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A note on the structure of the zeros of a polynomial and Sendov's conjecture

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ABSTRACT

In this note we prove a result that highlights an interesting connection between the structure of the zeros of a polynomial $p(z)$ and Sendov's conjecture.

RESUMEN

En esta nota demostramos un resultado que da luces sobre una conexión interesante entre la estructura de los ceros de un polinomio $p(z)$ y la conjetura de Sendov.

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1 Introduction

Let $p(z) := \sum_{j=0}^{n} a_j z^j$, where $a_j \in \mathbb{C}$ be a polynomial with complex coefficients. If we plot the zeros of a polynomial $p(z)$ and the zeros of its derivative $p'(z)$ (the critical points of $p(z)$) in the complex plane, there are interesting geometric relations between the two sets of points. To start with they have the same centroid. We also have the Gauss-Lucas Theorem which states that the critical points of a polynomial p lie in the convex hull of its zeros. Regarding the distribution of critical points of p within the convex hull of its zeros the well known Sendov's Conjecture asserts:

"If all the zeros of a polynomial p lie in $|z| \leq 1$ and if z_0 is any zero of $p(z)$, then there is a critical point of p in the disk $|z - z_0| \leq 1$."

The conjecture was posed by Bulgarian mathematician Blagovest Sendov in 1958, but is often attributed to Ilieff because of a reference in Hayman's *Research Problems in Function Theory* [6] in 1967. A large number of papers have been published on this conjecture (for details see [9]) but the general conjecture remains open. Rubinstein [10] in 1968 proved the conjecture for all polynomials of degree 3 and 4. In 1969 Schmeisser [11] showed that, if the convex hull containing all zeros of p has its vertices on $|z| = 1$, then p satisfies the conjecture (for the proof see [9, Theorem 7.3.4]). Later Schmeisser [12] also proved the conjecture for the Cauchy class of polynomials. In 1996 Borcea [2] showed that the conjecture holds true for polynomials with atmost six distinct zeros and in 1999 Brown and Xiang [3] proved the conjecture for polynomials of degree up to eight. Dégot [5] proved that for every zero (say) z_0 of a polynomial p there exists lower bound N_0 depending upon the modulus of z_0 such that $|z - z_0| \leq 1$ contains a critical point of p if $deg(p) > N_0$. Chalebgwa [4] gave an explicit formula for such a N_0. More recent work in this area includes that of Kumar [7], Sofi, Ahanger and Gardner [14], and Sofi and Shah [13]. As for the latest, Terence Tao [15] following on the work of Dégot [5], proved that the Sendov's conjecture holds for polynomials with sufficiently high degree.

In this paper we prove an interesting connection between the geometric structure of the zeros of a polynomial and Sendov's conjecture.

2 Statement and proof of the theorem

Theorem 2.1. *Let $p(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \geq 2$ with all its zeros $z_1, z_2, \ldots, z_n$ lying inside the closed unit disk. Suppose that for all $j = 1, 2, \ldots, n$*

$$\sum_{i=1, i\neq j}^{n} \left|1 - \frac{1}{z_j - z_i}\right|^2 \leq \sum_{i=1, i\neq j}^{n} \left|\frac{1}{z_j - z_i}\right|^2 \tag{2.1}$$

then $|z - z_j| \leq 1$ contains some critical point of p, that is, Sendov's conjecture holds for p.

[One prime (but not the only) example of polynomials satisfying the hypotheses of Theorem 2.1 are the polynomials whose zeros lie on a circle within the closed unit disk. In this case we may assume without loss of generality that $|z_i| = |z_j|$ for all $1 \leq i, j \leq n$ and that for a fixed but arbitrary $1 \leq j \leq n$, $0 < z_j \leq 1$. Hence for all $1 \leq i \leq n$

$$|z_i - (z_j - 1)| \leq |z_i| + |z_j - 1| = |z_i| + 1 - z_j = 1$$

and the required condition

$$\sum_{i=1, i\neq j}^{n} \left|1 - \frac{1}{z_j - z_i}\right|^2 \leq \sum_{i=1, i\neq j}^{n} \left|\frac{1}{z_j - z_i}\right|^2$$

is satisfied.]

Proof. Let $\zeta_1, \zeta_2 \ldots, \zeta_{n-1}$ be the critical points of p and assume to the contrary. Then there exists a zero of p say z_1 such that $|z_1 - \zeta_i| > 1$ for $1 \leq i \leq n-1$. We note that z_1 cannot be a repeated zero of p and hence $z_1 - z_i \neq 0$ for all $i = 2, 3, \ldots, n$ and

$$\frac{1}{|z_1 - \zeta_i|} < 1 \quad \text{for all } 1 \leq i \leq n-1.$$

Also we can write

$$p'(z) = na_n \prod_{i=1}^{n-1} (z - \zeta_i)$$

so that

$$\frac{p''(z)}{p'(z)} = \sum_{i=1}^{n-1} \frac{1}{z - \zeta_i}.$$

This gives

$$\frac{p''(z_1)}{p'(z_1)} = \sum_{i=1}^{n-1} \frac{1}{z_1 - \zeta_i}.$$

Hence

$$\left|\frac{p''(z_1)}{p'(z_1)}\right| = \left|\sum_{i=1}^{n-1} \frac{1}{z_1 - \zeta_i}\right| \leq \sum_{i=1}^{n-1} \frac{1}{|z_1 - \zeta_i|} < n-1.$$

That is

$$\left|\frac{p''(z_1)}{p'(z_1)}\right| < n-1. \tag{2.2}$$

Now suppose

$$p(z) = a_n(z - z_1)q(z), \quad \text{where } q(z) = \prod_{i=2}^{n} (z - z_i).$$

This gives

$$\frac{q'(z)}{q(z)} = \sum_{i=2}^{n} \frac{1}{z - z_i}$$

so that

$$\frac{q'(z_1)}{q(z_1)} = \sum_{i=2}^{n} \frac{1}{z_1 - z_i}.$$

Also

$$p'(z_1) = q(z_1) \quad \text{and} \quad p''(z_1) = 2q'(z_1).$$

Therefore from (2.2), we obtain

$$\left|\frac{2q'(z_1)}{q(z_1)}\right| = \left|\frac{p''(z_1)}{p'(z_1)}\right| < n - 1$$

and hence

$$\left|\frac{q'(z_1)}{q(z_1)}\right| < \frac{n-1}{2}.$$

Thus

$$\left|\sum_{i=2}^{n} \frac{1}{z_1 - z_i}\right| < \frac{n-1}{2}. \tag{2.3}$$

Now

$$\mathfrak{Re}\left(\frac{1}{z_1 - z_i}\right) = \frac{1}{2} + \frac{1 - |z_1 - z_i - 1|^2}{2|z_1 - z_i|^2}$$

for all $i = 2, 3, \ldots, n$. This gives

$$\begin{aligned}
\sum_{i=2}^{n} \mathfrak{Re}\left(\frac{1}{z_1 - z_i}\right) &= \frac{n-1}{2} + \sum_{i=2}^{n} \frac{1 - |z_1 - z_i - 1|^2}{2|z_1 - z_i|^2} \\
&= \frac{n-1}{2} + \frac{1}{2}\left(\sum_{1=2}^{n} \left|\frac{1}{z_1 - z_i}\right|^2 - \sum_{i=2}^{n} \left|\frac{z_1 - z_i - 1}{z_1 - z_i}\right|^2\right) \\
&= \frac{n-1}{2} + \frac{1}{2}\left(\sum_{i=2}^{n} \left|\frac{1}{z_1 - z_i}\right|^2 - \sum_{1=2}^{n} \left|1 - \frac{1}{z_1 - z_i}\right|^2\right)
\end{aligned}$$

Now from (2.1)

$$\left(\sum_{i=2}^{n} \left|\frac{1}{z_1 - z_i}\right|^2 - \sum_{i=2}^{n} \left|1 - \frac{1}{z_1 - z_i}\right|^2\right) \geq 0$$

Therefore

$$\mathfrak{Re}\left(\sum_{i=2}^{n} \frac{1}{z_1 - z_i}\right) = \sum_{i=2}^{n} \mathfrak{Re}\left(\frac{1}{z_1 - z_i}\right) \geq \frac{n-1}{2}$$

and hence

$$\left|\sum_{i=2}^{n} \frac{1}{z_1 - z_i}\right| \geq \frac{n-1}{2}$$

which contradicts (2.3) and the contradiction proves the result. □

3 Declarations

Ethical Approval:

Not Applicable.

Conflict of Interest:

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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