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Osculating varieties and their joins: $\mathbb{P}^1 \times \mathbb{P}^1$

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ABSTRACT

Let $X \subset \mathbb{P}^r$ be an integral projective variety. We study the dimensions of the joins of several copies of the osculating varieties $J(X,m)$ of X. Our methods are general, but we give a full description in all cases only if X is a linearly normal embedding of $\mathbb{P}^1 \times \mathbb{P}^1$. For these embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ we give several examples and then study the joins of one copy of $J(X,m)$ and an arbitrary number of copies of X.

RESUMEN

Sea $X \subset \mathbb{P}^r$ una variedad proyectiva entera. Estudiamos la dimensión de las adjunciones de varias copias de las variedades osculantes $J(X,m)$ de X. Nuestros métodos son generales, pero damos una descripción completa en todos los casos solo si X es un embebimiento linealmente normal de $\mathbb{P}^1 \times \mathbb{P}^1$. Para estos embebimientos de $\mathbb{P}^1 \times \mathbb{P}^1$ damos varios ejemplos y luego estudiamos las adjunciones de una copia de $J(X,m)$ y un número arbitrario de copias de X.

Keywords and Phrases: Osculating space; joins of projective varieties; secant varieties; quadric surface.

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1 Introduction

Let $X \subset \mathbb{P}^r$ be an integral projective variety defined over a fixed algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. We consider the classical problem about the dimension of joins of varieties related to X. Let $J(X, m) \subseteq \mathbb{P}^r$, $m \geq 0$, denote the m-osculating variety of X, *i.e.* the closure in $\mathbb{P}^r$ of the union of all m-osculating spaces to the smooth points of X. With our convention of m-osculating linear spaces we have $J(X, 0) = X$, while $J(X, 1)$ is the tangential variety of X, *i.e.* the with the convention and the dimension of the joins of several $J(X, m_i)$, i varying in a finite set. Our notation calls $J(X, 1)$ the tangential variety $\tau(X) \subseteq \mathbb{P}^r$ of X, *i.e.* the closure in $\mathbb{P}^r$ of the union in $\mathbb{P}^r$ of the tangent spaces T_pX of X at all $p \in X_{\mathrm{reg}}$. For us $J(X, m)$ is the closure in $\mathbb{P}^r$ of the union of the m-osculating spaces at all p in a non-empty open subset of X_{reg} at which these m-osculating spaces have constant dimension.

Take integral varieties $T, Y \subset \mathbb{P}^r$. The join $J(T, Y)$ of T and Y is defined in the following way. If $T = Y$ and Y is a point, p, then $J(\{p\}, \{p\}) = \{p\}$. In all other cases $J(T, Y)$ is the closure of the union of all lines spanned by a point of T and a different point of Y. The algebraic set $J(T, Y)$ is always an irreducible variety and $\dim(T, Y) \leq \min\{r, \dim T + \dim Y + 1\}$ if $\dim T > 0$. The integer $\min\{r, \dim T + \dim Y + 1\}$ is the *expected dimension* of $J(T, Y)$. One defines inductively the join $J(T_1, \dots, T_s)$ of $s \geq 3$ integral varieties $T_i \subset \mathbb{P}^r$ by the formula $J(T_1, \dots, T_s) := J(J(T_1, \dots, T_{s-1}), T_s)$ ([1]). If $\dim T_1 > 0$ we have $\dim J(T_1, \dots, T_s) \leq \min\{r, \dim T_1 + \cdots + \dim T_s + s - 1\}$. If $\dim J(T_1, \dots, T_s) = \min\{r, \dim T_1 + \cdots + \dim T_s + s - 1\}$ we say that $J(T_1, \dots, T_s)$ has the *expected dimension.* The most famous and useful join is the case $T_i = T_1$ for all i, *i.e.*, the s-secant variety of T_1. However, other cases appear. For instance when X is the Veronese variety the join of the tangential variety $J(X, 1)$ of X and $s - 1$ copies of X is related to a certain additive decomposition of forms ([4]).

By the Terracini lemma for joins ([1, Corollary 1.11]) to compute the dimension of the join of s varieties $J(X, m_i)$, $1 \leq i \leq s$, it is sufficient to compute the dimension of the linear span of the tangent spaces $T_{Q_i}J(X, m_i)$ at a general $Q_i \in J(X, m_i)$. Obviously, we first need to compute $\dim T_{Q_i}J(X, m_i)$, but in all our examples these integers are known and hence the only problem is to see how linearly independent are these linear spaces $T_{Q_i}J(X, m_i)$. Fix a general $Q_i \in J(X, m_i)$ and let $p_i \in X_{\mathrm{reg}}$ the point of X_{reg} corresponding to Q_i. A key property of the osculating spaces $T_{Q_i}J(X, m_i)$, is that even for $m > 1$ there is a zero-dimensional scheme $Z_i \subset X$ such that $(Z_i) = \{p_i\}$ and $T_{Q_i}J(X, m_i)$ is the linear span of Z_i (Remark 2.1). If $m > 0$ the scheme is not unique, it is associated to the choice of a line of $T_{p_i}X$ containing p_i (Remark 3.4). Fix a general $(p_1, \dots, p_s) \in X^s_{\mathrm{reg}}$. For each i with $m_i > 0$ choose a "general" Z_i. As always in this type of problems ([3, 6, 7, 8, 9, 10, 11, 12]) it is sufficient to find the schemes $Z_i \subset X$, $1 \leq i \leq s$, and then to prove that the dimension of the linear span of $Z_1 \cup \cdots \cup Z_s$ is the expected one, $\sum_{i=1}^{s} \deg(Z_i) - 1$.

Set $n := \dim X$. For the joins of several copies of X it is sufficient to take as schemes the first infinitesimal neighborhood $2p$, $p \in X_{\mathrm{reg}}$, *i.e.* the closed subscheme of X with $(\mathcal{I}_p)^2$ as its ideal sheaf (this is the classical Terracini lemma for secant varieties [1, Corollary 1.11]); in this case the scheme has degree $n+1$. We call it the case $m=0$. For the tangential variety the scheme Z_1 has degree $2n+1$ and it was used in several papers ([3, 6, 7, 8, 10, 11, 12]), some of them also considering the general case with any $m_i > 0$. Contrary to the case $m=0$ the schemes $Z_i \subset X_{\mathrm{reg}}$ are not uniquely determined by the point $p \in X_{\mathrm{reg}}$ such that $(Z_i) = \{p\}$. For any $m > 0$ the scheme $W(m,p)$ associated to $J(X,m)$ at p has degree $n + \binom{n+m}{n}$ and it is implicitly computed in [6] (and by the classical algebraic geometers quoted in [5, 6]) and given in full generality in [7, 8, 10] at least for the Veronese embeddings of projective spaces. It depends on the choice of some $p \in X_{\mathrm{reg}}$ and a line through p of the embedded tangent space of X at p (Remark 2.1).

Of course, to define the osculating spaces we also need to fix an embedding of X in a projective space or, more generally, a line bundle $\mathcal{L}$ on X and a linear subspace $V \subseteq H^0(\mathcal{L})$. This set-up was described in a modern language by R. Piene ([18]), first defining the bundles of principal parts $\mathbb{P}^m_X(\mathcal{L})$ of $\mathcal{L}$ and then considering an evaluation map $\mathcal{O}_X \otimes V \longrightarrow \mathbb{P}^m_X(L)$. Thus for a fixed $m \geq 0$ and a general $p \in X_{\mathrm{reg}}$ we may choose an irreducible family of zero-dimensional schemes $\mathcal{Z}(m,p)$ such that for each $Z \in \mathcal{Z}(m,p)$ we have $Z = \{p\}$ and $\deg(Z) = n + \binom{n+m}{m}$. Moreover, for any $s > 0$ and any $m_i \geq 0$, the join of $J(X,m_1), \dots, J(X,m_s)$ has dimension $\langle W(m_1,p_1) \cup \cdots \cup W(m_s,p_s) \rangle$, where $\langle\ \rangle$ denote the linear span and $(p_1, \dots, p_s)$ is general in X^s.

The freedom in the choice to define $W(m,p)$ for $m > 0$ will be used several times in our proofs.

We only consider the case $X = \mathbb{P}^1 \times \mathbb{P}^1$ with all its Segre-Veronese embedding. We prove the following result.

Theorem 1.1. *Fix integers $c \geq 0$, $m \geq 0$ and $a \geq b \geq m+3$. Let $W \subset X$ be a general union of one element of $\mathcal{Z}(m)$ and c 2-points. Then either $h^0(\mathcal{I}_W(a,b)) = 0$ or $h^1(\mathcal{I}_W(a,b)) = 0$.*

The following result may also be proved using the tools in [9, 12].

Proposition 1.2. *Fix integer $c \geq 0$, $m \geq 2$ and $a \geq b \geq m$. Let $W \subset X$ be a general union of one m-point and c 2-points. Then either $h^0(\mathcal{I}_W(a,b)) = 0$ or $h^1(\mathcal{I}_W(a,b)) = 0$, except in the case $m=2$, $b=2$, a even and $c = a/2$.*

In section 3 (again with $X = \mathbb{P}^1 \times \mathbb{P}^1$) we give several examples of our tools and tricks to compute the dimensions of several joins.

2 General tools

In this section we collect the necessary tools lifted from the literature and add some remarks which greatly improve their use to compute the dimensions of joins.

For all $p \in X$ and all integers $m > 0$ let mp denote the closed subscheme of X with $(\mathcal{I}_p)^m$ as its ideal sheaf. For any $Y \subseteq X$ and any $p \in Y_{\mathrm{reg}} \cap X_{\mathrm{reg}}$ set $(mp, Y) := mp \cap Y$. Since p is a smooth point of both X and Y, (mp, Y) is the closed subscheme of Y whose ideal sheaf is $(\mathcal{I}_{p,Y})^m \subset \mathcal{O}_Y$.

Let X be an integral projective variety, $\mathcal{L}$ a line bundle on X and $V \subseteq H^0(\mathcal{L})$ a linear subspace. Let $Z \subseteq W \subset X$ be a zero-dimensional scheme. Obviously if $V \cap H^0(\mathcal{I}_Z \otimes \mathcal{L}) = 0$, then $V \cap H^0(\mathcal{I}_W \otimes \mathcal{L}) = 0$. Since W is zero-dimensional, the restriction map $H^0(\mathcal{O}_W \otimes \mathcal{L}) \longrightarrow H^0(\mathcal{O}_Z \otimes \mathcal{L})$ is surjective. Thus $h^1(\mathcal{I}_Z \otimes \mathcal{L}) \le h^1(\mathcal{I}_W \otimes \mathcal{L})$.

Remark 2.1. *Let X be an integral projective variety. Set $n := \dim(X)$. The schemes $Z \in \mathcal{Z}(X,m)$, $m \ge 0$, used to detect the tangent space $T_Q J(X,n)$ at a general $Q \in J(X,m)$ are all schemes obtained in the following way. Set $\mathcal{Z}(X,0) := \{2p\}_{p \in X_{\mathrm{reg}}}$. Now assume $m > 0$. Set $\mathcal{Z}(X,m) := \cup_{p \in X_{\mathrm{reg}}} \mathcal{Z}(X,p,m)$, where each $\mathcal{Z}(X,p,m)$ is defined in the following way. Fix $p \in X_{\mathrm{reg}}$. Any zero-dimensional scheme $Z \in \mathcal{Z}(X,p,m)$ will have $Z = \{p\}$ and hence to define each element Z of $\mathcal{Z}(X,p,m)$ it is sufficient to define the ideal $\mathcal{J}$ of the local ring $\mathcal{O}_{X,p}$ such that $\mathcal{O}_Z = \mathcal{O}_{X,p}/\mathcal{J}$. Let μ be the maximal ideal of $\mathcal{O}_{X,p}$. The ideal $\mathcal{J}$ is constructed taking a germ at p of a smooth curve contained in a neighborhood of p in X and containing p, i.e. taking a regular system of parameters $t_1, \dots, t_n$ of the local ring $\mathcal{O}_{X,p}$, i.e. any system of n generators $t_1, \dots, t_n$ of the maximal ideal μ of $\mathcal{O}_{X,p}$ and taking any germ of curve with $(t_2, \dots, t_n)$ as its ideal in $\mathcal{O}_{X,p}$. As ideal of Z we take $\mu^{m+2} + t_1^{m+1}\mu$. With obvious conventions (i.e. taking as L_Z the germ of X at p) this ideal gives the ideal μ^{m+2} if $n = 1$, i.e. for $n = 1$ it gives the correct answer $\mathcal{J} = \mu^{m+2}$. The scheme Z is uniquely determined by the choice of a one-dimensional linear subspace of the n-dimensional vector space μ/μ^2, i.e. by the choice of a non-zero element of μ/μ^2. We will say that Z depends on the choice of a tangent vector L_Z of X at p. Each $Z \in \mathcal{Z}(X,m)$ has as its reduction a unique $p \in X_{\mathrm{reg}}$. We have*

$$(m+1)p \subset Z \subset (m+2)p, \quad \deg(Z) = n + \deg((m+1)p) = n + \binom{m+n}{n}.$$

We say that Z is defined by p and the tangent vector L_Z, because L_Z is uniquely determined by a connected degree 2 scheme $E \subset X$ such that $E = \{p\}$.

We often write $\mathcal{Z}(m)$ (resp. $\mathcal{Z}(p,m)$) instead of $\mathcal{Z}(X,m)$ (resp. $\mathcal{Z}(X,p,m)$).

Remark 2.2. *Let X be an integral projective variety and D an effective Cartier divisor of X. For any zero-dimensional scheme $Z \subset X$ the residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed zero-dimensional subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal scheme. We have $\mathrm{Res}_D(Z) \subseteq Z$,*

$\deg(Z) = \deg(\mathrm{Res}_D(Z)) + \deg(Z \cap D)$ *and for every line bundle* $\mathcal{L}$ *on* X *there is an exact sequence*

$$0 \longrightarrow \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D) \longrightarrow \mathcal{I}_Z \otimes \mathcal{L} \longrightarrow \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D} \longrightarrow 0 \tag{2.1}$$

For any Z *and* $\mathcal{L}$ *we will say that* (2.1) *is the residual exact sequence of* D. *Fix* $Z \in \mathcal{Z}(X,m)$, $m > 0$, *and set* $\{p\} := Z$. *Let* L_Z *be the tangent vector of* X *at* p *defining* Z *and call* $t_1,\dots,t_n$ *a regular system of generators of the maximal ideal of* μ *of* $\mathcal{O}_{X,p}$ *such that* L_V *is defined by* $t_2 = \cdots = t_n = 0$, $t_1^2 = 0$. *Now assume* $p \in D_{\mathrm{reg}}$.

- *(a) Assume that* L_Z *is not contained in the tangent space of* D *at* p. *Then* $D \cap Z = ((m+1)p, D)$ *and hence* $\deg(\mathrm{Res}_D(Z)) = n + \binom{m+n}{n} - \binom{m+n-1}{n-1} = n + \binom{m+n-1}{n}$. *Moreover,* $\mathrm{Res}_D(Z) \in \mathcal{Z}(X, m-1)$ *and if* $m \geq 2$ *the scheme* $\mathrm{Res}_D(Z)$ *is defined by the same tangent vector* L_Z.
- *(b) Assume that* L_Z *is contained in the tangent space of* D *at* p. *Then* $D \cap Z \in \mathcal{Z}(D,m)$ *and hence* $\deg(\mathrm{Res}_D(Z)) = n + \binom{m+n}{n} - n + 1 - \binom{m+n-1}{n-1} = 1 + \binom{m+n-1}{n}$. *We have* $\mathrm{Res}_D(Z) \supset mp$ *and* $\deg(\mathrm{Res}_D(Z)) = \deg(mp) + 1$. *The scheme* $\mathrm{Res}_D(Z)$ *is the union of* mp *and the scheme* $t_1^{m+2} = t_2 = \cdots = t_n = 0$.

In both cases the scheme Z *is vertically graded with respect to* D *in the sense of [2] and hence we may apply the Differential Horace Horace Lemma to* Z *([2]).*

For any line bundle $\mathcal{L}$ on X, any closed subscheme B of X and any vector space $V \subseteq H^0(\mathcal{L})$ set $V(-B) := V \cap H^0(\mathcal{I}_B \otimes \mathcal{L})$.

We describe the case of 2-points of the Differential Horace Lemma ([2]). The reader will find in that paper explicitly the case of points with higher multiplicities and the case (vertically graded subschemes) sufficient to handle all $Z \in \mathcal{Z}(m)$. Let X be an integral projective n-dimensional variety, D an effective Cartier divisor of X, $\mathcal{L}$ a line bundle on X, $V \subseteq H^0(\mathcal{L})$ a linear subspace. Let V_D be the image of V by the restriction map $\rho : H^0(\mathcal{L}) \longrightarrow H^0(D, \mathcal{L}_{|D})$. Set $n := \dim X$. Let $V(-D)$ be the set of all $f \in H^0(\mathcal{L}(-D))$ such that $zf \in V$, where $z \in H^0(\mathcal{O}_X(D))$ is the equation of D. Take a general $p \in X_{\mathrm{reg}} \cap D_{\mathrm{reg}}$. To prove that $\dim V(-Z-2q) = \max\{0, \dim V(-Z) - n - 1\}$ for a general $q \in X_{\mathrm{reg}}$ it is sufficient to prove that one of the following sets of conditions is satisfied:

(a) $\dim V_D(-Z \cap D) \leq 1$ and

$$\dim V(-D)(-\mathrm{Res}_D(Z) - (2p, D)) = \max\{0, \dim W(-\mathrm{Res}_D(Z)) - n\};$$

(b) $\dim V_D(-Z \cap D) > 0$ and

$$\dim V(-D)(-\mathrm{Res}_D(Z) - (2p, D)) = \dim V(-D)(-\mathrm{Res}_D(Z)) - n.$$

Remark 2.3. *Take any projective variety X, any line bundle $\mathcal{L}$ on X and any vector space $V \subseteq H^0(\mathcal{L})$. Fix $(u,v) \in \mathbb{N}^2$. Let $B \subset X$ be a general union of u tangent vectors of X_{reg} and v points of X. By [14] we have* $\dim V(-B) = \max\{0, \dim V - 2u - v\}$.

Remark 2.3 is useful because it applies to non-complete linear systems, too. We will use this key feature in the proof of the next lemma.

Lemma 2.4. *Take a projective variety X, a line bundle $\mathcal{L}$ on X and an integral Cartier divisor $D \subset X$. Assume $h^1(\mathcal{L}) = h^1(\mathcal{L}(-D)) = h^1(D, \mathcal{L}_{|D}) = 0$. Fix $(u,v) \in \mathbb{N}^2$. Let $Z \subset X$ be a zero-dimensional scheme. Let $B \subset D$ be a general union of u tangent vectors of D_{reg} and v points of X. Assume $h^1(\mathcal{I}_Z \otimes \mathcal{L}) = 0$, $h^1(D, \mathcal{I}_{D\cap Z,D} \otimes \mathcal{L}_{|D}) = 0$ and $h^0(D, \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D}) \geq 2u + v$. Then $h^0(\mathcal{I}_{Z\cup B} \otimes \mathcal{L}) = \max\{0, h^0(\mathcal{I}_Z \otimes \mathcal{L}) - 2u - v\}$.*

Proof. Remark 2.3 applied to D, $\mathcal{L}_{|D}$, $H^0X, (\mathcal{I}_Z \otimes \mathcal{L})$ and $H^0(D, \mathcal{I}_{Z\cap D} \otimes \mathcal{L}_{|D})$ gives

$$h^0(D, \mathcal{I}_{(Z\cap D)\cup B,D} \otimes \mathcal{L}_{|D}) = h^0(D, \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D}) - 2u - v.$$

Use twice the residual exact sequence of D, first with $\mathcal{I}_Z \otimes \mathcal{L}$ in the middle and then with $\mathcal{I}_{Z\cup B} \otimes \mathcal{L}$ in the middle. Use that $\mathrm{Res}_D(Z \cup B) = \mathrm{Res}_D(Z)$, because $B \subset D$ (as schemes). □

If we take the set-up and assumptions of Lemma 2.4 except the inequality on $h^0(D, \mathcal{I}_{Z\cup D,D} \otimes \mathcal{L}_{|D})$ and we have $h^0(D, \mathcal{I}_{Z\cup D,D} \otimes \mathcal{L}_{|D}) \leq 2u + v$, then Remark 2.3 gives $h^0(D, \mathcal{I}_{(Z\cap D)\cup B,D} \otimes \mathcal{L}_{|D}) = 0$. Thus the residual exact sequence of D gives $h^0(\mathcal{I}_{Z\cup B} \otimes \mathcal{L}) = h^0(\mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D))$.

Remark 2.5. *Fix a line bundle $\mathcal{L}$ on an integral projective variety X. Let $Z_1 \subsetneq Z_2$ be zero-dimensional schemes. Note that $h^1(\mathcal{I}_{Z_1} \otimes \mathcal{L}) \leq h^1(\mathcal{I}_{Z_2} \otimes \mathcal{L})$. If $h^0(\mathcal{I}_{Z_2} \otimes \mathcal{L}) = h^0(\mathcal{L}) - \deg(Z_2)$, then $h^0(\mathcal{I}_{Z_1} \otimes \mathcal{L}) = h^0(\mathcal{L}) - \deg(Z_1)$. Set $n := \dim X$. Let $\mathcal{U}$ (resp. $\mathcal{V}$) be the set of all triples $(e,f,g) \in \mathbb{N}^3$ such that $h^0(\mathcal{L}) \leq e(n+\binom{n+2}{n}) + f(2n+1) + g(n+1)$ (resp. $h^0(\mathcal{L}) \geq e(n+\binom{n+2}{n}) + f(2n+1) + g(n+1)$). Fix $(e,f,g) \in \mathcal{U}$. Let $Z \subset X$ be a general union of e elements of $\mathcal{Z}(2)$, f elements of $\mathcal{Z}(1)$ and g 2-points. Suppose you want to prove that $h^0(\mathcal{I}_Z \otimes \mathcal{L}) = h^0(\mathcal{L}) - e(n+\binom{n+2}{n}) - f(2n+1) - g(n+1)$. It is sufficient to show that $h^0(\mathcal{I}_Z \otimes \mathcal{L}) = h^0(\mathcal{L}) - e(n+\binom{n+2}{n}) - f(2n+1) - g'(n+1)$ for some integer $g' \geq g$, where Z' is the union of Z and $g' - g$ general 2-points. Thus to check for all $(e,f,g) \in \mathcal{U}$ that a general union of e elements of $\mathcal{Z}(2)$, f element of $\mathcal{Z}(1)$ and g 2-points imposes independent conditions to $h^0(\mathcal{L})$ it is sufficient to check all $(e,f,g) \in \mathbb{N}^3$ such that*

$$h^0(\mathcal{L}) - n \leq e\left(n + \binom{n+2}{n}\right) + f(2n+1) + g(n+1) \tag{2.2}$$

Suppose you want to prove that $h^0(\mathcal{I}_W \otimes \mathcal{L}) = 0$ for all $(u,v,w) \in \mathcal{V}$, where W is a general union of u elements of $\mathcal{Z}(2)$, v elements of $\mathcal{Z}(1)$ and w 2-points. Decreasing if necessary the zero-dimensional scheme, it is sufficient to check all $(u,v,w) \in \mathbb{N}^3$ satisfying one of the following sets *of conditions:*

$$h^0(\mathcal{L}) \le e\left(n + \binom{n+2}{n}\right) + f(2n+1) + g(n+1) \le h^0(\mathcal{L}) + n \tag{2.3}$$

$$g = 0,\ h^0(\mathcal{L}) \le e\left(n + \binom{n+2}{n}\right) + f(2n+1) \le h^0(\mathcal{L}) + 2n \tag{2.4}$$

$$f = g = 0,\ h^0(\mathcal{L}) \le e\left(n + \binom{n+2}{n}\right) + f(2n+1) \le h^0(\mathcal{L}) + n - 1 + \binom{n+2}{n} \tag{2.5}$$

With minimal modifications the interested reader may state similar statements for general unions of prescribed numbers of osculating spaces and multiple points with arbitrary multiplicities and for m-points instead of just 2*-points (see Remark 3.2).*

Fix a linear subspace $V \subseteq H^0(\mathcal{L})$*. Suppose* $\dim V = 1$*. Thus* $V(-p) = 0$ *for a general* $p \in X$*. Hence* $V(-2p) = 0$ *for a general* $p \in X_{\text{reg}}$*. Suppose* $\dim V = 2$*. By [14]* $V(-A) = 0$ *for a general tangent vector* A *of* X_{reg}*. Thus* $V(-2p) = 0$ *for a general* $p \in X_{\text{reg}}$*. Thus in* (2.3) *it is not necessary to check all cases with* $e > 0$ *and* $e(n + \binom{n+2}{n}) + f(2n+1) + g(n+1) \in \{h^0(\mathcal{L}) + n - 1, h^0(\mathcal{L}) + n\}$ *(Remark 3.2).*

Remark 2.6. *Let* X *be an integral projective variety,* L *a line bundle on* X *and* $V \subseteq H^0(L)$*. Set* $n := \dim X$*. Fix a general* $p \in X_{\text{reg}}$*. The function* $f : \mathbb{N} \longrightarrow \mathbb{N}$ *defined by the formula* $f(m) := \dim V(-mp)$ *is non-increasing. Since we take* p *general in* X_{reg}*, the semicontinuity theorem for cohomology shows that this function does not depend upon the choice of the general* p*. Consider its first difference* $g : \mathbb{N} \longrightarrow \mathbb{N}$*, i.e. set* $g(0) := f(0) = \dim V$ *and* $g(m) = f(m-1) - f(m)$ *for all* $m > 0$*.*

Observation 1: *If* $f(m) \neq 0$*, then* $g(m+1) > 0$*, i.e.* $f(m+1) < f(m)$*, unless* $f(m) = 0$ *([13, Proposition 2.3]).*

Now we fix an arbitrary $o \in X_{\text{reg}}$*, set* $R := \mathcal{O}_{X,o}$ *and call* μ *the maximal ideal of the local ring* R*. Thus* $R/\mu \cong \mathbb{K}$ *and, since* X *is smooth at* o*, the graded ring* $GR_o := \bigoplus_{t \ge 0} \mu^t/\mu^{t+1}$ *(with the convention* $\mu^0 = R$*) is isomorphic to a polynomial ring in* n *variables over* $\mathbb{K}$*. Taking a regular system of parameters* $t_1, \dots, t_n$*, we may see each* μ^m/μ^{m+1} *as the* $\mathbb{K}$*-vector space of all degree* m *homogeneous polynomials in the variables* $t_1, \dots, t_n$*. Thus* $\dim_{\mathbb{K}} \mu^m/\mu^{m+1} = \binom{n+m-1}{n-1}$*. Set* $f_o(m) := \dim V(-mo)$ *and* $g_o(m) := f_o(m+1) - f_o(m)$*. There is an evaluation map* $e_{o,m} : V(-m)/V(-(m+1)o) \longrightarrow \mu^m/\mu^{m+1}$ *and* $g_o(m)$ *is the rank of the evaluation map* $e_{o,m}$*. For a general* o *we write* e_m *instead of* $e_{o,m}$*. For any integer* v *such that* $0 \le v \le \binom{n+m-1}{n-1}$ *let* $G(v, \mu^m/\mu^{m+1})$ *denote the Grassmannian of all* v*-dimensional linear subspaces of* μ^m/μ^{m+1}*. Call* $\pi : \mu^m \longrightarrow \mu^{m+1}$ *the quotient map. Fix* v *and* $W \in G(v, \mu^m/\mu^{m+1})$*. Set* $I_W := \pi^{-1}(o) \subset R$ *and* $Z_W := \operatorname{Spec}(R/I_W)$*. Note that* Z_W *is a connected degree* 0 *subscheme of* X *of degree* $\binom{n+m}{n} + \dim W$*. The integer* $\dim V(-Z_W)$ *is the number of conditions that* Z_W *imposes to* V*.*

*We have $\mu^{k+1} \subseteq I_Z \subset \mu^{k+1}$. The integer $\dim V(-Z_W)$ depends on the integers $g_o(m)$ and $\dim W$ and on the position of W with respect to the linear subspace $\mathrm{Im}(e_{m,o})$. Concerning the integer $\dim V(-Z_W)$ we only know the trivial inequalities coming from the Grassmann formula. Since the Grassmannian is an irreducible variety, it makes sense to speak about the general element of $G(v, \mu^m/\mu^{m+1})$. For such a general W we have $\dim V(-Z_W) = \dim V(-mo) - \min\{v, g_o(m)\}$. For a general o we have $g_m(o) > 0$. In this case any W of positive dimension imposes at least one condition to $V(-mo)$. The m-**spread** $\mathrm{sp}_m(X,V)$ of (X,V) at its general point is the minimal integer x such that $0 \le x \le n$ and there is an x-dimensional linear subspace $E \subseteq \mathbb{K}[t_1,\dots,t_n]$ such that $\mathrm{Im}(e_m) \subseteq S^m(E)$. Obviously $\mathrm{sp}_m(X,V) \le \min\{n, g(m)\}$. When $g(m) < n$ the pair (X,V) has a very particular behaviour ([5, Proposition 1]). We do not know (we lack an integrability condition) if something similar is true just assuming $\mathrm{sp}_m(X,V) < n$.*

3 Examples

In this section we take $X := \mathbb{P}^1 \times \mathbb{P}^1$. As a warming up for the next section we give 2 cases (Propositions 3.4 and 3.5) in which the congruence classes for some of the integer a, b of the line bundle $\mathcal{O}_X(a,b)$ greatly help and then a case (Proposition 3.6) which shows how to use the lucky cases to prove more general ones. We also show how to handle some zero-dimensional schemes with a very particular shape (Lemmas 3.7 and 3.8 and Proposition 3.9).

Remark 3.1. *Fix integers $a \ge b > 0$. Let $W \subset X$ be a general union of c 2-points. Then either $h^0(\mathcal{I}_W(a,b)) = 0$ or $h^1(\mathcal{I}_W(a,b)) = 0$, except in the case $b = 2$, a even and $c = a/2+1$ ([10, 16, 17]). In the exceptional case $h^0(\mathcal{I}_W(a,b)) = h^1(\mathcal{I}_W(a,b)) = 1$ and $|\mathcal{I}_W(a,2)| = \{2C\}$ where $C \cong \mathbb{P}^1$ and $\{C\} = |\mathcal{I}_W(a/2,1)|$.*

The following observation simplifies many proofs and it is essential to do by computer in a cheap way some small degrees cases to be used for inductive proofs for other joins.

Remark 3.2. *Fix positive integers a, b and w and a zero-dimensional scheme $W \subset X$ such that $\deg(W) = w$ and $h^1(\mathcal{I}_W(a,b)) > 0$. To prove that for all integers $c \in \mathbb{N}$ a general union Z of W and c 2-points satisfies either $h^1(\mathcal{I}_Z(a,b)) = 0$ or $h^0(\mathcal{I}_W(a,b)) = 0$ it is sufficient to check the integers $c \in \{\lfloor((a+1)(b+1)-w)/3\rfloor, \lceil((a+1)(b+1)-w)/3\rceil\}$. Hence it is sufficient to check all $c \in \mathbb{N}$ such that*

$$(a+1)(b+1) - 2 \le w + 3c \le (a+1)(b+1) + 2 \tag{3.1}$$

We can do better. Indeed, any 2-point at a general $p \in X$ contains a general connected degree 2 zero-dimensional scheme v. Thus for any $V \subseteq H^0(\mathcal{O}_X(a,b))$ we have $\dim V(-2p) \le \min\{0, \dim V - 2\}$. Thus it is sufficient to check all integers c such that

$$(a+1)(b+1) - 1 \le w + 3c \le (a+1)(b+1) + 1 \tag{3.2}$$

Now assume $w+3c=(a+1)(a+1)+1$ and that we know that $h^1(\mathcal{I}_{W\cup E}(a,b))=0$ for all unions of c' general 2-points with c' satisfying $w+3c'\le(a+1)(b+1)$. Let $E\subset X$ be a general union of $c-1$ 2-points. Thus $h^0(\mathcal{I}_{W\cup E}(a,b))=2$. Thus a general union E' of E and a general 2-point satisfies $h^0(\mathcal{I}_{W\cup E'}(a,b))=0$. Thus it is sufficient to check all integers c such that

$$(a+1)(b+1)-1\le w+3c\le(a+1)(b+1) \tag{3.3}$$

Thus it is sufficient to check the integer $c:=\lfloor((a+1)(b+1)-w)/3\rfloor$.

Lemma 3.3. *Fix $(a,b)\in\mathbb{N}^2$, $L\in|\mathcal{O}_X(1,0)|$ and a zero-dimensional scheme $W\subset X$ such that $h^1(\mathcal{I}_W(a,b))=0$. Set $u:=\deg(W\cap L)$. Let $E\subset L$ be a zero-dimensional scheme such that $E\cap W=\emptyset$ and set $x:=\deg(E)$. Assume $h^1(\mathcal{I}_{\mathrm{Res}_L(W)}(a-1,b))\le b+1-u$. Then $h^1(\mathcal{I}_{W\cup E}(a,b))=\max\{0,h^0(\mathcal{I}_W(a,b))-x\}$.*

Proof. First assume $x=b+1-u$. Thus $h^i(L,\mathcal{I}_{L\cap(W\cup E)}(a,b))=0$, $i=0,1$. Hence $h^i(\mathcal{I}_{W\cup E}(a,b))=h^i(\mathcal{I}_{\mathrm{Res}_L(W)}(a-1,b))$, $i=0,1$.

If $x<b+1-u$, then we reduce the proof to the case just proved taking instead of E the union of E and $b+1-u-x$ points.

Now assume $x>b+1$. Instead of E we use any subscheme $E'\subset E$ such that $\deg(E')=b+1-u$. □

Proposition 3.4. *Fix positive integers a and b such that a is odd and $b\equiv 4 \pmod 5$. Then for all $c>0$ the join of c copies of $J(1)$ has the expected dimension $\min\{(a+1)(b+1)-1,5c-1\}$.*

Proof. Let $Z\subset X$ be a general union of c elements of $\mathcal{Z}(1)$. It is sufficient to do the case $c=(a+1)(b+1)/5$ and prove that $h^i(\mathcal{I}_Z(a,b))=0$, $i=0,1$. We fix $L\in|\mathcal{O}_X(1,0)|$. Let $Z'\subset X$ be a general union of $(a-1)(b+1)/5$ elements of $\mathcal{Z}(1)$ with the convention $Z'=\emptyset$ if $a=1$. Take a general $A\cup B\subset L$ such that $\#A=\#B=(b+1)/5$ and $A\cap B=\emptyset$. Let W be the union of Z' and the scheme Z'' obtained in the following way. We degenerate $(b+1)/5$ connected components of Z to elements of $Z(p,1)$, $p\in A$, with respect to a tangent vector not tangent to L and $(b+1)/5$ connected components of Z to elements of $Z(p,1)$, $p\in B$, with respect to the tangent vector of L. Remark 2.2 gives $\deg(Z''\cap L)=b+1$ and $\deg(\mathrm{Res}_L(Z)\cap L)=b+1$. Thus using twice the residual exact sequence of L we get that it is sufficient to prove that $h^i(\mathcal{I}_{Z'}(a-1,b))=0$, $i=0,1$. This is true if $a=1$ (and hence $Z'=\emptyset$), while if $a\ge 3$ we use induction on a. □

Proposition 3.5. *Fix positive integers a and b such that $b\equiv 4 \pmod 5$ and $a\ge 3$. The join of $(b+1)/5$ copies of $J(2)$ and an arbitrary number, c, of copies of X has the expected dimension.*

Proof. Let Z_1 be a general union of c 2-points. Since each element of $J(2)$ has degree 8, it is sufficient to check the positive integers c such that $8(b+1)/5+3c\le(a+1)(b+1)$ (Remark 3.2). Fix $L\in|\mathcal{O}_X(1,0)|$. For every $p\in L$, let $E(p)$ be an element of $Z(p,2)$ with as tangent vector the one associated to L. We have $\deg(E(p)\cap L)=4$, $\deg(\mathrm{Res}_L(E(p))\cap L)=3$ and $\mathrm{Res}_L(\mathrm{Res}_L(E(p)))=\{p\}$. Set $E:=\bigcup_{p\in A}E(p)$. By semicontinuity to prove the proposition for the integer c it is sufficient to prove that $h^j(\mathcal{I}_{E\cup Z_1}(a,b))=0$.

Take a general $A\cup B\subset L$ such that $\#A=\#B=(b+1)/5$ and $A\cap B=\emptyset$. Let E be the union of all $E(p)$, $p\in A$. By semicontinuity to prove the proposition for the integer c it is sufficient to prove $h^1(\mathcal{I}_{E\cup Z_1}(a,b))=0$. Let $(2B,L)$ denote the union of all 2-points of L with a point of B as their reduction. We apply the Differential Horace Lemma for 2-points at each $p\in B$. Since $h^i(L,\mathcal{I}_{B\cup(E\cap L)}(a,b))=0$, $i=0,1$, the Differential Horace Lemma gives that to prove that $h^1(\mathcal{I}_{E\cup E'}(a,b))=0$ for a general union E' of $2(b+1)/5$ 2-points, it is sufficient to prove $h^1(\mathcal{I}_{\mathrm{Res}_L(E)\cup(2B,L)}(a-1,b))=0$. Since $\deg((\mathrm{Res}_L(E)\cup(2B,L))=b+1$, it is sufficient to prove $h^1(\mathcal{I}_A(a-2,b))=0$. Since $a\geq 2$, we proved the case $c\leq 2(b+1)/5$.

Now assume $c>2(b+1)/5$ and set $x:=c-2(b+1)/5$. Let $W\subset X$ be a general union of x 2-points. Either $h^0(\mathcal{I}_W(a-2,b))=0$ or $h^1(\mathcal{I}_W(a-2,b))=0$ or a and b are even, $x=b+1$ and $h^1(\mathcal{I}_W(a-2,b))=h^0(\mathcal{I}_W(a-2,b))=1$ (Remark 3.1).

If $h^0(\mathcal{I}_W(a-2,b))\leq 1$, then $h^0(\mathcal{I}_{E'\cup W}(a,b))=0$, concluding the proof in this case.

Now assume $h^1(\mathcal{I}_W(a-2,b))=0$ and hence $h^0(\mathcal{I}_W(a-2,b))=(a-1)(b+1)-3x$. To prove this case we need to prove that either $h^0(\mathcal{I}_{W\cup A}(a-2,b))=0$ or $h^1(\mathcal{I}_{W\cup A}(a-2,b)=0$. Since W is general, $W\cap L=\emptyset$. Since A is general in L and $\#A=(b+1)/5$, it is sufficient to prove that $h^0(\mathcal{I}_W(a-3,b))\leq\min\{0,(a-1)(b+1)-3x-(b+1)/5\}$.

If $a\geq 4$, the inequality $h^0(\mathcal{I}_W(a-3,b))\leq\min\{0,(a-1)(b+1)-3x-(b+1)/5\}$ is true, because either $h^0(\mathcal{I}_W(a-3,b))=0$ or $h^1(\mathcal{I}_W(a-3,b))=0$ or $h^0(\mathcal{I}_W(a-3,b))=h^1(\mathcal{I}_W(a-3,b))=1$ (Remark 3.1).

Now assume $a=3$. We have $h^0(\mathcal{I}_W(1,b))=2b+2-3x$ and $h^0(\mathcal{I}_W(0,b))=b+1<h^0(\mathcal{I}_W(1,b))-(b+1)/5$. □

Fix the bidegree a, b of the line bundle $\mathcal{O}_X(a,b)$. Instead of a prescribed number of copies of $J(2)$ we may use an arbitrary, but small with respect to b, number of copies of $J(2)$ as in the following statement (taking all possible $e\leq b-2$ with $e\equiv 4\pmod 5$).

Proposition 3.6. *Fix positive integers a, e and b such that $e\equiv 4\pmod 5$, $a\geq 3$ and $b\geq e+4$. The join of $(e+1)/5$ copies of $J(2)$ and an arbitrary number, c, of copies of X has the expected dimension.*

Proof. Let Z_1 be a general union of s 2-points. Since each element of $J(2)$ has degree 8, it is sufficient to check the positive integer c such that $8(e+1)/5+3c\leq(a+1)(b+1)$ (Remark 3.2).

Fix $L\in|\mathcal{O}_X(1,0)|$. For every $p\in L$ let $E(p)$ be an element of $Z(p,2)$ with as tangent vector the one associated to L. We have $\deg(E(p)\cap L)=4$, $\deg(\mathrm{Res}_L(E(p))\cap L)=3$ and $\mathrm{Res}_L(\mathrm{Res}_L(E(p)))=\{p\}$.

Set $E_1 := \bigcup_{p\in A} E(p)$. Set $e_1 := \lfloor (b-e)/2\rfloor$, $f_1 := b-e-2e_1$, $e_2 := \lfloor (b-e-e_1-2f_1)/2\rfloor$ and $f_2 := b-e-e_1-f_1-2f_1$. Note that $0 \le f_1 \le 1$ and $0 \le f_2 \le 1$. Since $b \ge e+2$, $(e_1, f_1) \ne (0,1)$ and hence $2e_1+f_1 \ge e_1+2f_1$. Since $b \ge e+4$, we have $e_1+2f_1 \ge 2$. Take a general $A\cup B\cup E\cup F \subset L$ such that $\#A = \#B = (e+1)/10$, $\#E = e_1$, $\#F = f_1$ and the sets A, B, E and F are pairwise disjoint. Let U be the union of all $E(p)$, $p \in A$. By semicontinuity to prove the proposition for the integer c it is sufficient to prove $h^j(\mathcal{I}_{U\cup Z_1}(a,b)) = 0$.

Let $(2B, L)$ (resp. $(2F, L)$, resp. $(2E, L)$) denote the union of all 2-points of L with a point of B (resp. F) as their reduction. We apply the Differential Horace Lemma for 2-points at each $p \in B\cup F$, while add all 2-points $2p$ of X with $p \in E$. Since $h^i(L, \mathcal{I}_{B\cup(U\cap L)\cup(2E,L)\cup F}(a,b)) = 0$, $i = 0,1$, the Differential Horace Lemma gives that to prove that $h^1(\mathcal{I}_{U\cup U'}(a,b)) = 0$ for a general union U' of $(e+1)/10+e_1+f_1$ 2-points, it is sufficient to prove $h^1(\mathcal{I}_{\mathrm{Res}_L(E_1)\cup(2B,L)\cup(E,L)\cup(2F,L)}(a-1,b)) = 0$. We have $\deg(\mathrm{Res}_L(E)\cap L) + \deg((2B,L)) + \deg((2F,L)) \le b+1$. Thus the intersection τ of $\mathrm{Res}_L(E)\cup(2B,L)\cup(E,L)\cup(2F,L)$ with L satisfies $h^1(L, \mathcal{I}_\tau(a-1,b)) = 0$, while its residue is A.

If $c \le (e+1)/5+e_1+f_1$, then we get that the join has the expected dimension $e+1+3c-1$.

Assume for the moment $c \ge (e+1)/5+e_1+f_1+e_2+f_2$. Fix a general $G\cup H \subset L$ such that $\#G = e_2$ and $\#H = f_2 \le 1$. We apply the Differential Horace Lemma to H (if $f_2 = 1$) and specialize e_2 2-points to the 2-points $2p$, $p \in G$. Let Z_2 be a general union of $c-(e_1+1)/5-e_1-f_1-e_2-f_2$. To prove that $h^1(\mathcal{I}_{U\cup Z_1}(a,b)) = 0$ it is sufficient to prove that $h^j(\mathcal{I}_{Z_2\cup A\cup F\cup(2H,L)}(a-2,b)) = 0$. This is done as in the proof of Proposition 3.5, even if $H \ne \emptyset$, by Remark 2.3 applied to L or by Lemma 3.3.

If $(e+1)/5+e_1+f_1 < c < (e+1)/5+e_1+f_1+e_2+f_2$ (and hence $c \le (e+1)/5+e_1+f_1+e_2$) instead of G and H we take G' with $\#G' = c-(e+1)/5-e_1-f_1-e_2$ and $H' = \emptyset$. □

We explain why a general union of 2 m-points (plus other objects) are easy to handle.

Lemma 3.7. *Let $Z \subset X$ be a general union of 2 m-points, $m \ge 2$. Then $h^i(\mathcal{I}_Z(m,m-1)) = 0$, $i = 0,1$.*

Proof. Fix $L \in |\mathcal{O}_X(0,1)|$ and $o, o' \in L$, $o \ne o'$. We take mo and apply the Differential Horace Lemma with respect to L and o'. Thus on L we add $\{o\}$, at the first residual with respect to L intersected with L we add $(2o, L)$ and so on. Thus the intersection with L of the union W of mo with this virtual scheme has degree $m+1$ and the same holds for the intersection of L with the first m residual with respect to L. Thus we get the lemma taking several residual exact sequences of L. □

In the same way we get the following result.

Lemma 3.8. *Let $Z \subset X$ be a general union of one m-point, $m > 0$, and one $(m+1)$-point. Then $h^i(\mathcal{I}_Z(m,m)) = 0$, $i = 0,1$.*

Proposition 3.9. *Fix integer $m \geq 3$, $c \geq 0$, and $a \geq b \geq m+3$. Let $Z \subset X$ be a general union of 2 m-points and c 2-points. Then either $h^0(\mathcal{I}_Z(a,b)) = 0$ or $h^1(\mathcal{I}_Z(a,b)) = 0$.*

Proof. It is sufficient to check the positive integers c such that $(m+1)m + 3c \leq (a+1)(b+1)$ (Remark 3.2). Fix $L \in |\mathcal{O}_X(0,1)|$ and $o, o' \in L$, $o \neq o'$. We take mo and apply the Differential Horace Lemma with respect to L and o'. Thus on L we add $\{o\}$, at the first residual with respect to L intersected with L we add $(2o, L)$ and so on. Thus the intersection with L of the union W of mo with this virtual scheme has degree $m+1$ and the same holds for the intersection of L with the first m residual with respect to L. We call W this virtual degeneration of a general union of 2 m-points. Recall (Lemma 3.7) that $h^i(\mathcal{I}_W(m, m-1)) = 0$, $i = 0, 1$. We set $W_0 := W$ and for each $i \geq 1$ define recursively the virtual scheme W_i by the formula $W_i := \mathrm{Res}_L(W_{i-1})$. Thus $W_j = \emptyset$ for all $j \geq m$ and $\deg(W_i \cap L) = m+1$ for all $i < m$. The proof of Lemma 3.7 gives $h^i(\mathcal{I}_{W_j}(m, m-1-j)) = 0$, $0 \leq j \leq m$.

Set $e := \lfloor (a-m)/2 \rfloor$ and $f := a - m - 2e$. Fix a general $A \cup B \subset L$ such that $\#A = e$, $\#B = f$ and $A \cap B = \emptyset$.

We call H_i, $0 \leq i \leq m$, the assertion that a general union of W_i and an arbitrary number of 2-points has the expected postulation with respect to $\mathcal{O}_X(a, b-i)$. The case $i = m$ is true by Remark 3.1. Since H_0 proves the proposition for c, we prove all H_i by descending induction on i, Thus (changing b and m) we may assume H_1.

Assume for the moment $c \geq e + f$. Let $E \subset X$ be a general union of $c - e - f$ 2-points. We take as e of the 2-points the 2-points $2p$, $p \in A$. If $f \neq 0$ we apply the Differential Horace Lemma to F. Since $2e + f + \deg(W \cap L) = a + 1$, the Differential Horace Lemma shows that to show that to prove the proposition it is sufficient to prove that $h^j(\mathcal{I}_{W_1 \cup E \cup A \cup (2F,L)}(a, b-1)) = 0$.

Claim 1: $h^1(\mathcal{I}_{W_1 \cup E}(a, b-1)) = 0$.

Proof of Claim 1: By the inductive assumption either $h^1(\mathcal{I}_{W_1 \cup E}(a, b-1)) = 0$ or $h^0(\mathcal{I}_{W_1 \cup E}(a, b-1)) = 0$. Since $\deg(W_1 \cup E) - e - 2f = \deg(W) + 3c \leq (a+1)(b+1)$, $h^1(\mathcal{I}_{W_1 \cup E}(a, b-1)) = 0$.

Claim 1 gives $h^0(\mathcal{I}_{W_1 \cup E}(a, b-1)) = (a+1)b - m(m+1) - 3(c-e-f)$. Claim 1 and Remark 2.3 applied to L or Lemma 3.3 show that to prove H_0 it is sufficient to prove that $h^0(\mathcal{I}_{W_1 \cup E}(a, b-2)) \leq \max\{0, (a+1)b - m(m+1) - 3(c-e-f) - e - 2f\}$. Since $2e + f = a - m$ and $f \leq 1$, $3e + 3f + 2 \leq 2(a+1)$. Thus $h^0(\mathcal{I}_{W_2 \cup E}(a, b-2)) = 0$.

In the case $c < e + f$ (and hence $c \leq e$) instead of A and B we take $B = \emptyset$ and A with $\#A = c$. $\square$

4 The proofs

In this section we take $X = \mathbb{P}^1 \times \mathbb{P}^1$. Since $\dim X = 2$, for each $m \geq 0$, any $p \in X$ and any $Z \in \mathcal{Z}(m)$ we have $\deg((m+1)p) = \binom{m+2}{2}$ and $\deg(Z) = 2 + \binom{m+2}{2}$.

Remark 4.1. *Fix integers $a \geq b \geq 0$ and $z \geq 2$. Fix homogeneous coordinates x_0, x_1 and y_0, y_1 of $\mathbb{P}^1$. The vector space $H^0(\mathcal{O}_X(a,b))$ is formed by all $f \in \mathbb{K}[x_0, x_1, y_0, y_1]$ which are bihomogenous of bidegree (a,b), i.e. homogenous of degree a with respect to x_0, x_1 and homogeneous of degree b with respect to y_0, y_1. Thus $H^0(\mathcal{O}_X(a,b))$ has as a basis all monomials $x_0^{\alpha_0}x_1^{\alpha_1}y_0^{\beta_0}y_1^{\beta_1}$ such that $(\alpha_0, \alpha_1, \beta_0, \beta_1) \in \mathbb{N}^4$, $\alpha_0 + \alpha_1 = a$ and $\beta_0 + \beta_1 = b$. Fix $p \in X$ and choose bihomogeneous coordinates x_0, x_1, y_0, y_1 such that $p = ((1:0),(1:0))$. Set $x := x_1/x_0$ and $y := y_1/y_0$. The vector space $H^0(\mathcal{I}_{zp}(a,b))$ is isomorphic to the subspace of the polynomial ring $\mathbb{K}[x,y]$ with as a bases all monomials $x^u y^v$ with $u + v \geq z$, $0 \leq u \leq a$ and $0 \leq v \leq b$. Since $\deg(\mathcal{O}_{zp}) = \binom{z+1}{2}$ and $a \geq b$, $h^1(\mathcal{I}_{zp}(a,b)) = 0$ if and only if $b \geq z-1$. If $a > b = z-2$, then $h^1(\mathcal{I}_{zp}(a, z-2)) = 1$.*

Proposition 4.2. *Fix positive integers $a \geq b$ and m.*

(1) *If $b < m$, then $h^1(\mathcal{I}_Z(a,b)) > 0$ for all $Z \in \mathcal{Z}(m)$.*

(2) *If $b > m$, then $h^1(\mathcal{I}_Z(a,b)) = 0$ for all $Z \in \mathcal{Z}(m)$.*

(3) *If $a > m$, then $h^1(\mathcal{I}_Z(a,m)) = 0$ for a general $Z \in \mathcal{Z}(m)$.*

(4) *There is $Z \in \mathcal{Z}(m)$ such that $h^1(\mathcal{I}_Z(a,m)) > 0$.*

Proof. Fix $p \in X$. We consider $Z \in \mathcal{Z}(p,m)$. Thus $(m+1)p \subset Z \subset (m+2)p$. Parts (1) and (2) follow from Remark 4.1.

Let L denote the only element of $|\mathcal{O}_X(1,0)|$ passing through p.

(a) Now we prove part (4). Use L as L_Z to define the scheme Z. Note that $L \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_L(a,m)) = m$. Thus $h^0(L, \mathcal{O}_L(a,m)) = m+1$. By part (b) of Remark 2.2 we have $\deg(Z \cap L) = m+2$ and hence $h^1(L, \mathcal{I}_{Z\cap L,L}(a,m)) = 1$. Thus $h^1(\mathcal{I}_Z(a,m)) > 0$.

(b) Now we prove part (3). Thus $a > m$. By the semicontinuity theorem for cohomology it is sufficient to find one $Z \in \mathcal{Z}(p,m)$ such that $h^1(\mathcal{I}_Z(a,m)) = 0$. Take $Z \in \mathcal{Z}(p,m)$ whose tangent vector is not contained in L. We have $\deg(Z \cap L) = m+1$ and $\mathrm{Res}_L(Z) \in \mathcal{Z}(m-1)$ (even if $m = 1$) by Remark 2.2. Use the residual exact sequence of L and that $h^1(\mathcal{I}_{\mathrm{Res}_L(Z)}(a-1,m)) = 0$, because $a - 1 \geq m$. □

Proof of Proposition 1.2: The case $m = 2$, *i.e.* the case of $c+1$ general 2-points is described in Remark 3.1. Assume $m > 2$ and that the proposition is true for smaller multiplicities. In step (b) we check the case $m = 3$ and see that the exceptional case $b = 2$ and a even of the case $m = 2$ gives no problem for the inductive proof.

(a) Fix $L \in |\mathcal{O}_X(0,1)|$, $p \in L$, and take mp. For any $x \in \mathcal{Z}$ we have $h^0(\mathcal{O}_L(a,x)) = a+1$. Note that $\deg(mp \cap L) = m$ and $\mathrm{Res}_L(mp) = (m-1)p$. Since $\deg(mp) = (m+1)m/2$, it is sufficient we may assume $3c \le (a+1)(b+1) - (m+1)m/2$. Thus we need to prove that $h^1(\mathcal{I}_{mp\cup G}(a,b)) = 0$ for a general union of c 2-points. Set $e := \lfloor (a+1-m)/2 \rfloor$ and $f := a+1-m-2e$. Thus $0 \le f \le 1$. Assume for the moment $c \ge e+f$. Let $E \subset X$ be a general union of $c-e-f$ 2-points. Take a general $A \cup B \subset L$ such that $\#A = e$, $\#B = f$ and $A \cap B = \emptyset$. We degenerate e 2-points to the 2-points $2q$, $q \in A$ and, if $f = 1$, apply the Differential Horace Lemma to F. Since $m+2e+f = a+1$, $\mathrm{Res}_L(mp) = (m-1)p$ and $\mathrm{Res}_L(q) = \{q\}$ for all $q \in A$, the Differential Horace Lemma shows that to prove $h^1(\mathcal{I}_{mp\cup G}(a,b)) = 0$ it is sufficient to prove $h^1(\mathcal{I}_{(m-1)p\cup E\cup A\cup(2B,L)}(a,b-1)) = 0$.

Claim 1: $h^1(\mathcal{I}_{(m-1)p\cup E}(a,b-1)) = 0$.

Proof of Claim 1: Since $b-1-(m-1) = b-m$, we may use the inductive assumption. We have $\deg(mp) + \deg(G) - \deg((m-1)p) - 2e + f = h^0(\mathcal{O}_X(a,b)) - h^0(\mathcal{O}_X(a,b-1))$. Thus to prove Claim 1 it is sufficient to observe that $e+2f \le 2e+f$, which is true because $a+1-m \ge 2$ and $0 \le f \le 1$.

Claim 1 implies $h^0(\mathcal{I}_{(m-1)p\cup E}(a,b-1)) = (a+1)b - m(m-1)/2 - 3(c-e-f)$. Note that $\mathrm{Res}_L((m-1)p) = (m-2)p$ and that $\deg(A\cup(2B,L)) = e+2f$. By Lemma 2.3 to prove that $h^1(\mathcal{I}_{(m-1)p\cup E\cup A\cup(2B,L)}(a,b-1)) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_{(m-2)p\cup E}(a,b-2)) \le \max\{0,(a+1)b - m(m-1)/2 - 3(c-e-f) - e - 2f\}$. Recall that $(m+1)m/2 + 3c \le (a+1)(b+1)$. We have $\deg((m-1)p) + \deg(E) - \deg((m-2)p) - \deg(E) = m$. We have $a+1-(m-1)-e-2f \ge 0$, because $e > 0$, $f \le 1$ and hence $2e+f \ge e+2f$. Since $m-2-b+2 = m-2$, we may use the inductive assumption (or Remark 3.1 for $m = 4$ or that $h^0(\mathcal{I}_{p\cup E}(a,b-2)) = \max\{0, h^0(\mathcal{I}_E(a,b-2)) - 1\}$ for a general $p \in X$ if $m = 3$).

If $c < e+f$ (and hence $c \le e$) the proof works taking $B = \emptyset$ and $\#A = c$.

(b) Assume $m = 3$. Take L, p, A, B and E as in step (a). We first check Claim 1. Taking a general L and a general $p \in L$ and then taking a general E we see that $2p \cup E$ is a general union of $c-e-f+1$ 2-points of X. We have $a > b > 0$ and $\deg(2p\cup E) = 3c+3-3e-3f$ with $6+3c \le (a+1)(b+1)$ and $2e+f = a-2$. To conclude the proof of Claim 1 it is sufficient to use Remark 3.1 and that $e+2f \ge 2$. By Claim 1 to conclude it is sufficient to check that $h^0(\mathcal{I}_{p\cup E}(a,b-2)) \le \max\{0,(a+1)b - 3 - 3(c-e-f) - e - 2f\}$. The generality of p gives $h^0(\mathcal{I}_{p\cup E}(a,b-2)) = \max\{0, h^0(\mathcal{I}_E(a,b-2)) - 1\}$. Since $m = 3$, $b-2 > 0$. Remark 3.1 gives that either $h^0(\mathcal{I}_E(a,b-2)) \le 1$ or $h^0(\mathcal{I}_E(a,b-2)) = (a+1)(b-1) - 3(c-e-f)$. □

Proof of Theorem 1.1: Since the case $m = 0$ is true by Remark 3.1, we may use induction on m even if $m = 1$. Fix $L \in |\mathcal{O}_X(0,1)|$ and $o \in L$. Take $U \in \mathcal{Z}(o,m)$ with L not as its tangent vector. By (3.3) it is sufficient to take a positive integer c such that $\binom{m+1}{2} + 2 + 3c \le (a+1)(b+1)$ and prove that $h^1(\mathcal{I}_{U\cup W}(a,b)) = 0$ for a general union W of c 2-points. By part (a) of Remark 2.2 $W := \mathrm{Res}_L(U) \in \mathcal{Z}(o, m-1)$, L is not the tangent vector of W and $\deg(W \cap L) = m+1$. Set $e := \lfloor (a-m)/2 \rfloor$ and $f = a - m - 2e$. We have $0 \le f \le 1$. Since $a \ge m+2$, $e > 0$ and hence $2e + f \ge e + 2f$.

Claim 1: $h^1(\mathcal{I}_{G\cup E}(a, b-1)) = 0$.

Proof of Claim 1: We have $b - m = (b-1) - (m-1)$ and hence it is sufficient to use the inductive assumption. We have $\deg(U \cup W) - \deg(G \cup E) = m + 1 - 3(e+f)$ and $h^0(\mathcal{O}_X(a,b)) - h^0(\mathcal{O}_X(a, b-1)) = a+1$. Since $a + 1 = m + 1 + 2e + f$, Claim 1 follows from the inductive assumption.

Claim 1 implies $h^0(\mathcal{I}_{G\cup E}(a, b-1)) = (a+1)b - \binom{m}{2} - 2 - 3(c - e - f)$. We have $G' = \mathrm{Res}_L(G) \in \mathcal{Z}(o, m-2)$ if $m \ge 2$ and $G' = \{o\}$ if $m = 1$. We use Lemma 2.4 applied to the image of the restriction map $H^0(\mathcal{I}_{G\cup E\cup A\cup(2B,L)}(a, b-1)) \longrightarrow H^0(\mathcal{O}_L(a, b-1))$. To conclude the proof for m, c, a and b using it is sufficient to prove that $h^0(\mathcal{I}_{G'\cup E}(a, b-2)) \le \max\{0, (a+1)b - \binom{m}{2} - 2 - 3(c - e - f) - e - 2f\}$ and that $\deg(G' \cup E) \le (a+1)(b-1)$. The first inequality follows from the inductive assumption, while the second one follows from the following facts: $\deg(U) + 3c \le (a+1)(b+1)$, $\deg(U) - \deg(G') = 2m+1$, $3c - \deg(E) = 3e + 3f$, $m + 1 + 2e + f = a + 1$ and $f \le 1$.

If $c < e + f$ (and hence $c \le e$) the proof works taking $B = \emptyset$ and $\#A = c$. □

5 Competing interests

The author has no competing interest.

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7 Data availability

All proofs are contained in the body of the paper.

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