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Curvature properties of α-cosymplectic manifolds with $*$-η-Ricci-Yamabe solitons

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ABSTRACT

In this research article, we study $*$-η-Ricci-Yamabe solitons on an α-cosymplectic manifold by giving an example in the support and also prove that it is an η-Einstein manifold. In addition, we investigate an α-cosymplectic manifold admitting $*$-η-Ricci-Yamabe solitons under some conditions. Lastly, we discuss the concircular, conformal, conharmonic, and W_2-curvatures on the said manifold admitting $*$-η-Ricci-Yamabe solitons.

RESUMEN

En el presente artículo, estudiamos solitones $*$-η-Ricci-Yamabe en una variedad α-cosimpléctica dando un ejemplo que lo soporta y también probamos que es una variedad η-Einstein. Adicionalmente, investigamos una variedad α-cosimpléctica que admite solitones $*$-η-Ricci-Yamabe bajo ciertas condiciones. Finalmente, discutimos las curvaturas concircular, conforme, con-armónica y W_2 en dicha variedad admitiendo solitones $*$-η-Ricci-Yamabe.

Keywords and Phrases: $*$-η-Ricci-Yamabe soliton, α-cosymplectic manifold, curvature, η-Einstein manifold.

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1 Introduction

In the year 1982, R. S. Hamilton [9] investigated the concept of Ricci flow on a smooth Riemannian manifold (shortly, RM). A self-similar solution to the Ricci flow is nothing but a Ricci soliton if it moves only by a one parameter family of diffeomorphism and scaling. After introducing the idea of Ricci flow, the theory of Yamabe flow was also initiated by Hamilton in [10] to construct Yamabe metrics on a compact RM. A Yamabe soliton is again corresponded to a self-similar solution of the Yamabe flow.

S. Guler and M. Crasmareanu gave a new class of geometric flow of type (ρ, q), known as Ricci-Yamabe flow in [7]. They proposed the idea of Ricci-Yamabe soliton (shortly, RYS) if it moves only by one parameter group of diffeomorphism and scaling. The metric of the RM (M^n, h), $n > 2$, is said to be RYS (h, V, Λ, ρ, q) if it satisfies the following [20]:

$$\pounds_V h + 2\rho Ric = [2\Lambda - qr]h, \tag{1.1}$$

where Lie derivative operator of the metric h along the vector field V represented by $\pounds_V h$, the Ricci curvature tensor by Ric (the Ricci operator Q defined by $Ric(A, B) = h(QA, B)$ for $A, B \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M), the scalar curvature by r and the real scalars by Λ, ρ, q. According to Λ, RYS will be expanding, steady or shrinking if Λ is negative, zero or positive, respectively.

The concept of η-Ricci-Yamabe solitons (η-RYS) was defined by M. D. Siddiqi, *et al.* [20] in 2020 as a new generalization of RYS and it is defined as

$$\pounds_V h + 2Ric + [2\Lambda - qr]\, h + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where μ is a constant and η is a 1-form on M.

On the other hand, S. Dey and S. Roy [5] inaugurated a new generalization of η-Ricci soliton (η-RS) [3], namely $*$-η-Ricci soliton ($*$-η-RS), defined below:

$$\pounds_V h + 2Ric^* + 2\Lambda h + 2\mu\eta \otimes \eta = 0, \tag{1.3}$$

where $*$-Ricci tensor (shortly, $*$-RT) is denoted by Ric^*.

Tachibana [22] brought up the concept of $*$-RT on almost Hermitian manifolds and afterwards Hamada [8] studied $*$-RT on real hypersurfaces of non-flat complex space forms. Such geometrical works inspired S. Roy, *et al.* to come up with new idea $*$-η-Ricci-Yamabe soliton (shortly, $*$-η-RYS) of type (ρ, q), which is RM and fulfilling [18]

$$\pounds_V h + 2\rho Ric^* + [2\Lambda - qr^*]h + 2\mu\eta \otimes \eta = 0, \tag{1.4}$$

where $r^*(= trace(Ric^*))$ is the $*$-scalar curvature and Λ, ρ, q, μ are real scalars. The $*$-η-RYS is shrinking, steady or expanding if Λ is negative, zero or positive respectively. And they discussed $*$-η-RYS on α-cosymplectic manifolds with a quarter-symmetric metric (shortly, QSM) connection.

Further, A. Haseeb, R. Prasad and F. Mofarreh [12] obtained some interesting results on an α-Sasakian manifold admitting $*$-η-RYS with the potential vector field ζ satisfying conditions $Rim(\zeta, X).Ric = 0$, $Q(h, Ric) = 0$ and pseudo-Ricci symmetric and also showed that α-Sasakian admitting $*$-η-RYS is an η-EM.

In last few years, numerous authors have worked on the characterizations of Ricci, Ricci-Yamabe, η-Ricci-Yamabe and $*$-η-Ricci-Yamabe solitons (respectively, RS, RYS, η-RYS and $*$-η-RYS) in contact geometry. First, the study of RS in contact geometry was proposed by Sharma in [19]. After the initial work on Ricci solitons, some notable classes of contact geometry explored by H. I. Yoldas in [25, 26] where Ricci solitons have been investigated. Later on, D. Dey [2] provided the idea of an almost Kenmotsu metric as RYS. Also, P. Zhang *et al.* [27] have studied conformal RYS on perfect fluid space-time. New type of soliton namely $*$-RYS on contact geometry introduced by M. D. Siddiqi and Akyol in [20] and they have discussed the notion of η-RYS for geometrical structure on Riemannian submersions admitting η-RYS with the potential field. In recent years, a Kenmotsu metric in terms of η-RYS was measured by Yoldas in [23]. Next, the notion of $*$-η-RYS was studied by many authors on different odd dimensional Riemannian manifolds. It should be noted that the geometry of $*$-k-RYS and gradient $*$-k-RYS on Kenmotsu manifolds were given by S. Dey and S. Roy in [4].

We organize this paper as follows: In section 2, we review some basic definitions and tools of an α-cosymplectic manifold M. The main results are stated in section 3. In fact, we prove that an n-dimensional M admitting a $*$-η-RYS is an η-Einstein manifold. Then some curvature tensor conditions are studied on M with $*$-η-RYS. Finally, in section 4, we discuss some results on M when it is ζ-concircularly flat, ζ-conharmonically flat, ζ-W_2 flat and ζ-conformal.

2 Preliminaries

On an $n(= 2m + 1)$-dimensional RM M, if an almost contact metric structure (Φ, ζ, η, h) satisfies the following relations, then M is called an almost contact metric manifold:

$$\Phi^2 A = A - \eta(A)\zeta \tag{2.1}$$

$$\eta(\zeta) = 1, \quad \Phi(\zeta) = 0, \quad \eta(\Phi\zeta) = 0 \tag{2.2}$$

$$h(A, \Phi B) = -h(\Phi A, B), \tag{2.3}$$

$$h(A, \zeta) = \eta(A), \quad h(\Phi A, \Phi B) = h(A, B) - \eta(A)\eta(B), \tag{2.4}$$

for all $A, B \in \chi(M)$, where Φ denotes a $(1,1)$ tensor field, ζ is a vector field, η is a 1-form and h is the compatible Riemannian metric.

The fundamental form ϕ on M is defined as [1]

$$\phi(A, B) = h(\Phi A, B), \tag{2.5}$$

for all $A, B \in \chi(M)$.

If the Nijenhuis tensor field of Φ on M satisfies $N_\Phi(A, B) + 2d\eta(A, B)\zeta = 0$, then M is called a normal almost contact metric manifold. Here

$$N_\Phi(A, B) = \Phi^2[A, B] + [\Phi A, \Phi B] - \Phi[A, \Phi B] - \Phi[\Phi A, B],$$

for any $A, B \in \chi(M)$.

Under the following conditions, a normal almost contact metric manifold M is known as an α-cosymplectic manifold (shortly, α-CM):

(1) $d\eta = 0$,

(2) $d\phi = 2\alpha\eta \wedge \phi$,

for $\alpha \in \mathbb{R}$.

We note that an α-CM can be

(1) a cosymplectic manifold provided that $\alpha = 0$,

(2) an α-Kenmotsu manifold provided that $\alpha \neq 0$.

For an α-CM M, we have

$$(\nabla_A \Phi)B = \alpha(h(\Phi A, B)\zeta - \eta(B)\Phi A) \tag{2.6}$$

and

$$\nabla_A \zeta = -\alpha\Phi^2 A = \alpha[A - \eta(A)\zeta], \tag{2.7}$$

where ∇ is the Levi-Civita connection associated with h.

The main examples and curvature characteristics of α-CM were firstly obtained in [11,14,15]. Also, we have the following relations for the Riemannian curvature tensor Rim and the Ricci curvature tensor Ric of M:

$$Rim(A,B)\zeta = \alpha^2\left[\eta(A)B - \eta(B)A\right], \tag{2.8}$$

$$Rim(\zeta,A)B = \alpha^2\left[\eta(B)A - h(A,B)\zeta\right], \tag{2.9}$$

$$Rim(\zeta,A)\zeta = \alpha^2\left[A - \eta(A)\zeta\right], \tag{2.10}$$

$$\eta(Rim(A,B)C) = \alpha^2\left[\eta(B)h(A,C) - \eta(A)h(B,C)\right], \tag{2.11}$$

$$Ric(A,\zeta) = -\alpha^2(n-1)\eta(A), \tag{2.12}$$

for all $A, B, C \in \chi(M)$.

In [11], the $*$-RT Ric^* of type $(0,2)$ on an n-dimensional α-CM M is obtained as

$$Ric^*(B,C) = Ric(B,C) + \alpha^2(n-2)h(B,C) + \alpha^2\eta(B)\eta(C), \tag{2.13}$$

for any $B, C \in \chi(M)$.

Let $\{E_i|i = 1, 2, \ldots, n\}$ be an orthonormal basis of $T_p(M)$, $p \in M$. We set $B = C = E_i$ and it is easy to derive the $*$-scalar curvature $r^* = trace(Ric^*)$ as

$$r^* = r + \alpha^2(n-1)^2. \tag{2.14}$$

On the other hand, α-CM M is said to be an η-EM if the Ricci curvature tensor has the following form [24]:

$$Ric(A,B) = uh(A,B) + v\eta(A)\eta(B), \tag{2.15}$$

for $A, B \in \chi(M)$, where u and v being constants.

For this paper, we need some curvature tensors on a RM (M^n, h), which are given below [17]:

$$\overline{\mathbb{C}}(A,B)C = Rim(A,B)C - \frac{r}{n(n-1)}[h(B,C)A - h(A,C)B], \tag{2.16}$$

$$H(A,B)C = Rim(A,B)C - \frac{1}{n-2}[h(B,C)QA - h(A,C)QB + Ric(B,C)A - Ric(A,C)B], \tag{2.17}$$

$$W_2(A,B)C = Rim(A,B)C + \frac{1}{n-1}[h(A,C)QB - h(B,C)QA], \tag{2.18}$$

$$\begin{aligned}\mathbb{C}^*(A,B)C = {} & Rim(A,B)C - \frac{1}{n-2}[Ric(B,C)A - Ric(A,C)B + h(B,C)QA \\ & - h(A,C)QB] + \frac{r}{(n-1)(n-2)}[h(B,C)A - h(A,C)B],\end{aligned} \tag{2.19}$$

where $\overline{\mathbb{C}}$, H, W_2 and $\mathbb{C}^*$ represent the concircular curvature tensor [16], the conharmonic curvature tensor [13], the W_2-curvature tensor [16] and the conformal curvature tensor [6].

3 On α-CM admitting $*$-η-RYS

Let us take a $*$-η-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$ on an n-dimensional α-CM M, which is given by

$$(\pounds_\zeta h)(A,B) + 2\rho Ric^*(A,B) + [2\Lambda - qr^*]\, h(A,B) + 2\mu\eta(A)\eta(B) = 0, \tag{3.1}$$

for any $A, B \in \chi(M)$.

Theorem 3.1. *An n-dimensional α-CM M admitting $*$-η-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$ is an η-EM of the constant scalar curvature r. Moreover, the scalars Λ and μ are related by*

$$\Lambda + \mu = \frac{qr}{2} + \frac{q\alpha^2(n-1)^2}{2}. \tag{3.2}$$

Proof. From (2.4) and (2.7), we arrive at

$$(\pounds_\zeta h)(A,B) = h(\nabla_A \zeta, B) + h(A, \nabla_B \zeta) = 2\alpha\Big(h(A,B) - \eta(A)\eta(B)\Big). \tag{3.3}$$

Substitute (3.3)) into (3.1) to get

$$Ric^*(A,B) = -\frac{1}{\rho}\left(\Lambda - \frac{qr^*}{2} + \alpha\right) h(A,B) - \frac{(\mu-\alpha)}{\rho}\eta(A)\eta(B). \tag{3.4}$$

By using (2.13) and (2.14) in (3.4), we obtain

$$Ric(A,B) = \left[-\frac{1}{\rho}\left(\Lambda - \frac{qr}{2} - \frac{q\alpha^2(n-1)^2}{2} + \alpha\right) - \alpha^2(n-2)\right] h(A,B) - \left(\frac{(\mu-\alpha)}{\rho} + \alpha^2\right)\eta(A)\eta(B), \tag{3.5}$$

that is,

$$Ric(A,B) = \sigma_1 h(A,B) + \sigma_2 \eta(A)\eta(B), \tag{3.6}$$

where

$$\sigma_1 = -\frac{1}{\rho}\left(\Lambda - \frac{qr}{2} - \frac{q\alpha^2(n-1)^2}{2} + \alpha\right) - \alpha^2(n-2), \quad \sigma_2 = -\Big(\frac{(\mu-\alpha)}{\rho} + \alpha^2\Big).$$

Now, if we fix $B = \zeta$ in (3.6), then we can easily get the following relation:

$$Ric(A,\zeta) = \left[-\frac{1}{\rho}\left(\Lambda - \frac{qr}{2} - \frac{q\alpha^2(n-1)^2}{2} + \mu\right) - \alpha^2(n-1)\right]\eta(A). \tag{3.7}$$

Using (2.12) and values of σ_1 and σ_2 in (3.7), we can have (3.2). Also, on contracting (3.6) and using the values of σ_1 and σ_2, we find

$$r = (n-1)\left(\frac{\mu}{\rho} - \frac{\alpha}{\rho} - \alpha^2(n-1)\right), \tag{3.8}$$

where μ and $\rho(\neq 0)$ are constant.

Thus, (3.6) together with (3.2) and (3.8) give the relation of Λ and μ, which shows that $*$-η-RYS on α-CM is an η-EM. $\square$

Remark 3.2. *For the particular value of $\rho = 0$ in (3.1), an n-dimensional α-CM M endowed with $*$-η-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$ furnishes the scalar quantities as $\Lambda = -\alpha + \frac{qr^*}{2}$ and $\mu = \alpha$.*

First we give the more general construction of α-cosymplectic manifold:

Example 3.3. *Let $(N, J, \tilde{h})$ be a Kähler manifold. Denote by $\mathbb{R}\times_\sigma N$ the manifold $(\mathbb{R}\times N, \Phi, \zeta, \eta, h)$, where Φ is the tensor field such that*

$$\Phi\left(\frac{d}{dt}\right) = 0, \quad \Phi(A) = J(A), \quad A \in TN,$$
$$\zeta = \frac{d}{dt}, \quad \eta = dt, \quad h = dt \otimes dt + \exp(2\alpha t)\tilde{h}, \quad \alpha \in \mathbb{R}.$$

Putting $\sigma = \exp(\alpha t)$, h is the warped product metric of the Euclidean metric and $\tilde{h}$ by means of the function σ. Then $\mathbb{R}\times_\sigma N$ is α-cosymplectic and $(N, \tilde{h})$ is a totally umbilical submanifold with mean curvature vector $-\alpha\zeta$. Assume that $\alpha \neq 0$. Applying well-known curvature formulas, one relates the Ricci tensors of N and $\mathbb{R}\times_\sigma N$. But here we consider the flat Kähler manifold $\mathbb{R}^4$ endowed with the canonical complex structure and then the α-cosymplectic manifold $\mathbb{R}\times_\sigma \mathbb{R}^4$. If $\alpha = 0$, one has $\sigma = 1$, $\mathbb{R}\times_\sigma N = \mathbb{R}\times N$ is cosymplectic and N is totally geodetic. In this case the Ricci tensors are related by:

$$Ric(A, B) = \tilde{Ric}(A - \eta(A)\zeta, B - \eta(B)\zeta). \tag{3.9}$$

It follows that if N is an Einstein manifold, then $\mathbb{R}\times N$ is η-Einstein.

Next, by giving the following example we can show the existence of this soliton in α-cosymplectic manifold:

Example 3.4. *Recall an example of 5-dimensional α-CM in [11], that is,*

$$M = \{(x_1, x_2, y_1, y_2, z) \in \mathbb{R}^5, \Phi, \zeta, \eta, h\},$$

where (x_1, x_2, y_1, y_2, z) are the standard coordinates in $\mathbb{R}^5$.

The linearly independent vector fields on M are denoted by $E_1 = \exp^{\alpha z}\partial x_1$, $E_2 = \exp^{\alpha z}\partial x_2$, $E_3 = \exp^{\alpha z}\partial y_1$, $E_4 = \exp^{\alpha z}\partial y_2$ and $E_5 = \zeta = -\partial z$ for $i = \{1, 2\}$. Thus, h and Φ are respectively defined as

$$h(E_i, E_i) = 1, \quad h(E_i, E_j) = 0, \quad i \neq j = \{1, 2, 3, 4, 5\}$$

and

$$\Phi E_1 = -E_2, \quad \Phi E_2 = E_1, \quad \Phi E_3 = -E_4, \quad \Phi E_4 = E_3, \quad \Phi E_5 = \Phi\zeta = 0.$$

By the linearity of these tensors, it is quite easy to compute (2.1)-(2.4). *Also,* (2.6) *and* (2.7) *are verified in [11].*

By applying Koszul's formula, Rim of M (see [11]) can be obtained easily and hence the components Ric of Ricci tensor of M are: $Ric(E_i, E_i) = -4\alpha^2$ *for* $i = \{1, 2, 3, 4, 5\}$. *Since* $r = \sum_{i=1}^{5} Ric(E_i, E_i)$, *so we have* $r = -20\alpha^2$.

Now, we use (3.7) *and find*

$$Ric(E_5, E_5) = Ric(\zeta, \zeta) = \left[-\frac{1}{\rho}\left(\Lambda + 2q\alpha^2 + \mu\right) - 4\alpha^2 \right].$$

By equating the values of $Ric(\zeta, \zeta)$, *we arrive at a relation:* $\Lambda + \mu = -2q\alpha^2$. *We also verify this relation for* $n = 5$ *by using* (3.2). *Thus, h gives an* $*$-η-*RYS* $(h, \zeta, \Lambda, \mu, \rho, q)$ *on an* α-*cosymplectic manifold M of dimension* 5.

On the other hand, suppose that an n-dimensional α-CM M admitting $*$-η-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$ satisfies

$$Q(h, Ric)(A, B, C, D) = 0, \tag{3.10}$$

where $Q(h, Ric)(A, B, C, D) = (h(A, B).Ric)(C, D)$, for all vector fields A, B, C, D on M. This can be expressed as

$$\begin{aligned} Q(h, Ric)(A, B, C, D) =& h(B, C)Ric(A, D) - h(A, C)Ric(B, D) \\ &+ h(B, D)Ric(A, C) - h(A, D)Ric(B, C). \end{aligned} \tag{3.11}$$

Theorem 3.5. *If* $*$-η-*RYS on an* α-*CM M satisfies* $Q(h, Ric) = 0$, *then*

$$\Lambda = \frac{q}{2}\left(r + \alpha^2(n-1)^2\right) - \alpha\left(1 - \alpha\rho\right), \tag{3.12}$$

$$\mu = \alpha(1 - \alpha\rho). \tag{3.13}$$

Proof. From the expressions (3.6), (3.10) and (3.11), we have

$$\sigma_2[h(B, C)\eta(A)\eta(D) - h(A, C)\eta(B)\eta(D) + h(B, D)\eta(A)\eta(C) - h(A, D)\eta(B)\eta(C)] = 0. \tag{3.14}$$

Above equation follows that $\sigma_2 = 0$, which implies that

$$\mu = \alpha(1 - \alpha\rho).$$

We obtain the following from (3.2)

$$\Lambda = \frac{q}{2}\left(r + \alpha^2(n-1)^2\right) - \alpha\left(1 - \alpha\rho\right). \tag{3.15}$$

□

Now, by using these values of σ_1, σ_2 and Λ as well as μ in (3.6), we calculate

$$Ric(A, B) = -(\alpha^2(n-1))h(A, B). \tag{3.16}$$

Thus, from above we can state the following result:

Corollary 3.6. *If $*$-η-RYS on an α-CM M satisfies $Q(h, Ric) = 0$, then M is an EM.*

Next, we have

$$Rim(\zeta, A).Ric = 0, \tag{3.17}$$

then we have

$$Ric(Rim(\zeta, A)B, C) + Ric(B, Rim(\zeta, A)C) = 0, \tag{3.18}$$

for all vector fields A, B, C on M.

Theorem 3.7. *If $*$-η-RYS on an α-CM M satisfies $Rim(\zeta, A).Ric = 0$, then either M becomes CM or we have*

$$\Lambda = \frac{q}{2}\left(r + \alpha^2(n-1)^2\right) - \alpha(1 - \alpha\rho) \tag{3.19}$$

$$\mu = \alpha(1 - \alpha\rho). \tag{3.20}$$

Proof. In view of (3.6) and (3.18), we compute

$$\alpha^2\sigma_2\big(2\eta(A)\eta(B)\eta(C) - \eta(C)h(A, B) - \eta(B)h(A, C)\big) = 0. \tag{3.21}$$

Putting $C = \zeta$ into (3.21) and using (2.4), it is quite easy to see

$$\alpha^2\sigma_2 h(\Phi A, \Phi B) = 0, \tag{3.22}$$

which implies either $\alpha = 0$ or $\sigma_2 = 0$. Further, from later case we find $\mu = \alpha(1 - \alpha\rho)$ and hence from (3.2), we calculate the value of Λ. From the first case we can also say that M is CM. This is the desired result. □

Next, by using the values of Λ as well μ in (3.6), we have

$$Ric(A, B) = -(\alpha^2(n-1))h(A, B). \tag{3.23}$$

Thus, we can state the following:

Corollary 3.8. *If $* - \eta$-RYS on an α-CM M satisfies $Rim(\zeta, A).Ric = 0$ then M is either an EM or CM.*

The non-flat manifold M of n-dimension is named pseudo Ricci symmetric, if $Ric\,(\neq 0)$ of M satisfies the condition:

$$(\nabla_C Ric)\,(A,B) = 2\kappa(C)Ric(A,B) + \kappa(A)Ric(C,B) + \kappa(B)Ric(C,A), \tag{3.24}$$

where κ is a non-zero 1-form. In particular, M is said to be Ricci symmetric if $\kappa = 0$.

Theorem 3.9. *If an α-CM M admitting $*$-η-RYS is pseudo-Ricci-symmetric, then M is either Ricci symmetric or CM.*

Proof. The covariant derivative of (3.6) leads

$$(\nabla_C Ric)\,(A,B) = \nabla_C\left[\sigma_1 h(A,B) + \sigma_2\eta(A)\eta(B)\right] = \alpha\sigma_2\left[h(\Phi A, \Phi C)\eta(B) + \eta(A)h(\Phi B, \Phi C)\right]. \tag{3.25}$$

Further, we use the relations (3.6), (3.24), (3.25) and obtain

$$\begin{aligned} &2\kappa(C)\left[\sigma_1 h(A,B) + \sigma_2\eta(A)\eta(B)\right] + \kappa(A)\left[\sigma_1 h(C,B) + \sigma_2\eta(C)\eta(B)\right] \\ &+ \kappa(B)\left[\sigma_1 h(C,A) + \sigma_2\eta(C)\eta(A)\right] = \alpha\sigma_2\left[h(\Phi A, \Phi C)\eta(B) + \eta(A)h(\Phi B, \Phi C)\right]. \end{aligned} \tag{3.26}$$

Taking $C = B = \zeta$ in (3.26), we get

$$(\sigma_1 + \sigma_2)\big(\kappa(A) + 3\eta(A)\kappa(\zeta)\big) = 0,$$

which gives either

$$\kappa(A) = -3\eta(A)\kappa(\zeta) \tag{3.27}$$

or

$$\sigma_1 + \sigma_2 = 0. \tag{3.28}$$

Putting $A = \zeta$ in (3.27), we have $\kappa(\zeta) = 0$, which further implies that $\kappa(A) = 0$. Also, from (3.28) and (3.2), we can have $\alpha^2(n-1) = 0$. This implies that $\alpha = 0$ because $n \neq 1$. Thus, we arrive at our desired result. □

4 Some curvature properties on α-CM admitting $*$-η-RYS

This section deals with the curvature properties on M admitting $*$-η-RYS. We mainly discuss the conditions that M is ζ-concircularly flat, ζ-conharmonically flat, ζ-W_2 flat and ζ-conformal flat.

Theorem 4.1. *Let M be an n-dimensional α-CM admitting $*$-η-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$, where ζ being the Reeb vector field on M. Then M is*

(1) *ζ-concircularly flat if and only if $\mu = \alpha - \rho\alpha^2$.*

(2) *ζ-conformal curvature flat.*

(3) *ζ-conharmonically flat if and only if $\mu = \alpha + (n-1)\alpha^2\rho$.*

(4) *$\zeta - W_2-$curvature flat if and only if $\mu = \alpha - \rho\alpha^2$.*

Proof. By using the property $h(QA, B) = Ric(A, B)$ in (3.6), we arrive at

$$QB = \sigma_1 B + \sigma_2 \eta(B)\zeta, \tag{4.1}$$

where $\sigma_1 = -\frac{1}{\rho}\left(\Lambda - \frac{qr}{2} - \frac{q\alpha^2(n-1)^2}{2} + \alpha\right) - \alpha^2(n-2)$ and $\sigma_2 = -\left(\frac{\mu-\alpha}{\rho} + \alpha^2\right)$.

Firstly, we put $C = \zeta$ into (2.16) and use the relations (2.4), (2.8) and (3.8), we have

$$\overline{\mathbb{C}}(A, B)\zeta = \frac{1}{n}\left(\frac{\mu}{\rho} - \frac{\alpha}{\rho} + \alpha^2\right)(\eta(A)B - \eta(B)A), \tag{4.2}$$

which gives $\overline{\mathbb{C}}(A, B)\zeta = 0$ if and only if $\mu = \alpha - \rho\alpha^2$.

Secondly, if we put $C = \zeta$ and use (2.8), (2.12), (4.1) in (2.19), then we have

$$\mathbb{C}^*(A, B)\zeta = \left(\frac{\alpha^2 - \sigma_1}{n-2} + \frac{r}{(n-1)(n-2)}\right)(\eta(B)A - \eta(A)B). \tag{4.3}$$

Again, using the value of σ_1, (3.2) and (3.8), we have

$$\mathbb{C}^*(A, B)\zeta = 0. \tag{4.4}$$

Thirdly, we take $C = \zeta$ in (2.17) and make use of (2.8), (4.1) and (2.12), we get

$$H(A, B)\zeta = \left(\frac{\sigma_1 - \alpha^2}{n-2}\right)(\eta(A)B - \eta(B)A). \tag{4.5}$$

This implies

$$H(A, B)\zeta = 0$$

if and only if

$$\sigma_1 = \alpha^2.$$

Thus,

$$H(A,B)\zeta = 0$$

if and only if

$$\mu = \alpha + (n-1)\alpha^2\rho.$$

Lastly, by taking $C = \zeta$ and using (2.8) and (4.1) in (2.18), we conclude

$$W_2(A,B)\zeta = \left(\alpha^2 + \frac{\sigma_1}{n-1}\right)(\eta(A)B - \eta(B)A). \tag{4.6}$$

From (4.6),

$$W_2(A,B)\zeta = 0$$

if and only if

$$\alpha^2 + \frac{\sigma_1}{n-1} = 0.$$

This further implies that

$$W_2(A,B)\zeta = 0$$

if and only if

$$\mu = \alpha - \rho\alpha^2.$$

□

Remark 4.2. *We observe that above results are true only for α-Kenmotsu manifolds because Λ and μ are depending on α. But for $\alpha = 0$, one puts in* (3.1) $B = \xi$ *obtains*

$$\Lambda + \mu = \frac{1}{2}qr.$$

Then (3.1) *implies*

$$Ric = \frac{\mu}{\rho}(h - \eta \otimes \eta).$$

So, according to the cases μ is zero or non-zero, M is Ricci-flat or η-Einstein for $\alpha = 0$. This is consistent with the formula (3.9)*, when N is Einstein.*

Remark 4.3. *If M is a cosymplectic manifold, then we have*

$$\overline{\mathbb{C}}(A,B)\zeta = -\left(\frac{\mu}{n-\rho}\right)(\eta(B)A - \eta(A)B).$$

and similar relations can be obtained for $H(A,B)\zeta$ and $W_2(A,B)\zeta$, while

$$\mathbb{C}^*(A,B)\zeta = 0.$$

By the above formulas, one has $\mu = 0$ *if and only if* $\overline{\mathbb{C}}(A, B)\zeta = 0$ *if and only if* $H(A, B)\zeta = 0$ *if and only if* $W_2(A, B)\zeta = 0$.

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References

[1] D. E. Blair, *Contact manifolds in Riemannian geometry*, ser. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1976, vol. 509.

[2] J. T. Cho and M. Kimura, "Ricci solitons and real hypersurfaces in a complex space form," *Tohoku Math. J. (2)*, vol. 61, no. 2, pp. 205–212, 2009, doi: 10.2748/tmj/1245849443.

[3] D. Dey, "Almost Kenmotsu metric as Ricci-Yamabe soliton," 2020, *arXiv:2005.02322*.

[4] S. Dey, P. L.-i. Laurian-ıoan, and S. Roy, "Geometry of $*$-k-ricci-yamabe soliton and gradient $*$-k-ricci-yamabe soliton on kenmotsu manifolds," *Hacettepe Journal of Mathematics and Statistics*, vol. 52, no. 4, p. 907–922, 2023, doi: 10.15672/hujms.1074722.

[5] S. Dey and S. Roy, "$*$-η-Ricci soliton within the framework of Sasakian manifold," *J. Dyn. Syst. Geom. Theor.*, vol. 18, no. 2, pp. 163–181, 2020, doi: 10.1080/1726037X.2020.1856339.

[6] L. P. Eisenhart, *Riemannian Geometry.* Princeton University Press, Princeton, NJ, 1949, 2d printing.

[7] S. Güler and M. Crasmareanu, "Ricci-Yamabe maps for Riemannian flows and their volume variation and volume entropy," *Turkish J. Math.*, vol. 43, no. 5, pp. 2631–2641, 2019, doi: 10.3906/mat-1902-38.

[8] T. Hamada, "Real hypersurfaces of complex space forms in terms of Ricci $*$-tensor," *Tokyo J. Math.*, vol. 25, no. 2, pp. 473–483, 2002, doi: 10.3836/tjm/1244208866.

[9] R. S. Hamilton, "Three-manifolds with positive Ricci curvature," *J. Differential Geometry*, vol. 17, no. 2, pp. 255–306, 1982.

[10] R. S. Hamilton, "The Ricci flow on surfaces," in *Mathematics and general relativity (Santa Cruz, CA, 1986)*, ser. Contemp. Math. Amer. Math. Soc., Providence, RI, 1988, vol. 71, pp. 237–262, doi: 10.1090/conm/071/954419.

[11] A. Haseeb, D. G. Prakasha, and H. Harish, "$*$-conformal η-Ricci solitons on α-cosymplectic manifolds," *IJAA*, vol. 19, no. 2, pp. 165–179, 2021, Art. ID 5718736, doi: 10.28924/2291-8639.

[12] A. Haseeb, R. Prasad, and F. Mofarreh, "Sasakian manifolds admitting $*$-η-Ricci-Yamabe solitons," *Adv. Math. Phys.*, 2022, Art. ID 5718736, doi: 10.1155/2022/5718736.

[13] Y. Ishii, "On conharmonic transformations," *Tensor (N.S.)*, vol. 7, pp. 73–80, 1957.

[14] Z. Olszak, "Locally conformal almost cosymplectic manifolds," *Colloq. Math.*, vol. 57, no. 1, pp. 73–87, 1989, doi: 10.4064/cm-57-1-73-87.

[15] Z. Olszak and R. Roşca, "Normal locally conformal almost cosymplectic manifolds," *Publ. Math. Debrecen*, vol. 39, no. 3-4, pp. 315–323, 1991, doi: 10.5486/pmd.1991.39.3-4.12.

[16] G. P. Pokhariyal and R. S. Mishra, "Curvature tensors' and their relativistics significance," *Yokohama Math. J.*, vol. 18, pp. 105–108, 1970.

[17] S. Roy, S. Dey, and A. Bhattacharyya, "Some results on η-Yamabe solitons in 3-dimensional trans-Sasakian manifold," *Carpathian Math. Publ.*, vol. 14, no. 1, pp. 158–170, 2022, doi: 10.15330/cmp.14.1.158-170.

[18] S. Roy, S. Dey, A. Bhattacharyya, and M. D. Siddiqi, "$*$-η-Ricci-Yamabe solitons on α-cosymplectic manifolds with a quarter-symmetric metric connection," 2021, *arXiv:2109.04700.*

[19] R. Sharma, "Certain results on K-contact and (k, μ)-contact manifolds," *J. Geom.*, vol. 89, no. 1-2, pp. 138–147, 2008, doi: 10.1007/s00022-008-2004-5.

[20] M. D. Siddiqi and M. A. Akyol, "η-Ricci-Yamabe soliton on Riemannian submersions from Riemannian manifolds," 2021, *arXiv:2004.14124*.

[21] A. N. Siddiqui and M. D. Siddiqi, "Almost Ricci-Bourguignon solitons and geometrical structure in a relativistic perfect fluid spacetime," *Balkan J. Geom. Appl.*, vol. 26, no. 2, pp. 126–138, 2021.

[22] S.-i. Tachibana, "On almost-analytic vectors in almost-Kählerian manifolds," *Tohoku Math. J. (2)*, vol. 11, pp. 247–265, 1959, doi: 10.2748/tmj/1178244584.

[23] H. I. Yoldaş, "On Kenmotsu manifolds admitting η-Ricci-Yamabe solitons," *Int. J. Geom. Methods Mod. Phys.*, vol. 18, no. 12, 2021, Art. ID 2150189, doi: 10.1142/S0219887821501899.

[24] H. I. Yoldaş, "Some results on α-cosympletic manifolds," *Bull. Transilv. Univ. Braşov Ser. III. Math. Comput. Sci.*, vol. 1(63), no. 2, pp. 115–128, 2021, doi: 10.31926/but.mif.2021.1.63.2.10.

[25] H. I. Yoldaş, "Some soliton types on Riemannian manifolds," *Rom. J. Math. Comput. Sci.*, vol. 11, no. 2, pp. 13–20, 2021.

[26] H. I. Yoldaş, "Some classes of Ricci solitons on Lorentzian α-Sasakian manifolds," *Differ. Geom. Dyn. Syst.*, vol. 24, pp. 232–244, 2022, doi: 10.3390/e24020244.

[27] P. Zhang, Y. Li, S. Roy, S. Dey, and A. Bhattacharyya, "Geometrical structure in a perfect fluid spacetime with conformal ricci–yamabe soliton," *Symmetry*, vol. 14, no. 3, 2022, Art. ID 594, doi: 10.3390/sym14030594 .