixed point approach", *Int. J. Nonlinear Anal. Appl.*, vol. 5, no. 2, pp. 7-15, 2014. DOI: https://doi.org/10.22075/IJNAA.2014.121.

[16] V. Govindan, C. Park, S. Pinelas and Th. M. Rassias, "Hyers-Ulam stability of an additive-quadratic functional equation", *Cubo*, vol. 22, no. 2, pp. 233–255, 2020. DOI: http://dx.doi.org/10.4067/S0719-06462020000200233.

[17] D. H. Hyers, "On the stability of the linear functional equation", *Proc. Nat. Acad. Sci. U.S.A.*, vol. 27, pp. 222–224, 1941. DOI: https://doi.org/10.1073/pnas.27.4.222.

[18] R. Kerner, "Ternary algebraic structures and their applications in physics", Pierre et Marie Curie University, Paris; Proc. BTLP, 23rd International Conference on Group Theoretical Methods in Physics, Dubna, Russia, 2000. DOI: https://doi.org/10.48550/arXiv.math-ph/0011023.

[19] F. Lü and H. Guo, "On meromorphic solutions of the Fermat-type functional equation $f(z)^n + f(z+c)^m = e^{\alpha z+\beta}$", *Mediterr. J. Math.*, vol. 19, no. 3, Art. ID 118, 2022. DOI: https://doi.org/10.1007/s00009-022-02054-x.

[20] D. Mihet and V. Radu, "On the stability of the additive Cauchy functional equation in random normed spaces", *J. Math. Anal. Appl.*, vol. 343, no. 1, pp. 567–572, 2008. DOI: https://doi.org/10.1016/j.jmaa.2008.01.100.

[21] Y. Nambu, "Generalized Hamiltonian mechanics", *Phys. Rev.*, vol. 7, no. 8, pp. 2405–2412, 1973. DOI: https://doi.org/10.1103/PhysRevD.7.2405.

[22] S. Paokanta, M. Dehghanian, C. Park and Y. Sayyari, "A system of additive functional equations in complex Banach algebras", *Demonstr. Math.*, vol. 56, no. 1, Art. ID 20220165, 2023. DOI: https://doi.org/10.1515/dema-2022-0165.

[23] C. Park, "Homomorphisms between Poisson JC^*-algebras", *Bull. Braz. Math. Soc. (N.S.)*, vol. 36, no. 1, pp. 79–97, 2005. DOI: https://doi.org/10.1007/s00574-005-0029-z.

[24] C. Park, "Fixed point method for set-valued functional equations", *J. Fixed Point Theory Appl.*, vol. 19, no. 4, pp. 2297–2308, 2017. DOI: https://doi.org/10.1007/s11784-017-0418-0.

[25] C. Park, "The stability of an additive (ρ_1, ρ_2)-functional inequality in Banach spaces", *J. Math. Inequal.*, vol. 13, no. 1, pp. 95–104, 2019. DOI: https://doi.org/10.7153/jmi-2019-13-07.

[26] C. Park, "Derivation-homomorphism functional inequality", *J. Math. Inequal.*, vol. 15, no. 1, pp. 95–105, 2021. DOI: https://doi.org/10.7153/jmi-2021-15-09.

[27] Y. Sayyari, M. Dehghanian and Sh. Nasiri, "Solution of some irregular functional equations and their stability", *J. Linear Topol. Algebra*, vol. 11, no. 4, pp. 271–277, 2022. DOI: https://doi.org/10.30495/jlta.2023.699062.

[28] Y. Sayyari, M. Dehghanian and C. Park, "A system of biadditive functional equations in Banach algebras", *Appl. Math. Sci. Eng.*, vol. 31, no. 1, Art. ID 2176851, 2023. DOI: https://doi.org/10.1080/27690911.2023.2176851.

[29] Y. Sayyari, M. Dehghanian, C. Park and J. Lee, "Stability of hyper homomorphisms and hyper derivations in complex Banach algebras", *AIMS Math.*, vol. 7, no. 6, pp. 10700–10710, 2022. DOI: https://doi.org/10.3934/math.2022597.

[30] N. Širovnik, "On certain functional equation in semiprime rings and standard operator algebras", *Cubo*, vol. 16, no. 1, pp. 73–80, 2014. DOI: http://dx.doi.org/10.4067/S0719-06462014000100007.