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Laeng-Morpurgo-type uncertainty inequalities for the Weinstein transform

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ABSTRACT

In this work, by combining Carlson-type and Nash-type inequalities for the Weinstein transform $\mathscr{F}_W$ on $\mathbb{K} = \mathbb{R}^{d-1} \times [0, \infty)$, we show Laeng-Morpurgo-type uncertainty inequalities. We establish also local-type uncertainty inequalities for the Weinstein transform $\mathscr{F}_W$, and we deduce a Heisenberg-Pauli-Weyl-type inequality for this transform.

RESUMEN

En este trabajo, combinando desigualdades de tipo Carlson y de tipo Nash para la transformada de Weinstein $\mathscr{F}_W$ en $\mathbb{K} = \mathbb{R}^{d-1} \times [0, \infty)$, demostramos desigualdades de incertidumbre de tipo Laeng-Morpurgo. Establecemos también desigualdades de incertidumbre de tipo local para la transformada de Weinstein $\mathscr{F}_W$, y deducimos una desigualdad de tipo Heisenberg-Pauli-Weyl para esta transformada.

Keywords and Phrases: Laeng-Morpurgo-type inequality; local-type inequality; Heisenberg-Pauli-Weyl-type inequality.

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1 Introduction

Uncertainty principles are mathematical arguments that give limitations on the simultaneous concentration of a function and its Fourier transform. They have implications in quantum physics and signal analysis. They also play an important role in harmonic analysis, many of them have already been studied from several points of view for the Fourier transform, Heisenberg-Pauli-Weyl inequality and local uncertainty [9, 10]. Laeng-Morpurgo and Morpurgo [4, 7] obtained Heisenberg inequality involving a combination of L^1 and L^2 norms.

In this paper, we consider the Weinstein transform $\mathscr{F}_W$ [2, 5, 6] defined on $L^1(\mathbb{K}, \nu_k)$ by

$$\mathscr{F}_W(f)(\xi) := \int_{\mathbb{K}} f(x)\Psi_\xi(x)\mathrm{d}\nu_k(x), \quad \xi = (\xi', \xi_d) \in \mathbb{K},$$

where $\mathbb{K} := \mathbb{R}^{d-1} \times [0, \infty)$, $\mathrm{d}\nu_k(x) := \frac{x_d^{2k+1}}{\pi^{(d-1)/2}2^{k+(d-1)/2}\Gamma(k+1)}\,\mathrm{d}x'\mathrm{d}x_d$ and

$$\Psi_\xi(x) = e^{-i\langle x', \xi'\rangle} j_k(x_d\xi_d), \quad x = (x', x_d) \in \mathbb{K}.$$

Here j_k is the spherical Bessel function.

Many uncertainty principles have already been proved for the Weinstein transform $\mathscr{F}_W$ on $\mathbb{K}$, namely Mejjaoli and Salhi are the first that describe the uncertainty principles for the Weinstein transform [6]. Next, Ben Salem and Nasr obtained Heisenberg-type inequalities [3] for the Weinstein transform $\mathscr{F}_W$. Saoudi [11] proved a variation of L^p uncertainty principles for the Weinstein transform $\mathscr{F}_W$. In this work, by using Carlson-type inequality and Nash-type inequality [2, 8] for the Weinstein transform $\mathscr{F}_W$ on $L^1 \cap L^2(\mathbb{K}, \nu_k)$; we deduce uncertainty inequalities of Heisenberg-type for the Weinstein transform $\mathscr{F}_W$ on $L^1 \cap L^2(\mathbb{K}, \nu_k)$. Next, due to a local uncertainty inequality for the Weinstein transform $\mathscr{F}_W$ on $L^2(\mathbb{K}, \nu_k)$, we show uncertainty inequality of Heisenberg-Pauli-Weyl-type for the transform $\mathscr{F}_W$ on $L^2(\mathbb{K}, \nu_k)$.

The analog uncertainty inequalities are also proved, for the Dunkl transform $\mathscr{F}_k$ on $\mathbb{R}^d$ by Soltani [12, 13].

This paper is organized as follows. In Section 2, we recall some results about the Weinstein transform $\mathscr{F}_W$ on $\mathbb{K}$. In Section 3, we prove uncertainty inequalities of Heisenberg-type for the Weinstein transform $\mathscr{F}_W$ on $L^1 \cap L^2(\mathbb{K}, \nu_k)$. We show also uncertainty inequality of Heisenberg-Pauli-Weyl-type for the transform $\mathscr{F}_W$ on $L^2(\mathbb{K}, \nu_k)$. In the last section, we summarize the obtained results and describe the future work.

2 Weinstein transform

In this section we recall some basic results related to the Weinstein analysis.

We consider the Weinstein operator Δ_W [1, 3, 8] defined on $\mathbb{R}^{d-1} \times (0, \infty)$ by

$$\Delta_W := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \frac{2k+1}{x_d} \frac{\partial}{\partial x_d} = \Delta_{d-1} + L_k, \quad d \geq 2, \quad k > -1/2,$$

where Δ_{d-1} is the Laplacian operator in $\mathbb{R}^{d-1}$ and L_k is the Bessel operator with respect to the variable x_d defined on $(0, \infty)$ by

$$L_k := \frac{\partial^2}{\partial x_d^2} + \frac{2k+1}{x_d} \frac{\partial}{\partial x_d}.$$

The Weinstein operator (also called Laplace-Bessel operator) has several applications in pure and applied mathematics. The harmonic analysis associated to this operator is studied in [1, 2, 3, 5, 6, 8] and references therein.

Throughout this subsection, let $k > -1/2$ and $\mathbb{K} := \mathbb{R}^{d-1} \times [0, \infty)$. We denote by $L^p(\mathbb{K}, \nu_k)$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{K}$, such that

$$\begin{aligned} \|f\|_{L^p(\mathbb{K},\nu_k)} &:= \left(\int_{\mathbb{K}} |f(x', x_d)|^p \mathrm{d}\nu_k(x', x_d) \right)^{1/p} < \infty, \quad p \in [1, \infty), \\ \|f\|_{L^\infty(\mathbb{K},\nu_k)} &:= \operatorname*{ess\,sup}_{(x', x_d) \in \mathbb{K}} |f(x', x_d)| < \infty, \end{aligned}$$

where

$$\mathrm{d}\nu_k(x) := \mathrm{d}\nu_k(x', x_d) = \frac{x_d^{2k+1}}{\pi^{(d-1)/2} 2^{k+(d-1)/2} \Gamma(k+1)} \mathrm{d}x' \mathrm{d}x_d,$$

and $\mathrm{d}x' = \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{d-1}$.

Let $r > 0$, the measure ν_k satisfies [3]:

$$\nu_k(|x| < r) = c r^\alpha, \tag{2.1}$$

where

$$c = \frac{1}{2^{\frac{\alpha}{2}} \Gamma(\frac{\alpha}{2} + 1)} \quad \text{and} \quad \alpha = 2k + d + 1. \tag{2.2}$$

For all $\xi \in \mathbb{K}$, the system

$$L_k u(x) = -\xi_d^2 u(x), \quad \frac{\partial^2 u}{\partial x_j^2}(x) = -\xi_j^2 u(x), \quad j = 1, \dots, d-1,$$
$$u(0) = 1, \quad \frac{\partial u}{\partial x_d}(0) = 0, \quad \frac{\partial u}{\partial x_j}(0) = -i\xi_j, \quad j = 1, \dots, d-1,$$

admits a unique solution $\Psi_\xi(x)$, given by

$$\Psi_\xi(x) = e^{-i\langle x', \xi' \rangle} j_k(x_d \xi_d), \quad x \in \mathbb{K},$$

where j_k is the spherical Bessel function given by

$$j_k(x) := \Gamma(k+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{2n}.$$

For all $x, \xi \in \mathbb{K}$, the Weinstein kernel $\Psi_\xi(x)$ satisfies

$$|\Psi_\xi(x)| \leq 1.$$

The Weinstein (or Laplace-Bessel) transform $\mathscr{F}_W$ [2, 5, 6] is defined for $f \in L^1(\mathbb{K}, \nu_k)$ by

$$\mathscr{F}_W(f)(\xi) := \int_{\mathbb{K}} f(x) \Psi_\xi(x) \, \mathrm{d}\nu_k(x), \quad \xi \in \mathbb{K}.$$

The transform $\mathscr{F}_W$ initially defined on $L^1 \cap L^2(\mathbb{K}, \nu_k)$ extends uniquely to an isometric isomorphism on $L^2(\mathbb{K}, \nu_k)$, that is,

$$\|\mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} = \|f\|_{L^2(\mathbb{K},\nu_k)}, \quad f \in L^2(\mathbb{K}, \nu_k). \tag{2.3}$$

Moreover if $f \in L^1(\mathbb{K}, \nu_k)$, then

$$\|\mathscr{F}_W(f)\|_{L^\infty(\mathbb{K},\nu_k)} \leq \|f\|_{L^1(\mathbb{K},\nu_k)}. \tag{2.4}$$

Finally, if f and $\mathscr{F}_W(f)$ are both in $L^1(\mathbb{K}, \nu_k)$, the inverse Weinstein transform is defined by

$$f(x) = \int_{\mathbb{K}} \mathscr{F}_W(f)(\xi) \Psi_{-\xi}(x) \, \mathrm{d}\nu_k(\xi), \quad \text{a.e} \quad x \in \mathbb{K}.$$

3 Heisenberg-type uncertainty principles

Similar results have been appeared in the literature by Soltani [13], he proved a Laeng-Morpurgo-type uncertainty inequalities for the Dunkl transform $\mathscr{F}_k$ on $\mathbb{R}^d$. In the following, we will give Laeng-Morpurgo-type uncertainty inequalities for the Weinstein transform $\mathscr{F}_W$ on $\mathbb{K}$.

Proposition 3.1 ([2, 8]).

(i) (Carlson-type inequality). Let $a > 0$. There exists a constant $A(a, \alpha) > 0$ such that for every $f \in L^1 \cap L^2(\mathbb{K}, \nu_k)$, we have

$$\|f\|_{L^1(\mathbb{K},\nu_k)} \leq A(a,\alpha)\|f\|_{L^2(\mathbb{K},\nu_k)}^{\frac{2a}{\alpha+2a}} \| \, |x|^a f\|_{L^1(\mathbb{K},\nu_k)}^{\frac{\alpha}{\alpha+2a}}. \tag{3.1}$$

(ii) (Nash-type inequality). Let $b > 0$. There exists a constant $B(b, \alpha) > 0$ such that for every $f \in L^1 \cap L^2(\mathbb{K}, \nu_k)$, we have

$$\|f\|_{L^2(\mathbb{K},\nu_k)} \leq B(b,\alpha)\|f\|_{L^1(\mathbb{K},\nu_k)}^{\frac{2b}{\alpha+2b}} \| \, |\xi|^b \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)}^{\frac{\alpha}{\alpha+2b}}. \tag{3.2}$$

Thanks to the above proposition, by combining and multiplying the two relations (3.1) and (3.2) we obtain the following uncertainty inequalities of Laeng-Morpurgo-type [4, 7] for the Weinstein transform $\mathscr{F}_W$ on $L^1 \cap L^2(\mathbb{K}, \nu_k)$.

Theorem 3.2. *Let $a, b > 0$. There exist three constants $C(a, b, \alpha) > 0$, $N(a, b, \alpha) > 0$ and $D(a, b, \alpha) > 0$ such that for every $f \in L^1 \cap L^2(\mathbb{K}, \nu_k)$, we have*

(i) $\|f\|_{L^2(\mathbb{K},\nu_k)}^{\alpha+2a+2b} \leq C(a,b,\alpha)\| \, |x|^a f\|_{L^1(\mathbb{K},\nu_k)}^{2b} \| \, |\xi|^b \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)}^{\alpha+2a}$,

(ii) $\|f\|_{L^1(\mathbb{K},\nu_k)}^{\alpha+2a+2b} \leq N(a,b,\alpha)\| \, |x|^a f\|_{L^1(\mathbb{K},\nu_k)}^{\alpha+2b} \| \, |\xi|^b \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)}^{2a}$,

(iii) $\|f\|_{L^1(\mathbb{K},\nu_k)}^{\alpha+2a} \|f\|_{L^2(\mathbb{K},\nu_k)}^{\alpha+2b} \leq D(a,b,\alpha)\| \, |x|^a f\|_{L^1(\mathbb{K},\nu_k)}^{\alpha+2b} \| \, |\xi|^b \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)}^{\alpha+2a}$.

By application of the two relations (3.1) and (3.2) we deduce also the following results which are a local-type uncertainty inequalities for the Weinstein transform $\mathscr{F}_W$ on $L^1 \cap L^2(\mathbb{K}, \nu_k)$.

Theorem 3.3. *Let E be a measurable subset of $\mathbb{K}$ such that $0 < \nu_k(E) < \infty$, and let $a, b > 0$. If $f \in L^1 \cap L^2(\mathbb{K}, \nu_k)$, then*

(i) $\|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} \leq A(a,\alpha)(\nu_k(E))^{1/2}\|f\|_{L^2(\mathbb{K},\nu_k)}^{\frac{2a}{\alpha+2a}} \| \, |x|^a f\|_{L^1(\mathbb{K},\nu_k)}^{\frac{\alpha}{\alpha+2a}}$, *where $A(a, \alpha)$ is the constant given by Proposition 3.1 (i).*

(ii) $\|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^1(\mathbb{K},\nu_k)} \leq B(b,\alpha)(\nu_k(E))^{1/2}\|f\|_{L^1(\mathbb{K},\nu_k)}^{\frac{2b}{\alpha+2b}} \| \, |\xi|^b \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)}^{\frac{\alpha}{\alpha+2b}}$, *where $B(b, \alpha)$ is the constant given by Proposition 3.1 (ii).*

Being $\mathbf{1}_E$ the characteristic function of the set E.

Proof. Let $f \in L^1 \cap L^2(\mathbb{K}, \nu_k)$ and $a, b > 0$.

(i) From (2.4) we have

$$\|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} \leq (\nu_k(E))^{1/2} \|\mathscr{F}_W(f)\|_{L^\infty(\mathbb{K},\nu_k)} \leq (\nu_k(E))^{1/2} \|f\|_{L^1(\mathbb{K},\nu_k)}.$$

The desired result follows from Proposition 3.1 (i).

(ii) From (2.3) we have

$$\|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^1(\mathbb{K},\nu_k)} \leq (\nu_k(E))^{1/2} \|\mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} \leq (\nu_k(E))^{1/2} \|f\|_{L^2(\mathbb{K},\nu_k)}.$$

The desired result follows from Proposition 3.1 (ii). □

Soltani [12] proved a Heisenberg-Pauli-Weyl uncertainty principle for the Dunkl transform $\mathscr{F}_k$ on $\mathbb{R}^d$. In the following, we will give Heisenberg-Pauli-Weyl uncertainty principle for the Weinstein transform $\mathscr{F}_W$ on $L^2(\mathbb{K}, \nu_k)$.

Proposition 3.4. *(local-type inequality). Let $a > 0$ and let $f \in L^2(\mathbb{K}, \nu_k)$. If E be a measurable subset of $\mathbb{K}$ such that $0 < \nu_k(E) < \infty$, then*

$$\|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} \leq A(a, \alpha)(\nu_k(E))^{\frac{a}{\alpha+2a}} \|f\|_{L^2(\mathbb{K},\nu_k)}^{\frac{2a}{\alpha+2a}} \| \, |x|^a f\|_{L^2(\mathbb{K},\nu_k)}^{\frac{\alpha}{\alpha+2a}}, \tag{3.3}$$

where $A(a, \alpha)$ is the constant given by Proposition 3.1 (i).

Proof. Let $f \in L^2(\mathbb{K}, \nu_k)$ and $a > 0$. The inequality holds if $\| \, |x|^a f\|_{L^2(\mathbb{K},\nu_k)} = \infty$. Assume that $\| \, |x|^a f\|_{L^2(\mathbb{K},\nu_k)} < \infty$. For all $r > 0$, we have

$$\begin{aligned} \|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} &\leq \|\mathbf{1}_E \mathscr{F}_W(\mathbf{1}_{B_r} f)\|_{L^2(\mathbb{K},\nu_k)} + \|\mathbf{1}_E \mathscr{F}_W((1 - \mathbf{1}_{B_r})f)\|_{L^2(\mathbb{K},\nu_k)} \\ &\leq (\nu_k(E))^{1/2} \|\mathscr{F}_W(\mathbf{1}_{B_r} f)\|_{L^\infty(\mathbb{K},\nu_k)} + \|\mathscr{F}_W((1 - \mathbf{1}_{B_r})f)\|_{L^2(\mathbb{K},\nu_k)}. \end{aligned}$$

Hence it follows from (2.3) and (2.4) that

$$\|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} \leq (\nu_k(E))^{1/2} \|\mathbf{1}_{B_r} f\|_{L^1(\mathbb{K},\nu_k)} + \|(1 - \mathbf{1}_{B_r})f\|_{L^2(\mathbb{K},\nu_k)}. \tag{3.4}$$

On the other hand, by Hölder's inequality and (2.1), we obtain

$$\|\mathbf{1}_{B_r} f\|_{L^1(\mathbb{K},\nu_k)} \leq (cr^\alpha)^{1/2} \|f\|_{L^2(\mathbb{K},\nu_k)}, \tag{3.5}$$

where c is the constant given by (2.2).

Moreover,

$$\|(1-\mathbf{1}_{B_r})f\|_{L^2(\mathbb{K},\nu_k)} \leq r^{-a}\| \, |x|^a f\|_{L^2(\mathbb{K},\nu_k)}. \tag{3.6}$$

Combining the relations (3.4), (3.5) and (3.6), we deduce that

$$\|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} \leq (\nu_k(E))^{1/2}(cr^\alpha)^{1/2}\|f\|_{L^2(\mathbb{K},\nu_k)} + r^{-a}\| \, |x|^a f\|_{L^2(\mathbb{K},\nu_k)}.$$

By choosing

$$r = \left(\frac{2a\| \, |x|^a f\|_{L^2(\mathbb{K},\nu_k)}}{\alpha c^{1/2}\|f\|_{L^2(\mathbb{K},\nu_k)}}\right)^{\frac{2}{\alpha+2a}} (\nu_k(E))^{-\frac{1}{\alpha+2a}},$$

we obtain the desired inequality. □

We shall use the local uncertainty principle to obtain uncertainty principle of Heisenberg-Pauli-Weyl-type for the Weinstein transform $\mathscr{F}_W$ on $L^2(\mathbb{K},\nu_k)$. We note that the following theorem is given in [3] but in the proof, the approach is not the same.

Theorem 3.5. *Let $a, b > 0$. There exists a constant $K(a,b,\alpha) > 0$ such that for every $f \in L^2(\mathbb{K},\nu_k)$, we have*

$$\|f\|^{a+b}_{L^2(\mathbb{K},\nu_k)} \leq K(a,b,\alpha)\| \, |x|^a f\|^b_{L^2(\mathbb{K},\nu_k)}\| \, |\xi|^b \mathscr{F}_W(f)\|^a_{L^2(\mathbb{K},\nu_k)}.$$

Proof. Let $a, b > 0$ and let $r > 0$. Then

$$\|f\|^2_{L^2(\mathbb{K},\nu_k)} = \|\mathbf{1}_{B_r}\mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)} + \|(1-\mathbf{1}_{B_r})\mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)}. \tag{3.7}$$

Firstly,

$$\|(1-\mathbf{1}_{B_r})\mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)} \leq r^{-2b}\| \, |\xi|^b \mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)}. \tag{3.8}$$

From (2.1) and (3.3), we get

$$\|\mathbf{1}_{B_r}\mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)} \leq (A(a,\alpha))^2(cr^\alpha)^{\frac{2a}{\alpha+2a}}\|f\|^{\frac{4a}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)}\| \, |x|^a f\|^{\frac{2\alpha}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)}, \tag{3.9}$$

where c is the constant given by (2.2).

Combining the relations (3.7), (3.8) and (3.9), we obtain

$$\|f\|^2_{L^2(\mathbb{K},\nu_k)} \leq (A(a,\alpha))^2(cr^\alpha)^{\frac{2a}{\alpha+2a}}\|f\|^{\frac{4a}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)}\| \, |x|^a f\|^{\frac{2\alpha}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)} + r^{-2b}\| \, |\xi|^b \mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)}.$$

By setting

$$r = \left(\frac{b(\alpha+2a)\| \, |\xi|^b \mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)}}{a\alpha(A(a,\alpha))^2 c^{\frac{2a}{\alpha+2a}}\|f\|^{\frac{4a}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)}\| \, |x|^a f\|^{\frac{2\alpha}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)}}\right)^{\frac{\alpha+2a}{2a\alpha+2b(\alpha+2a)}},$$

we get the inequality with

$$K(a,b,\alpha) = (A(a,\alpha))^{2b(\alpha+2a)} c^{2ab} \left(\frac{b(\alpha+2a)}{a\alpha}\right)^{a\alpha} \left(1 + \frac{a\alpha}{b(\alpha+2a)}\right)^{a\alpha+b(\alpha+2a)}.$$

This completes the proof of the theorem. □

4 Conclusion and perspective

The manuscript deals with some uncertainty inequalities associated with the Weinstein transform $\mathscr{F}_W$. Especially, we studied Laeng-Morpurgo type uncertainty inequalities for this transform. As it is well known, uncertainty inequalities are of great interest in harmonic analysis, in applied mathematics and in several areas of mathematical physics. The results given in Section 3 are complements to those given in references [3, 6, 8] and others. They also represent our contribution in the study of local-type uncertainty inequalities and the Heisenberg type inequality for the Weinstein transform $\mathscr{F}_W$. Finally, in a future paper, we have the idea to study the Weinstein-Stockwell transform $\mathscr{S}_g$, $g \in L^2(\mathbb{K}, \nu_\alpha)$, in which we will prove some uncertainty inequalities for this transform analogous to those proven for the Weinstein transform $\mathscr{F}_W$ in this paper.

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