CUBO, A Mathematical Journal
Vol. 26, no. 2, pp. 191–215, August 2024
DOI: 10.56754/0719-0646.2602.191

Some properties of solutions of a linear set-valued differential equation with conformable fractional derivative

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ABSTRACT

The article explores a linear set-valued differential equation featuring both conformable fractional and generalized conformable fractional derivatives. It presents conditions for the existence of solutions and provides analytical expressions for the shape of solution sections at different time points. Model examples are employed to illustrate the results.

RESUMEN

Este artículo explora una ecuación diferencial lineal con valores en conjuntos que exhibe a la vez derivadas fraccionales conformables y conformables generalizadas. Se presentan condiciones para la existencia de soluciones y se proveen expresiones analíticas para la forma de secciones solución en diferentes puntos de tiempo. Se emplean ejemplos modelo para ilustrar los resultados.

Keywords and Phrases: Conformable fractional derivative, set-valued differential equation, Hukuhara derivative, generalized derivative.

2020 AMS Mathematics Subject Classification: 26A33, 34K37, 26E25, 34A08, 34A30

Published: 26 June, 2024
Accepted: 28 March, 2024
Received: 11 April, 2023

1 Introduction

Set-valued differential equations have recently been studied within the framework of an independent theory - set-valued equations, but they are widely used for ordinary differential inclusions and fuzzy differential equations and inclusions [7,26,29,30,36,37,46,48,53].

In 1967, M. Hukuhara introduced integral and derivative concepts for set-valued mappings and explored their relationship [20]. The proposed derivative and integral extend the conventional single-valued function derivative and integral to the set-valued context. However, the Hukuhara derivative has a notable limitation: if a mapping is Hukuhara differentiable, its cross-section diameter behaves as a non-decreasing function. To overcome this drawback, alternative derivative concepts were proposed: T. F. Bridgland introduced the Huygens derivative [6], while Yu. N. Tyurin [54] and H. T. Banks, M. Q. Jacobs [5] proposed the π-derivative using Radstrom's embedding theorem [52], and A. V. Plotnikov introduced the T-derivative [39,48]. Additionally, Ş. E. Amrahov, A. Khastan, N. Gasilov, A. G. Fatullayev [3,28] and A. V. Plotnikov, N. V. Skripnik [28,44,45] introduced generalized derivatives for set-valued mappings. Each of these derivatives has its own set of advantages and disadvantages [8,12,32,33,46,48]. In 2003, A. N. Vityuk introduced an analogue of the fractional Riemann-Liouville derivative [23,31] for set-valued mappings and established its properties [55,56]. Subsequently, in 2019, A. A. Martyniuk introduced an analogue of the conformable fractional derivative [22] for set-valued mappings and proved its properties [34,35]. The conformable fractional derivative for single-valued functions serves as a generalization of the ordinary derivative and, unlike fractional derivatives, adheres to all classical properties of the ordinary derivative [22]. Consequently, the Hukuhara conformable fractional derivative for set-valued mappings, introduced by A. A. Martyniuk, serves as a generalization of the Hukuhara derivative while preserving its properties [34,35].

In 1969, F. S. de Blasi and F. Iervolino explored differential equations involving the Hukuhara derivative [12]. Subsequently, many authors investigated the properties of solutions to such equations [26,29,30,36,43,46,48], integral and integro-differential equations [41,42], higher-order equations [38], as well as differential inclusions [11,24,48]. Furthermore, differential equations with the π-derivative [8,37,49], T-derivative [39,48], set-valued equations with a generalized derivative [28,40,44,45,47], nonlinear equations with the fractional Riemann-Liouville derivative [55,56], and conformable fractional derivative [34,35,57] have been explored. At first glance, such equations resemble their corresponding ordinary analogues; however, when studying and solving them, it is imperative to consider their set-valued nature. Consequently, traditional methods and approaches employed in studying and solving of single-valued systems may not always be applicable to set-valued systems, necessitating novel or alternative methods and approaches. It is also worth noting that due to set-valued nature, new properties emerge that warrant investigation.

This article delves into the Cauchy problem for a linear differential equation with the Hukuhara conformable fractional derivative, yielding analytical solutions in certain cases. Subsequently, we introduce a generalized conformable fractional derivative based on the generalized derivative for set-valued mappings [28, 44, 45], that allows us to expand the class of differentiable mappings. We then explore the Cauchy problem for a linear differential equation with the generalized conformable fractional derivative. Such a Cauchy problem boasts infinitely many solutions - two of which are termed basic [28, 44, 45], and we provide analytical forms for these solutions in selected cases. In conclusion, we demonstrate the feasibility of introducing conformable fractional derivatives akin to known conformable fractional derivatives for single-valued functions [1, 2, 4, 15, 17–19, 21, 22], alongside presenting analytical solutions for the corresponding Cauchy problems with these derivatives. The theoretical results are exemplified through model examples.

2 Preliminaries

In this section we recall some results from the publications that are of interest for our paper.

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ be the n-dimensional Euclidean space ($n \geq 2$). Denote by $conv(\mathbb{R}^n)$ the set of nonempty compact and convex subsets of $\mathbb{R}^n$ with the Hausdorff metric

$$h(X,Y) = \min\{r \geq 0 \,:\, X \subset Y + B_r(\mathbf{0}),\, Y \subset X + B_r(\mathbf{0})\},$$

where $X, Y \in conv(\mathbb{R}^n)$, $B_r(\mathbf{c}) = \{x \in \mathbb{R}^n \,:\, \|x - \mathbf{c}\| \leq r\}$ is the closed ball with radius $r > 0$ centered at the point $\mathbf{c} \in \mathbb{R}^n$ ($\|\cdot\|$ denotes the Euclidean norm), $\mathbf{0} = (0, \ldots, 0)^T$ is the zero vector.

In addition to the usual set-theoretic operations, the following operations in the space $conv(\mathbb{R}^n)$ are introduced: the sum of the sets, the product of the scalar on the set and the operation of the product of the matrix on the set:

$$X + Y = \bigcup_{x \in X,\, y \in Y} \{x + y\} \qquad \lambda X = \bigcup_{x \in X} \{\lambda x\}, \qquad AX = \bigcup_{x \in X} \{Ax\},$$

where $X, Y \in conv(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$, $A \in \mathbb{R}^{n \times n}$.

Lemma 2.1 ([51]). *The following properties hold:*

1) $(conv(\mathbb{R}^n), h)$ *is a complete metric space,*

2) $h(X + Z, Y + Z) = h(X, Y)$,

3) $h(\lambda X, \lambda Y) = |\lambda| h(X, Y)$ *for all* $X, Y, Z \in conv(\mathbb{R}^n)$ *and* $\lambda \in \mathbb{R}$.

However, $conv(\mathbb{R}^n)$ is not a linear space because it does not contain inverse elements for the addition, and therefore the difference is not well defined, *i.e.* if $X \in conv(\mathbb{R}^n)$ and $X \neq \{\mathbf{x}\}$, then

$X + (-1)X \neq \{\mathbf{0}\}$. As a consequence, alternative formulations for difference have been suggested [3,5,20,39,45,51]. One of these alternatives is the Hukuhara difference [20].

Definition 2.2 ([20]). *Let $X, Y \in conv(\mathbb{R}^n)$. A set $Z \in conv(\mathbb{R}^n)$ such that $X = Y + Z$ is called a Hukuhara difference (H-difference) of the sets X and Y and is denoted by $X \overset{H}{-} Y$.*

In this case $X \overset{H}{-} X = \{\mathbf{0}\}$ and $(X+Y) \overset{H}{-} Y = X$ for any $X, Y \in conv(\mathbb{R}^n)$, but obviously, $X \overset{H}{-} Y \neq X + (-1)Y$. The properties of this difference are studied in detail in [37,46,48,51]:

Lemma 2.3 ([27]). *If $X + Y = B_1(\mathbf{0})$, then $X = B_\mu(\mathbf{z_1})$ and $Y = B_\lambda(\mathbf{z_2})$, where $\mu + \lambda = 1$ and $\mathbf{z_1} + \mathbf{z_2} = \mathbf{0}$.*

Remark 2.4. *If the set X is subtracted from the ball $B_R(\mathbf{a})$ in the sense of Hukuhara and the difference $B_R(\mathbf{a}) \overset{H}{-} X$ exists, then the set X is the ball $B_r(\mathbf{b})$ and radius r does not exceed R.*

Theorem 2.5 ([14,16]). *For any real $(n \times n)$-matrix A there exist two orthogonal $(n \times n)$-matrices U and V such that $U^T A V = \Sigma$, where Σ is the diagonal matrix. We can also choose matrices U and V such that the diagonal elements of the matrix Σ satisfy the condition*

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0,$$

where r is the rank of the matrix A. That is, if A is a nondegenerate matrix, then $\sigma_1 \geq \cdots \geq \sigma_n > 0$.

Therefore, this matrix A can be represented as $A = U \Sigma V^T$. This decomposition is called **singular decomposition**. Columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of matrix U are called the **left singular vectors**, columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of matrix V are called the **right singular vectors**, and the numbers $\sigma_1, \ldots, \sigma_n$ are called the **singular numbers** of the matrix A.

By [14], the set $Y = \{Ax : x \in B_1(\mathbf{0}), A \in \mathbb{R}^{n \times n}\}$ is r-dimensional ellipsoid, its axis lengths are equal to the corresponding singular numbers of the matrix A, where $r = rank(A)$. Also, if $rank(A) = n$, then

$$B_{\sigma_n}(\mathbf{0}) \subset Y \subset B_{\sigma_1}(\mathbf{0}),$$

where $B_{\sigma_n}(\mathbf{0})$ is the inscribed ball in the set Y (*i.e.* the largest ball $B_r(\mathbf{0})$ that can fit inside the set Y), $B_{\sigma_1}(\mathbf{0})$ is the circumscribed ball of the set Y (*i.e.* the smallest ball $B_r(\mathbf{0})$, such that $Y \subseteq B_r(\mathbf{0})$).

It is also easy to see that if A is an orthogonal matrix, then $AB_r(\mathbf{0}) \equiv B_r(\mathbf{0})$ for all $r > 0$.

Let $X : [0,T] \to conv(\mathbb{R}^n)$ be a set-valued mapping.

Definition 2.6 ([34])**.** *Let $t \in (0,T)$ and $\alpha \in (0,1]$. If the Hukuhara differences $X\left(t+\varepsilon t^{1-\alpha}\right) \overset{H}{-} X\big(t\big)$ and $X\big(t\big) \overset{H}{-} X\left(t-\varepsilon t^{1-\alpha}\right)$ exist for all sufficiently small $\varepsilon > 0$ and there exists $Z \in conv(\mathbb{R}^n)$ such that the following equality holds:*

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\left(X\left(t+\varepsilon t^{1-\alpha}\right) \overset{H}{-} X\big(t\big)\right) = \lim_{\varepsilon \to 0} \varepsilon^{-1}\left(X\big(t\big) \overset{H}{-} X\left(t-\varepsilon t^{1-\alpha}\right)\right) = Z, \tag{2.1}$$

*then we say that the set-valued mapping $X(\cdot)$ has a **Hukuhara conformable fractional derivative of order α** at the point $t \in (0,T)$ and $D^\alpha X(t) = Z$.*

If $D^\alpha X(t)$ exists for all $t \in (0,T)$ and $\lim\limits_{t\to 0} D^\alpha X(t)$ exists, then we will assume that $D^\alpha X(0) = \lim\limits_{t\to 0} D^\alpha X(t)$.

Definition 2.7. *If the Hukuhara conformable fractional derivative $D^\alpha X(t)$ of order α exists for all $t \geq 0$, then we say that the set-valued mapping $X(\cdot)$ is **α-differentiable** on $\mathbb{R}_+$.*

Next, we give some properties of the Hukuhara conformable fractional derivative of order α.

Lemma 2.8 ([34])**.** *If the set-valued mapping $X(\cdot)$ is α-differentiable on $\mathbb{R}_+$, then the set-valued mapping $X(\cdot)$ is continuous on $\mathbb{R}_+$.*

Lemma 2.9 ([34])**.** *If the set-valued mapping $X(\cdot)$ is α-differentiable on $\mathbb{R}_+$, then the function $diam(X(\cdot))$ is a nondecreasing function on $\mathbb{R}_+$, where $diam(X) = \max\limits_{\psi \in S_1(\mathbf{0})} |c(X,\psi) + c(X,-\psi)|$, $S_1(\mathbf{0}) = \{\psi \in \mathbb{R}^n \,:\, \|\psi\| = 1\}$, $c(X,\psi) = \max\limits_{x\in X}\{x_1\psi_1 + \cdots + x_n\psi_n\}$.*

Lemma 2.10 ([34])**.** *If the set-valued mapping $X(t) \equiv X$ for all $t \geq 0$, then*

$$D^\alpha X(t) \equiv \{\mathbf{0}\},$$

and vice versa, if $D^\alpha X(t) \equiv \{\mathbf{0}\}$ for all $t \geq 0$ and $X(t') = X$, then $X(t) \equiv X$ for all $t \geq 0$, where $t' \geq 0$ is an arbitrary value.

Lemma 2.11 ([34])**.** *If the set-valued mappings $X(\cdot)$ and $Y(\cdot)$ are α-differentiable at $t > 0$, then*

$$D^\alpha(aX(t) + bY(t)) = aD^\alpha X(t) + bD^\alpha Y(t),$$

where $a, b \in \mathbb{R}_+$.

Lemma 2.12 ([34])**.** *If the set-valued mapping $X(\cdot)$ is α-differentiable at $t > 0$, then*

$$D^\alpha X(t) = t^{1-\alpha} D_H X(t),$$

where $D_H X(t)$ is the Hukuhara derivative [20].

Remark 2.13. *From Lemma 2.12 we have that the necessary and sufficient condition for the existence of a Hukuhara conformable fractional derivative* $D^{\alpha}X(t)$ *of order* α *for the set-valued mapping* $X(\cdot)$ *is the existence of the Hukuhara derivative* $D_H X(t)$.

Remark 2.14. *From Definition 2.6 and Lemma 2.12, we have that* $D^1 X(t)$ *coincides with the Hukuhara derivative* $D_H X(t)$.

Definition 2.15 ([34])**.** *The fractional integral associated with the Hukuhara conformable fractional derivative of order* α *is defined by*

$$I^{\alpha}X(t) = \int_0^t t^{\alpha-1}X(s)ds, \quad t \geq 0,$$

where the integral on the right-hand side is understood in the sense of the Hukuhara integral [20].

Lemma 2.16 ([34])**.** *If the set-valued mapping* $X(\cdot)$ *is continuous on* $\mathbb{R}_+$*, then*

$$D^{\alpha}I^{\alpha}X(t) = X(t), \quad t > 0.$$

Lemma 2.17 ([34])**.** *If the set-valued mapping* $X(\cdot)$ *is* α*-differentiable on* $\mathbb{R}_+$*, then*

$$I^{\alpha}D^{\alpha}X(t) = X(t) \overset{H}{-} X(0), \quad t > 0.$$

3 A linear set-valued differential equation with a Hukuhara conformable fractional derivative.

Consider the following Cauchy problem for linear set-valued differential equation with a Hukuhara conformable fractional derivative of order α

$$D^{\alpha}X(t) = AX(t), \quad X(0) = B_1(\mathbf{0}), \tag{3.1}$$

where $X : \mathbb{R}_+ \to conv(\mathbb{R}^2)$ is a set-valued mapping, $A \in \mathbb{R}^{2\times 2}$ is a nondegenerate matrix.

Definition 3.1. *A set-valued mapping* $X : \mathbb{R}_+ \to conv(\mathbb{R}^2)$ *is called a solution of Cauchy problem* (3.1) *if it is continuous and satisfies differential equation* (3.1) *for all* $t \geq 0$ *and* $X(0) = B_1(\mathbf{0})$.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$ such that $ad - bc \neq 0$.

It is easy to obtain that the singular numbers of the matrix A have the form

$$\sigma_1 = \sqrt{\frac{a^2+b^2+c^2+d^2+\sqrt{\delta}}{2}}, \quad \sigma_2 = \sqrt{\frac{a^2+b^2+c^2+d^2-\sqrt{\delta}}{2}},$$

where $\delta = (a^2+b^2+c^2+d^2)^2 - 4(ad-bc)^2$.

It is obvious that

$$\delta = (a^2+b^2+c^2+d^2)^2 - 4(ad-bc)^2 = (a^2-d^2)^2 + (b^2-c^2)^2 + 2(ab+cd)^2 + 2(ac+bd)^2,$$

i.e. $\delta \geq 0$.

Accordingly, if $d = a$ and $c = -b$ or $d = -a$ and $b = c$, *i.e.* if

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

then $\delta = 0$ and $\sigma_1 = \sigma_2 = \sigma = \sqrt{a^2+b^2}$. In other cases $\delta \neq 0$.

Theorem 3.2. *If matrix A satisfies the condition $\delta = 0$, then Cauchy problem* (3.1) *has the following solution*

$$X(t) = e^{\beta t^\alpha} B_1(\mathbf{0}),$$

where $t \geq 0$, $\beta = \dfrac{\sqrt{a^2+b^2}}{\alpha}$.

Proof. Let us prove that $X(\cdot)$ is a solution of Cauchy problem (3.1) by the direct substitution of the set-valued mapping $X(t) = e^{\beta t^\alpha} B_1(\mathbf{0})$ into differential equation (3.1) and by checking that the identity is satisfied:

$$D^\alpha \left(e^{\beta t^\alpha} B_1(\mathbf{0}) \right) \equiv A e^{\beta t^\alpha} B_1(\mathbf{0}).$$

Since $\beta > 0$, then $e^{\beta t^\alpha}$ is an increasing function and as

$$e^{\beta t^\alpha} B_1(\mathbf{0}) = B_{e^{\beta t^\alpha}}(\mathbf{0}),$$

then accordingly $diam(X(\cdot))$ is an increasing function. Then, according to Definition 2.6, it follows that $B_1(\mathbf{0})$ is a centrally symmetric body and $(-1)B_1(\mathbf{0}) = B_1(\mathbf{0})$, we have

$$\begin{aligned}\lim_{\varepsilon \to 0_+} \varepsilon^{-1} \big(X\left(t + \varepsilon t^{1-\alpha}\right) \stackrel{H}{-} X(t) \big) &= \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{\beta \left(t+\varepsilon t^{1-\alpha}\right)^\alpha} B_1(\mathbf{0}) \stackrel{H}{-} e^{\beta t^\alpha} B_1(\mathbf{0}) \right) \\ &= \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{\beta \left(t+\varepsilon t^{1-\alpha}\right)^\alpha} - e^{\beta t^\alpha} \right) B_1(\mathbf{0}) = \alpha \beta e^{\beta t^\alpha} B_1(\mathbf{0})\end{aligned}$$

and

$$\lim_{\varepsilon\to 0_+} \varepsilon^{-1}\left(X(t) \underline{H} X\left(t-\varepsilon t^{1-\alpha}\right)\right) = \lim_{\varepsilon\to 0_+} \varepsilon^{-1}\left(e^{\beta t^{\alpha}} B_1(\mathbf{0}) \underline{H} e^{\beta\left(t-\varepsilon t^{1-\alpha}\right)^{\alpha}} B_1(\mathbf{0})\right)$$
$$= \lim_{\varepsilon\to 0_+} \varepsilon^{-1}\left(e^{\beta t^{\alpha}} - e^{\beta\left(t-\varepsilon t^{1-\alpha}\right)^{\alpha}}\right) B_1(\mathbf{0}) = -\alpha\beta e^{\beta t^{\alpha}} B_1(\mathbf{0}) = \alpha\beta e^{\beta t^{\alpha}} B_1(\mathbf{0}).$$

That is,

$$D^{\alpha}X(t) = D^{\alpha}\left(e^{\beta t^{\alpha}} B_1(\mathbf{0})\right) = \alpha\beta e^{\beta t^{\alpha}} B_1(\mathbf{0}).$$

Since the singular numbers of the matrix A are equal ($\sigma_1 = \sigma_2 = \sigma$), then the singular decomposition of the matrix A has the form $A = U\Sigma V^T$, where U, V are orthogonal matrices and $\Sigma = \sigma I$, I is the identity matrix. Since $V^T B_r(\mathbf{0}) = B_r(\mathbf{0})$ and $UB_r(\mathbf{0}) = B_r(\mathbf{0})$ for all $r > 0$, then

$$Ae^{\beta t^{\alpha}} B_1(\mathbf{0}) = U\Sigma V^T e^{\beta t^{\alpha}} B_1(\mathbf{0}) = U\sigma I V^T e^{\beta t^{\alpha}} B_1(\mathbf{0})$$
$$= \sigma U I V^T e^{\beta t^{\alpha}} B_1(\mathbf{0}) = \sigma e^{\beta t^{\alpha}} U I V^T B_1(\mathbf{0}) = \sigma e^{\beta t^{\alpha}} B_1(\mathbf{0}).$$

As $\alpha\beta = \alpha\frac{\sqrt{a^2+b^2}}{\alpha} = \sqrt{a^2+b^2} = \sigma$, then we have

$$D^{\alpha}X(t) = D^{\alpha}\left(e^{\beta t^{\alpha}} B_1(\mathbf{0})\right) = \alpha\beta e^{\beta t^{\alpha}} B_1(\mathbf{0}) = \sigma e^{\beta t^{\alpha}} B_1(\mathbf{0}) \equiv \sigma e^{\beta t^{\alpha}} B_1(\mathbf{0}) = Ae^{\beta t^{\alpha}} B_1(\mathbf{0}) = AX(t),$$

i.e. $X(\cdot)$ is a solution of differential equation (3.1). The theorem is proved. □

Example 3.3. *Let* $A = \begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}$. *Then the singular numbers* σ_1 *and* σ_2 *of the matrix* A *are* $\sigma_1 = \sigma_2 = 2$. *Accordingly, Cauchy problem* (3.1) *has a solution* $X(t) = e^{2\alpha^{-1}t^{\alpha}} B_1(\mathbf{0})$. *That is,*

1) *if* $\alpha = 0.25$, *then at every moment of time* $t \geq 0$ *the cross section* $X(t)$ *is a circle of radius* $e^{8\sqrt[4]{t}}$ *(Figure 1);*

2) *if* $\alpha = 0.5$, *then at every moment of time* $t \geq 0$ *the cross section* $X(t)$ *is a circle of radius* $e^{4\sqrt{t}}$ *(Figure 2);*

3) *if* $\alpha = 0.75$, *then at every moment of time* $t \geq 0$ *the cross section* $X(t)$ *is a circle of radius* $e^{\frac{8}{3}\sqrt[4]{t^3}}$ *(Figure 3);*

4) *if* $\alpha = 1$, *then at every moment of time* $t \geq 0$ *the cross section* $X(t)$ *is a circle of radius* e^{2t} *(Figure 4).*

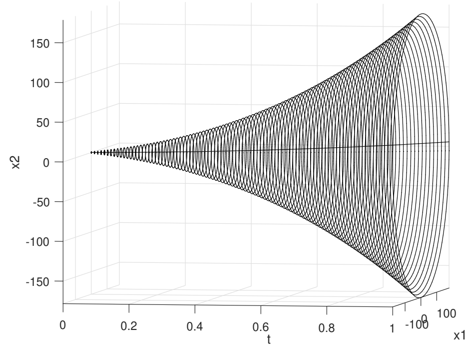

Figure 1: $\alpha = 0.25$, $X(t) = e^{8\sqrt[4]{t}}B_1(\mathbf{0})$.

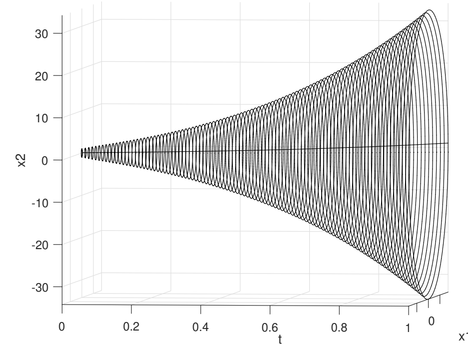

Figure 2: $\alpha = 0.5$, $X(t) = e^{4\sqrt{t}}B_1(\mathbf{0})$.

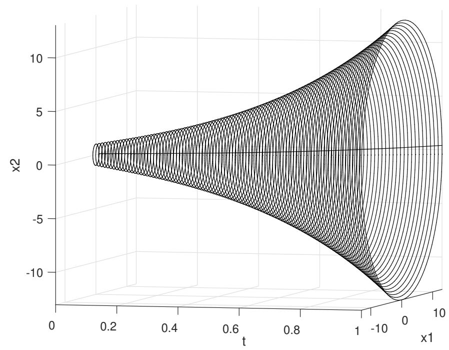

Figure 3: $\alpha = 0.75$, $X(t) = e^{\frac{8}{3}\sqrt[4]{t^3}}B_1(\mathbf{0})$.

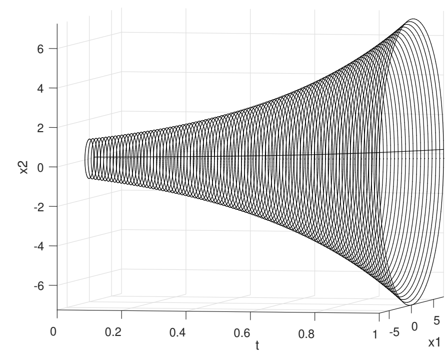

Figure 4: $\alpha = 1$, $X(t) = e^{2t}B_1(\mathbf{0})$.

Next, we consider the case when the matrix A satisfies the condition $\delta \neq 0$.

Theorem 3.4. *If matrix A is symmetric and $d \neq -a$, then Cauchy problem* (3.1) *has the following solution*

$$X(t) = Ue^{\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0}), \quad t \geq 0,$$

where $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$, $\sigma_{1,2} = |\lambda_{1,2}| = \left|\frac{a+d\pm\sqrt{(a-d)^2+4b^2}}{2}\right|$, $U = \begin{pmatrix} \frac{b}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{\lambda_2-d}{\sqrt{(\lambda_2-d)^2+b^2}} \\ \frac{\lambda_1-a}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{b}{\sqrt{(\lambda_2-d)^2+b^2}} \end{pmatrix}$.

Proof. Since the matrix A is symmetric and $d \neq -a$, it has the following form

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

It is known that the eigenvalues $\lambda_{1,2}$ of the symmetric matrix A are real, so in our case ($\delta \neq 0$), they will be different and not equal to zero. Let us consider all possible cases related to the eigenvalues of the matrix A, that is, three different cases are possible:

1) the eigenvalues $\lambda_{1,2} = \frac{a+d\pm\sqrt{D}}{2}$ of matrix A are positive, where $D = (a-d)^2+4b^2$, *i.e.* matrix A is a positive-definite matrix. In this case, the singular decomposition coincides with the spectral decomposition, *i.e.* $\sigma_1 = \lambda_1$, $\sigma_2 = \lambda_2$ and $U\Lambda U^T = U\Sigma U^T$, where

$$\Lambda = \Sigma = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad U = \begin{pmatrix} \frac{b}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{\lambda_2-d}{\sqrt{(\lambda_2-d)^2+b^2}} \\ \frac{\lambda_1-a}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{b}{\sqrt{(\lambda_2-d)^2+b^2}} \end{pmatrix}.$$

2) the eigenvalues $\lambda_{1,2}$ of matrix A are of different signs and $|\lambda_1| > |\lambda_2|$, *i.e.* matrix A is an indeterminate matrix. In this case, the singular decomposition is the following: $\sigma_1 = |\lambda_1|$, $\sigma_2 = |\lambda_2|$ and
$$U\Sigma W^T = U|\Lambda|DU^T,$$
where $W^T = DU^T$, $D = \begin{pmatrix} \frac{\lambda_1}{|\lambda_1|} & 0 \\ 0 & \frac{\lambda_2}{|\lambda_2|} \end{pmatrix}$.

3) the eigenvalues $\lambda_{1,2}$ of matrix A are negative and $|\lambda_1| > |\lambda_2|$, *i.e.* matrix A is a negative-definite matrix. In this case, the singular decomposition is $\sigma_1 = |\lambda_1|$, $\sigma_2 = |\lambda_2|$ and

$$U\Sigma W^T = U|\Lambda|DU^T,$$

where $W^T = DU^T$, $D = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

That is, in general, the singular decomposition of the matrix A has the form $A = U\Sigma W^T$, where $\Sigma = |\Lambda|$, $W = UD$.

We will prove that $X(\cdot)$ is a solution of Cauchy problem (3.1) by the direct substitution of the set-valued mapping $X(t) = Ue^{\alpha^{-1}t^\alpha\Sigma}B_1(0)$ into differential equation (3.1) and by checking that the identity is satisfied:

$$D^\alpha\left(Ue^{\alpha^{-1}t^\alpha\Sigma}B_1(\mathbf{0})\right) \equiv AUe^{\alpha^{-1}t^\alpha\Sigma}B_1(\mathbf{0}). \tag{3.2}$$

Since $\sigma_{1,2} > 0$, then $e^{\alpha^{-1}\sigma_1 t^\alpha}$ and $e^{\alpha^{-1}\sigma_2 t^\alpha}$ are the increasing functions and as

$$e^{\alpha^{-1}\sigma_1 t^\alpha} > e^{\alpha^{-1}\sigma_2 t^\alpha},$$

then accordingly $diam(X(t)) = 2e^{\alpha^{-1}\sigma_1 t^\alpha}$ is an increasing function. Then, according to Definition 2.6, it follows that $B_1(\mathbf{0})$ is a centrally symmetric body and, accordingly, $(-1)B_1(\mathbf{0}) = B_1(\mathbf{0})$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0_+}\varepsilon^{-1}\left(X\left(t+\varepsilon t^{1-\alpha}\right)\stackrel{H}{-}X(t)\right) &= \lim_{\varepsilon\to 0_+}\varepsilon^{-1}\left(Ue^{\alpha^{-1}\left(t+\varepsilon t^{1-\alpha}\right)^{\alpha}\Sigma}B_1(\mathbf{0})\stackrel{H}{-}Ue^{\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0})\right)\\ &= U\lim_{\varepsilon\to 0_+}\varepsilon^{-1}\left(e^{\alpha^{-1}\left(t+\varepsilon t^{1-\alpha}\right)^{\alpha}\Sigma}-e^{\alpha^{-1}t^{\alpha}\Sigma}\right)B_1(\mathbf{0})\\ &= U\lim_{\varepsilon\to 0_+}\varepsilon^{-1}\begin{pmatrix} e^{\alpha^{-1}\sigma_1\left(t+\varepsilon t^{1-\alpha}\right)^{\alpha}}-e^{\alpha^{-1}\sigma_1 t^{\alpha}} & 0\\ 0 & e^{\alpha^{-1}\sigma_2\left(t+\varepsilon t^{1-\alpha}\right)^{\alpha}}-e^{\alpha^{-1}\sigma_2 t^{\alpha}}\end{pmatrix}B_1(\mathbf{0})\\ &= U\begin{pmatrix}\sigma_1 e^{\alpha^{-1}\sigma_1 t^{\alpha}} & 0\\ 0 & \sigma_2 e^{\alpha^{-1}\sigma_2 t^{\alpha}}\end{pmatrix}B_1(\mathbf{0}) = U\Sigma e^{\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0})\end{aligned}$$

and similarly

$$\begin{aligned}\lim_{\varepsilon\to 0_+}\varepsilon^{-1}\left(X(t)\stackrel{H}{-}X\left(t-\varepsilon t^{1-\alpha}\right)\right) &= U\lim_{\varepsilon\to 0_+}\varepsilon^{-1}\left(e^{\alpha^{-1}t^{\alpha}\Sigma}-e^{\alpha^{-1}\left(t-\varepsilon t^{1-\alpha}\right)^{\alpha}\Sigma}\right)B_1(\mathbf{0})\\ &= U\Sigma e^{\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0}).\end{aligned}$$

That is,

$$D^{\alpha}X(t) = D^{\alpha}\left(e^{\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0})\right) = U\Sigma e^{\ \alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0}).$$

Since the singular matrix decomposition of the symmetric matrix A has the form $A = U\Sigma DU^T$, then

$$AUe^{\ \alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0}) = U\Sigma DU^TUe^{\ \alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0}) = U\Sigma e^{\ \alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0}).$$

It is obvious that identity (3.2) holds and, accordingly, $X(\cdot)$ is a solution of Cauchy problem (3.1). The theorem is proved. □

Example 3.5. *Let* $A = \begin{pmatrix}0.8 & 0.5\\ 0.5 & 0.3\end{pmatrix}$. *Then the singular decomposition of the matrix A has the following form* $U\Sigma U^T = \begin{pmatrix}0.8507 & -0.5257\\ 0.5257 & 0.8507\end{pmatrix}\begin{pmatrix}1.1090 & 0\\ 0 & 0.0090\end{pmatrix}\begin{pmatrix}0.8507 & 0.5257\\ -0.5257 & 0.8507\end{pmatrix}$. *Accordingly, Cauchy problem* (3.1) *has a solution* $X(t) = Ue^{\ \alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0})$. *That is,*

1) *if* $\alpha = 0.25$, *then at every moment of time* $t \geq 0$ *the cross section* $X(t)$ *is an ellipse with semi-axes* $e^{4.4361\sqrt[4]{t}}$ *and* $e^{0.0361\sqrt[4]{t}}$, *rotated at an angle* $\theta \approx 33^{\circ}$, *which is determined by the matrix* U *(Figure 5);*

2) *if* $\alpha = 0.5$, *then at every moment of time* $t \geq 0$ *the cross section* $X(t)$ *is an ellipse with semi-axes* $e^{2.2368\sqrt{t}}$ *and* $e^{0.0298\sqrt{t}}$, *rotated at an angle* $\theta \approx 33^{\circ}$ *(Figure 6);*

3) *if* $\alpha = 0.75$, *then at every moment of time* $t \geq 0$ *the cross section* $X(t)$ *is an ellipse with semi-axes* $e^{1.4787\sqrt[4]{t^3}}$ *and* $e^{0.0120\sqrt[4]{t^3}}$, *rotated at an angle* $\theta \approx 33^{\circ}$ *(Figure 7);*

4) if $\alpha = 1$, then at every moment of time $t \geq 0$ the cross section $X(t)$ is an ellipse with semi-axes $e^{1.1090t}$ and $e^{0.009t}$, rotated at an angle $\theta \approx 33^\circ$ (Figure 8).

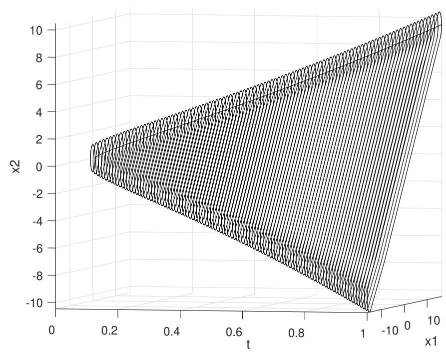

Figure 5: $\alpha = 0.25$, $X(t) = e^{4\sqrt[4]{t}\Sigma}B_1(\mathbf{0})$.

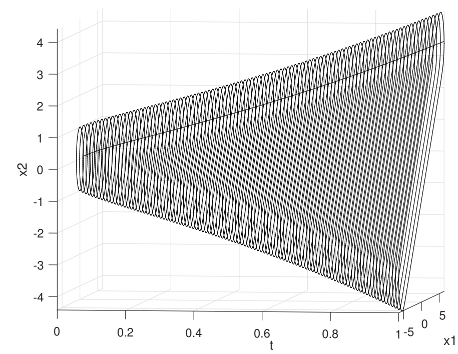

Figure 6: $\alpha = 0.5$, $X(t) = e^{2\sqrt{t}\Sigma}B_1(\mathbf{0})$.

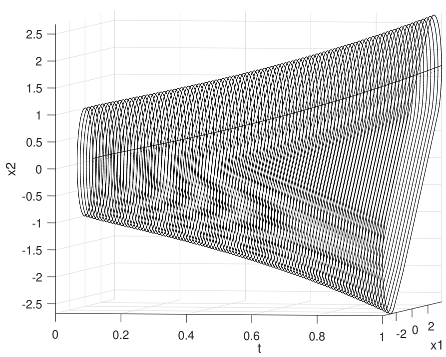

Figure 7: $\alpha = 0.75$, $X(t) = e^{\frac{4}{3}\sqrt[4]{t^3}\Sigma}B_1(\mathbf{0})$.

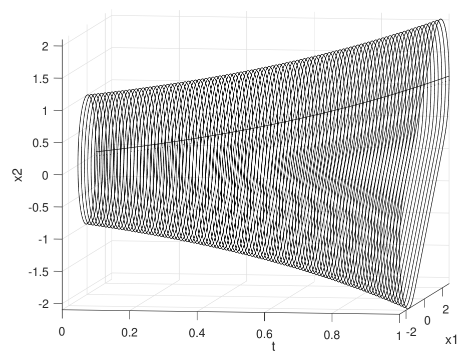

Figure 8: $\alpha = 1$, $X(t) = e^{t\Sigma}B_1(\mathbf{0})$.

4 A linear set-valued differential equation with a generalized conformable fractional derivative.

Let $X : [0,T] \to conv(\mathbb{R}^n)$ be a set-valued mapping.

Definition 4.1. *We say that a set-valued mapping $X(\cdot)$ has a **generalized conformable fractional derivative of order α** $D_g^\alpha X(t) \in conv(\mathbb{R}^n)$ at $t \in (0,T)$, if for all sufficiently small $\varepsilon > 0$ the Hukuhara differences and the limits exist in at least one of the following cases:*

i) $\lim\limits_{\varepsilon\to 0} \varepsilon^{-1}\left(X\left(t+\varepsilon t^{1-\alpha}\right) \overset{H}{-} X(t)\right) = \lim\limits_{\varepsilon\to 0} \varepsilon^{-1}\left(X(t) \overset{H}{-} X\left(t-\varepsilon t^{1-\alpha}\right)\right) = D_g^\alpha X(t)$,

ii) $\lim\limits_{\varepsilon\to 0} \varepsilon^{-1}\left(X(t) \overset{H}{-} X\left(t+\varepsilon t^{1-\alpha}\right)\right) = \lim\limits_{\varepsilon\to 0} \varepsilon^{-1}\left(X\left(t-\varepsilon t^{1-\alpha}\right) \overset{H}{-} X(t)\right) = D_g^\alpha X(t)$, *iii)* $\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon t^{1-\alpha}\right)\stackrel{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t-\varepsilon t^{1-\alpha}\right)\stackrel{H}{-}X(t)\right)=D_g^\alpha X(t)$,

iv) $\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\stackrel{H}{-}X\left(t+\varepsilon t^{1-\alpha}\right)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\stackrel{H}{-}X\left(t-\varepsilon t^{1-\alpha}\right)\right)=D_g^\alpha X(t)$.

Definition 4.2. *If a generalized conformable fractional derivative of order α $D_g^\alpha X(t)$ exists for all $t\geq 0$, then we will say that the set-valued mapping $X(\cdot)$ is* ***generalized α-differentiable*** *on $\mathbb{R}_+$.*

Remark 4.3. *Obviously, if the set-valued mapping $X(\cdot)$ is α-differentiable at a point $t>0$, then the set-valued mapping $X(\cdot)$ is generalized α-differentiable at a point $t>0$.*

Lemma 4.4. *If the set-valued mapping $X(\cdot)$ is generalized α-differentiable at a point $t>0$, then*

$$D_g^\alpha X(t)=t^{1-\alpha}D_gX(t),$$

where $D_gX(t)$ is the generalized derivative [25, 28, 45].

Proof. If the set-valued mapping $X(\cdot)$ is generalized α-differentiable at a point $t>0$, then at least one of the conditions of Definition 4.1 must be fulfilled. We will assume that the first condition is fulfilled, *i.e.*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon t^{1-\alpha}\right)\stackrel{H}{-}X\big(t\big)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\big(t\big)\stackrel{H}{-}X\left(t-\varepsilon t^{1-\alpha}\right)\right)=D_g^\alpha X(t).$$

Let $\theta=\varepsilon t^{1-\alpha}$. Then

$$\begin{aligned}D_g^\alpha X(t)&=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon t^{1-\alpha}\right)\stackrel{H}{-}X\big(t\big)\right)=\lim_{\theta\to 0}t^{1-\alpha}\theta^{-1}\left(X\left(t+\theta\right)\stackrel{H}{-}X\big(t\big)\right)\\&=t^{1-\alpha}\lim_{\theta\to 0}\theta^{-1}\left(X\left(t+\theta\right)\stackrel{H}{-}X\big(t\big)\right)=t^{1-\alpha}D_gX(t).\end{aligned}$$

Likewise,

$$\begin{aligned}D_g^\alpha X(t)&=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\big(t\big)\stackrel{H}{-}X\left(t-\varepsilon t^{1-\alpha}\right)\right)=\lim_{\theta\to 0}t^{1-\alpha}\theta^{-1}\left(X\big(t\big)\stackrel{H}{-}X\left(t-\theta\right)\right)\\&=t^{1-\alpha}\lim_{\theta\to 0}\theta^{-1}\left(X\big(t\big)\stackrel{H}{-}X\left(t-\theta\right)\right)=t^{1-\alpha}D_gX(t).\end{aligned}$$

It is similarly proved if the second, third or fourth conditions are fulfilled. The lemma is proved. □

Remark 4.5. *It follows from Lemma 4.4 that a necessary and sufficient condition for the existence of a generalized conformable fractional derivative $D_g^\alpha X(t)$ is the existence of a generalized derivative $D_gX(t)$.*

Also, it is easy to see that if $\alpha=1$, then $D_g^1X(t)=D_gX(t)$.

Consider the following Cauchy problem for linear set-valued differential equation with a generalized conformable fractional derivative of order α

$$D_g^{\alpha} X(t) = AX(t), \quad X(0) = B_1(\mathbf{0}), \tag{4.1}$$

where $X : \mathbb{R}_+ \to conv(\mathbb{R}^2)$ is a set-valued mapping, $A \in \mathbb{R}^{2\times 2}$ is a nondegenerate matrix.

Definition 4.6. *A set-valued mapping $X : \mathbb{R}_+ \to conv(\mathbb{R}^2)$ is called a solution of Cauchy problem* (4.1) *if it is continuous and satisfies differential equation* (4.1) *for all $t \geq 0$ and $X(0) = B_1(\mathbf{0})$.*

Remark 4.7. *It follows from Remark 4.3 that if the set-valued mapping $X(t)$ is a solution of equation* (3.1)*, then it is a solution of equation* (4.1)*.*

Remark 4.8. *In [25, 27, 28] a Cauchy problem for linear set-valued differential equation with a generalized derivative*

$$D_g X(t) = AX(t), \quad X(0) = B_1(\mathbf{0}) \tag{4.2}$$

was considered and the following results were obtained:

1) *Cauchy problem* (4.2) *has an infinite number of solutions, some (one or two) of which are called basic (their diameter are monotone functions), and others are mixed (their diameter are non-monotone functions). We also note that the first basic solution $X_1(\cdot)$ is the solution of Cauchy problem* (4.2)*, that satisfies the condition that $diam(X_1(t))$ is a nondecreasing function and is also the solution of the corresponding differential equation with the Hukuhara derivative. The second basic solution $X_2(\cdot)$ is called the solution of Cauchy problem* (4.2)*, that satisfies the condition that $diam(X_2(t))$ is a decreasing function;*

2) *if the singular numbers of the matrix A are such that $\sigma_1 = \sigma_2 = \sigma$, then Cauchy problem* (4.2) *has two basic solutions $X_1(t)$ and $X_2(t)$, whose cross-sections at each moment of time t are circles $B_{e^{\sigma t}}(0)$ and $B_{e^{-\sigma t}}(0)$, and if the singular numbers of the matrix A are such that $\sigma_1 \neq \sigma_2$, then Cauchy problem* (4.2) *has only the first basic solution $X_1(t)$, whose cross-section at each moment of time t is an ellipse with semiaxes equal to $e^{\sigma_1 t}$ and $e^{\sigma_2 t}$.*

Next, we obtain the results similar to Theorems 3.2 and 3.4.

Theorem 4.9. *If the matrix A satisfies the condition $\delta = 0$, then Cauchy problem* (4.1) *has two basic solutions $X_1(\cdot)$ and $X_2(\cdot)$ such that*

$$X_1(t) = e^{\,\beta t^{\alpha}} B_1(\mathbf{0}) \quad \text{and} \quad X_2(t) = e^{\,-\beta t^{\alpha}} B_1(\mathbf{0}),$$

where $t \geq 0$, $\beta = \dfrac{\sqrt{a^2+b^2}}{\alpha}$.

Proof. From Theorem 3.2, we have that the set-valued mapping $X_1(t)$ is a solution of Cauchy problem (3.1) and the function $diam(X(t))$ is non-decreasing. Then, taking into account Remark 4.3, $X_1(t)$ is the first basic solution of equation (4.1).

We will prove that $X_2(\cdot)$ is a solution of Cauchy problem (4.1) by the direct substitution of the set-valued mapping $X_2(t) = e^{-\beta t^\alpha} B_1(\mathbf{0})$ into differential equation (4.1) and by checking that the identity is satisfied:

$$D_g^\alpha \left(e^{-\beta t^\alpha} B_1(\mathbf{0}) \right) \equiv A e^{-\beta t^\alpha} B_1(\mathbf{0}).$$

Since $\beta > 0$, then $e^{-\beta t^\alpha}$ is a decreasing function, and as

$$e^{-\beta t^\alpha} B_1(\mathbf{0}) = B_{e^{-\beta t^\alpha}}(\mathbf{0}),$$

then, accordingly, the function $diam(X_2(\cdot))$ is a decreasing function. Then according to Definition 4.1 ii) and that the ball $B_1(\mathbf{0})$ is a centrally symmetric body and $(-1)B_1(\mathbf{0}) = B_1(\mathbf{0})$, we have

$$\begin{aligned} \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(X_2(t) \overset{H}{-} X_2\left(t + \varepsilon t^{1-\alpha}\right)\right) &= \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{-\beta t^\alpha} B_1(\mathbf{0}) \overset{H}{-} e^{-\beta\left(t+\varepsilon t^{1-\alpha}\right)^\alpha} B_1(\mathbf{0}) \right) \\ &= \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{-\beta t^\alpha} - e^{-\beta\left(t+\varepsilon t^{1-\alpha}\right)^\alpha} \right) B_1(\mathbf{0}) = -\alpha\beta e^{-\beta t^\alpha} B_1(\mathbf{0}) = \alpha\beta e^{-\beta t^\alpha} B_1(\mathbf{0}) \end{aligned}$$

and

$$\begin{aligned} \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(X_2\left(t - \varepsilon t^{1-\alpha}\right) \overset{H}{-} X_2(t)\right) &= \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{-\beta\left(t-\varepsilon t^{1-\alpha}\right)^\alpha} B_1(\mathbf{0}) \overset{H}{-} e^{-\beta t^\alpha} B_1(\mathbf{0}) \right) \\ &= \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{-\beta\left(t-\varepsilon t^{1-\alpha}\right)^\alpha} - e^{-\beta t^\alpha} \right) B_1(\mathbf{0}) = \alpha\beta e^{-\beta t^\alpha} B_1(\mathbf{0}). \end{aligned}$$

That is,

$$D_g^\alpha X_2(t) = D^\alpha \left(e^{-\beta t^\alpha} B_1(\mathbf{0}) \right) = \alpha\beta e^{-\beta t^\alpha} B_1(\mathbf{0}).$$

Since the matrix A satisfies the condition $\delta = 0$, the singular decomposition of the matrix A has the form $A = U\Sigma V^T$, where U, V are orthogonal matrices, $\Sigma = \sigma I$, $\sigma = \sqrt{a^2 + b^2}$. As $V^T B_r(\mathbf{0}) = B_r(\mathbf{0})$ and $U B_r(\mathbf{0}) = B_r(\mathbf{0})$ for all $r > 0$, then

$$\begin{aligned} A e^{-\beta t^\alpha} B_1(\mathbf{0}) &= U\Sigma V^T e^{-\beta t^\alpha} B_1(\mathbf{0}) = U\sigma E V^T e^{-\beta t^\alpha} B_1(\mathbf{0}) = \sigma U E V^T e^{-\beta t^\alpha} B_1(\mathbf{0}) \\ &= \sigma e^{-\beta t^\alpha} U E V^T B_1(\mathbf{0}) = \sigma e^{-\beta t^\alpha} B_1(\mathbf{0}). \end{aligned}$$

Since $\alpha\beta = \sigma$, we have

$$D_g^\alpha X_2(t) = \sigma e^{-\beta t^\alpha} B_1(\mathbf{0}) \equiv \sigma e^{-\beta t^\alpha} B_1(\mathbf{0}) = A X_2(t),$$

i.e. $X_2(\cdot)$ is the second basic solution of Cauchy problem (4.1). Thus the theorem is proved. □

Example 4.10. *Let* $A = \begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}$. *Then the singular numbers* σ_1 *and* σ_2 *of the matrix* A *are equal:* $\sigma_1 = \sigma_2 = \sigma = 2$.

Accordingly, Cauchy problem (4.1) *has solutions* $X_1(t) = e^{\,2\alpha^{-1}t^{\alpha}} B_1(\mathbf{0})$ *and* $X_2(t) = e^{\,-2\alpha^{-1}t^{\alpha}} B_1(\mathbf{0})$. *Below are the solutions for cases* $\alpha = 1$ *(Fig. 9, 10) and* $\alpha = 0.5$ *(Fig. 11, 12).*

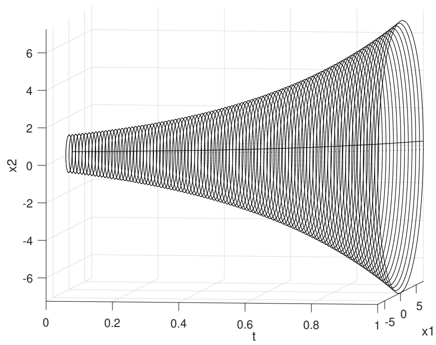

Figure 9: If $\alpha = 1$, then $X_1(t) = e^{2t} B_1(\mathbf{0})$.

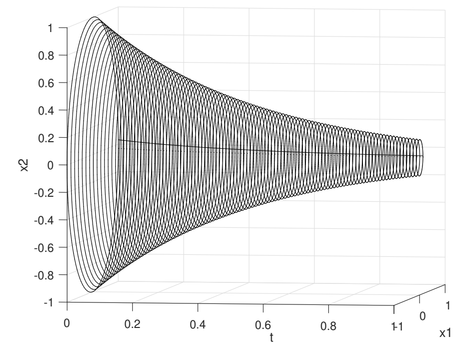

Figure 10: If $\alpha = 1$, then $X_2(t) = e^{-2t} B_1(\mathbf{0})$.

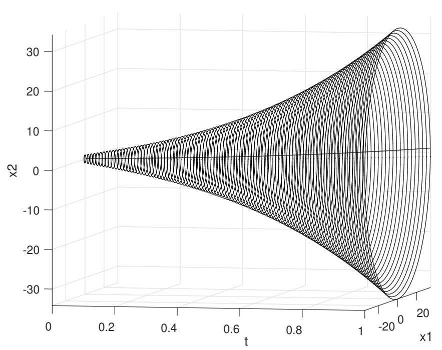

Figure 11: If $\alpha = 0.5$, then $X_1(t) = e^{4\sqrt{t}} B_1(\mathbf{0})$.

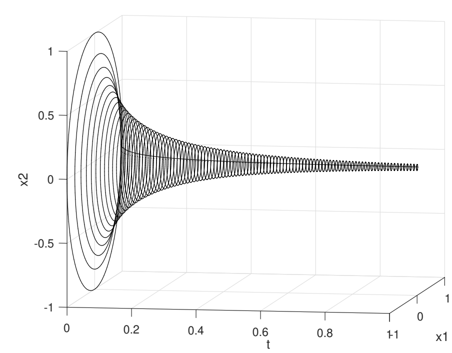

Figure 12: If $\alpha = 0.5$, then $X_2(t) = e^{-4\sqrt{t}} B_1(\mathbf{0})$.

Theorem 4.11. *If matrix* A *is symmetric and* $d \neq -a$, *then Cauchy problem* (4.1) *has only the first basic solution* $X_1(\cdot)$ *such that*

$$X_1(t) = U e^{\,\alpha^{-1}t^{\alpha}\Sigma} B_1(\mathbf{0}), \quad t \geq 0,$$

where $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$, $\sigma_{1,2} = |\lambda_{1,2}| = \left| \frac{a+d\pm\sqrt{(a-d)^2+4b^2}}{2} \right|$, $U = \begin{pmatrix} \frac{b}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{\lambda_2-d}{\sqrt{(\lambda_2-d)^2+b^2}} \\ \frac{\lambda_1-a}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{b}{\sqrt{(\lambda_2-d)^2+b^2}} \end{pmatrix}$.

Proof. According to Remark 4.7, the first basic solution of Cauchy problem (4.1) is also a solution of equation (3.1). Then, according to Theorem 3.4, the set-valued mapping $X_1(t) = Ue^{\alpha^{-1}t^\alpha\Sigma}B_1(\mathbf{0})$ is the first basic solution of Cauchy problem (4.1).

Now we will prove that the second basic solution $X_2(\cdot)$ of Cauchy problem (4.1) does not exist. We will prove it by contradiction. Let Cauchy problem (4.1) have the second basic solution $X_2(\cdot)$. Then $X_2(\cdot)$ satisfies the following integral equation

$$X_2(t) + A\int_0^t s^{\alpha-1}X_2(s)ds = B_1(\mathbf{0}).$$

Let us fix an arbitrary $T > 0$. Then $X_2(T) + A\int_0^T s^{\alpha-1}X_2(s)ds = B_1(\mathbf{0})$. From here,

$$B_1(0)\stackrel{H}{-}X_2(T) = A\int_0^T s^{\alpha-1}X_2(s)ds.$$

From Lemma 2.3, as $B_1(\mathbf{0})$ is a ball and Hukuhara difference $B_1(\mathbf{0})\stackrel{H}{-}X_2(T)$ exists, then $X_2(T)$ is a ball, *i.e.* $X_2(T) \equiv B_{r(T)}(\mathbf{0})$, where $0 \leq r(T) \leq 1$. As T is arbitrary, then $X_2(t) \equiv B_{r(t)}(\mathbf{0})$ for all $t \geq 0$. Hence,

$$\int_0^T s^{\alpha-1}X_2(s)ds = \int_0^T s^{\alpha-1}B_{r(s)}(\mathbf{0})ds = \int_0^T s^{\alpha-1}r(s)dsB_1(\mathbf{0}) = R(T)B_1(0) = B_{R(T)}(\mathbf{0}),$$

where $R(T) = \int_0^T s^{\alpha-1}r(s)ds$.

That is, we have

$$B_{r(T)}(\mathbf{0}) + AB_{R(T)}(\mathbf{0}) = B_1(\mathbf{0}). \tag{4.3}$$

Since the matrix A has two different singular numbers, then $AB_{R(T)}(0)$ is an ellipse. So, the set $B_{r(T)}(\mathbf{0}) + AB_{R(T)}(\mathbf{0})$ is not a ball. That is, equality (4.3) is not fulfilled and we have obtained a contradiction. The theorem is proved. □

Conclusion

In conclusion, we present some remarks.

Remark 4.12. *If in Definition 2.6 we replace equality* (2.1) *by the equality*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t\,e^{\varepsilon t^{-\alpha}}\right)\overset{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t\,e^{-\varepsilon t^{-\alpha}}\right)\right)=Z, \tag{4.4}$$

or

$$\lim_{\varepsilon\to 0}e^{\varepsilon^{-1}}\left(X\left(t+e^{-\varepsilon^{-1}}t^{1-\alpha}\right)\overset{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}e^{\varepsilon^{-1}}\left(X(t)\overset{H}{-}X\left(t-e^{-\varepsilon^{-1}}t^{1-\alpha}\right)\right)=Z, \tag{4.5}$$

then we obtain a generalization of the conformable fractional derivative of order α of a single-valued function [19] or [21] for set-valued mappings. Similarly, as it was done in [34], it is possible to prove the validity of Lemmas 2.8–2.17, which makes it possible to introduce the corresponding generalized conformable fractional derivative of order α, consider the corresponding differential equations, and prove theorems similar to Theorems 3.2–4.11, and since in this case $D^\alpha X(t)=t^{1-\alpha}D_H X(t)$, then the analytical formulas of the solutions will also be the same.

Remark 4.13. *If in Definition 2.6 we replace equality (2.1) by the equality*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon e^{(\alpha-1)t}\right)\overset{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t-\varepsilon e^{(\alpha-1)t}\right)\right)=Z, \tag{4.6}$$

then we obtain a generalization of the conformable fractional derivative of order α of a single-valued function [18] for set-valued mappings. Similarly, as it was done in [34], it is possible to prove the validity of Lemmas 2.8–2.17, which makes it possible to introduce the corresponding generalized conformable fractional derivative of order α, consider the corresponding differential equations, and prove theorems similar to Theorems 3.2–4.11. However, since in this case $D^\alpha X(t)=e^{(\alpha-1)t}D_H X(t)$, then the analytical formulas of solutions will have the following form:

Theorem 3.2: $X(t)=e^{\frac{\sigma}{1-\alpha}e^{(1-\alpha)t}}B_1(\mathbf{0})$*;*

Theorem 3.4: $X(t)=Ue^{\frac{1}{1-\alpha}e^{(1-\alpha)t}\Sigma}B_1(\mathbf{0})$*;*

Theorem 4.9: $X_1(t)=e^{\frac{\sigma}{1-\alpha}e^{(1-\alpha)t}}B_1(\mathbf{0})$, $X_2(t)=e^{\frac{\sigma}{\alpha-1}e^{(1-\alpha)t}}B_1(\mathbf{0})$*;*

Theorem 4.11: $X_1(t)=Ue^{\frac{1}{1-\alpha}e^{(1-\alpha)t}\Sigma}B_1(\mathbf{0})$.

Remark 4.14. *If in Definition 2.6 we replace equality* (2.1) *by the equality*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right)\overset{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t-\varepsilon\left(t+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right)\right)=Z, \quad (4.7)$$

where $\Gamma(\alpha)$ *is gamma function, then we obtain a generalization of the conformable fractional derivative of order* α *of a single-valued function [4] for set-valued mappings. Similarly, as it was done in [34], it is possible to prove the validity of Lemmas 2.8–2.17, which makes it possible to introduce the corresponding generalized conformable fractional derivative of order* α*, consider the corresponding differential equations, and prove theorems similar to Theorems 3.2–4.11. However, since in this case* $D^\alpha X(t)=\left(t+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}D_H X(t)$*, then the analytical formulas of solutions will have the following form:*

Theorem 3.2: $X(t)=e^{\frac{\sigma}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}B_1(\mathbf{0})$*;*

Theorem 3.4: $X(t)=Ue^{\frac{1}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\Sigma}B_1(\mathbf{0})$*;*

Theorem 4.9: $X_1(t)=e^{\frac{\sigma}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}B_1(\mathbf{0})$*,* $X_2(t)=e^{-\frac{\sigma}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}B_1(\mathbf{0})$*;*

Theorem 4.11: $X_1(t)=Ue^{\frac{1}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\Sigma}B_1(\mathbf{0})$.

Remark 4.15. *If in Definition 2.6 we replace equality* (2.1) *by the equality*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon k(t)^{1-\alpha}\right)\overset{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t-\varepsilon k(t)^{1-\alpha}\right)\right)=Z, \quad (4.8)$$

where $k(t)$ *is a continuous positive function for all* $t\geq 0$*, then we obtain a generalization of the conformable fractional derivative of order* α *of a single-valued function [2, 15] for set-valued mappings. Similarly, as it was done in [34], it is possible to prove the validity of Lemmas 2.8–2.17, which makes it possible to introduce the corresponding generalized conformable fractional derivative of order* α*, consider the corresponding differential equations, and prove theorems similar to Theorems 3.2–4.11. However, since in this case* $D^\alpha X(t)=k(t)^{1-\alpha}D_H X(t)$*, then the analytical formulas of solutions will have the following form:*

Theorem 3.2: $X(t)=e^{\sigma\int_0^t (k(s))^{\alpha-1}ds}B_1(\mathbf{0})$*;*

Theorem 3.4: $X(t)=Ue^{\int_0^t (k(s))^{\alpha-1}ds\Sigma}B_1(\mathbf{0})$

Theorem 4.9: $X_1(t)=e^{\sigma\int_0^t (k(s))^{\alpha-1}ds}B_1(\mathbf{0})$*,* $X_2(t)=e^{-\sigma\int_0^t (k(s))^{\alpha-1}ds}B_1(\mathbf{0})$*;*

Theorem 4.11: $X_1(t)=Ue^{\int_0^t (k(s))^{\alpha-1}ds\Sigma}B_1(\mathbf{0})$.

Remark 4.16. *If in Definition 2.6 we replace equality* (2.1) *by the equality*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+k(t)-k(t)e^{\varepsilon\frac{k(t)^{-\alpha}}{|k'(t)|}}\right)\overset{H}{-}X\left(t\right)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t\right)\overset{H}{-}X\left(t+k(t)-k(t)e^{-\varepsilon\frac{k(t)^{-\alpha}}{|k'(t)|}}\right)\right)=Z, \tag{4.9}$$

where $k(t)$ is a differentiable function for all $t\geq 0$ such that $k(t)>0$ and $k'(t)\neq 0$ for all $t\geq 0$, then we obtain a generalization of the conformable fractional derivative of order α of a single-valued function [1] for set-valued mappings. Similarly, as it was done in [34], it is possible to prove the validity of Lemmas 2.8–2.17, which makes it possible to also introduce the corresponding generalized conformable fractional derivative of order α, consider the corresponding differential equations, and prove theorems similar to Theorems 3.2–4.11. However, since in this case $D^\alpha X(t)=\frac{k(t)^{1-\alpha}}{k'(t)}D_H X(t)$, then the analytical formulas of solutions will have the following form:

Theorem 3.2: $X(t)=e^{\sigma\alpha^{-1}(k(t)^\alpha-k(0)^\alpha)}B_1(\mathbf{0})$;

Theorem 3.4: $X(t)=Ue^{\alpha^{-1}(k(t)^\alpha-k(0)^\alpha)\Sigma}B_1(\mathbf{0})$;

Theorem 4.9: $X_1(t)=e^{\sigma\alpha^{-1}(k(t)^\alpha-k(0)^\alpha)}B_1(\mathbf{0})$, $X_2(t)=e^{\sigma\alpha^{-1}(k(0)^\alpha-k(t)^\alpha)}B_1(\mathbf{0})$;

Theorem 4.11: $X_1(t)=Ue^{\sigma\alpha^{-1}(k(t)^\alpha-k(0)^\alpha)\Sigma}B_1(\mathbf{0})$.

Remark 4.17. *We also note that if in Definition 2.6 we replace equality* (2.1) *by the equality*

$$\lim_{\varepsilon\to 0}((t+\varepsilon)^\alpha-t^\alpha)^{-1}\left(X\left(t+\varepsilon\right)\overset{H}{-}X\big(t\big)\right)=\lim_{\varepsilon\to 0}(t^\alpha-(t-\varepsilon)^\alpha)^{-1}\left(X\big(t\big)\overset{H}{-}X\left(t-\varepsilon\right)\right)=Z, \tag{4.10}$$

then we obtain a generalization of the ***Chen-Hausdorff fractal derivative of order α*** *of a single-valued function [9, 10] for set-valued mappings. Similarly, as it was done in [34], it is possible to prove the validity of Lemmas 2.8–2.17, which makes it possible to introduce the corresponding generalized Chen-Hausdorff fractal derivative of order α, consider the corresponding differential equations, and prove theorems similar to Theorems 3.2–4.11. However, since in this case $D^\alpha X(t)=\alpha^{-1}t^{1-\alpha}D_H X(t)$, then the analytical formulas of solutions will have the following form:*

Theorem 3.2: $X(t)=e^{\sigma t^\alpha}B_1(\mathbf{0})$;

Theorem 3.4: $X(t)=Ue^{t^\alpha\Sigma}B_1(\mathbf{0})$;

Theorem 4.9: $X_1(t)=e^{\sigma t^\alpha}B_1(\mathbf{0})$, $X_2(t)=e^{-\sigma t^\alpha}B_1(\mathbf{0})$;

Theorem 4.11: $X_1(t)=Ue^{t^\alpha\Sigma}B_1(\mathbf{0})$.

References

[1] A. Akkurt, M. E. Yıldırım, and H. Yıldırım, “A new generalized fractional derivative and integral,” *Konuralp J. Math.*, vol. 5, no. 2, pp. 248–259, 2017.

[2] R. Almeida, M. Guzowska, and T. Odzijewicz, “A remark on local fractional calculus and ordinary derivatives,” *Open Math.*, vol. 14, no. 1, pp. 1122–1124, 2016, doi: 10.1515/math-2016-0104.

[3] Ş. E. Amrahov, A. Khastan, N. Gasilov, and A. G. Fatullayev, “Relationship between Bede-Gal differentiable set-valued functions and their associated support functions,” *Fuzzy Sets and Systems*, vol. 295, pp. 57–71, 2016, doi: 10.1016/j.fss.2015.12.002.

[4] A. Atangana and E. F. Doungmo Goufo, “Extension of matched asymptotic method to fractional boundary layers problems,” *Math. Probl. Eng.*, 2014, Art. ID 107535, doi: 10.1155/2014/107535.

[5] H. T. Banks and M. Q. Jacobs, “A differential calculus for multifunctions,” *J. Math. Anal. Appl.*, vol. 29, pp. 246–272, 1970, doi: 10.1016/0022-247X(70)90078-8.

[6] T. F. Bridgland, Jr., “Trajectory integrals of set valued functions,” *Pacific J. Math.*, vol. 33, pp. 43–68, 1970.

[7] S. Chakraverty, S. Tapaswini, and D. Behera, *Fuzzy differential equations and applications for engineers and scientists.* CRC Press, Boca Raton, FL, 2017.

[8] Y. Chalco-Cano, H. Román-Flores, and M. D. Jiménez-Gamero, “Generalized derivative and π-derivative for set-valued functions,” *Inform. Sci.*, vol. 181, no. 11, pp. 2177–2188, 2011, doi: 10.1016/j.ins.2011.01.023.

[9] W. Chen, “Time–space fabric underlying anomalous diffusion,” *Chaos, Solitons & Fractals*, vol. 28, no. 4, pp. 923–929, 2006, doi: 10.1016/j.chaos.2005.08.199.

[10] W. Chen, H. Sun, X. Zhang, and D. Korošak, “Anomalous diffusion modeling by fractal and fractional derivatives,” *Comput. Math. Appl.*, vol. 59, no. 5, pp. 1754–1758, 2010, doi: 10.1016/j.camwa.2009.08.020.

[11] F. S. de Blasi, V. Lakshmikantham, and T. G. Bhaskar, “An existence theorem for set differential inclusions in a semilinear metric space,” *Control Cybernet.*, vol. 36, no. 3, pp. 571–582, 2007.

[12] F. De Blasi and F. Iervolino, “Equazioni differenziali con soluzioni a valore compatto convesso,” *Boll. Unione Mat. Ital*, vol. 2, no. 4-5, pp. 491–501, 1969.

[13] E. C. de Oliveira and J. A. Tenreiro Machado, “A review of definitions for fractional derivatives and integral,” *Math. Probl. Eng.*, 2014, Art. ID 238459, doi: 10.1155/2014/238459.

[14] G. E. Forsythe and C. B. Moler, *Computer solution of linear algebraic systems.* Prentice-Hall, Inc., Englewood Cliffs, NJ, 1967.

[15] P. M. Guzmán, G. Langton, L. M. Lugo Motta Bittencurt, J. Medina, and J. E. Nápoles Valdes, “A new definition of a fractional derivative of local type,” *J. Math. Anal.*, vol. 9, no. 2, pp. 88–98, 2018.

[16] R. A. Horn and C. R. Johnson, *Matrix analysis*, 2nd ed. Cambridge University Press, Cambridge, 2013.

[17] M. Hukuhara, “Conformable fractional derivatives and it is applications for solving fractional differential equations,” *IOSR Journal of Mathematics*, vol. 13, no. 2, pp. 81–87, 2017, doi: 10.9790/5728-1302028187.

[18] A. Kajouni, A. Chafiki, K. Hilal, and M. Oukessou, “A new conformable fractional derivative and applications,” *Int. J. Differ. Equ.*, 2021, Art. ID 6245435, doi: 10.1155/2021/6245435.

[19] R. M. Kamble and S. S. Zampalwad, “New generalized definition of conformable fractional derivative,” *International Journal of Modern Developments in Engineering and Science*, vol. 1, no. 2, pp. 1–5, 2022.

[20] A. M. Kareem, “Intégration des applications mesurables dont la valeur est un compact convexe,” *Funkcial. Ekvac.*, vol. 10, pp. 205–223, 1967.

[21] U. N. Katugampola, “A new fractional derivative with classical properties,” 2014, *arXiv:1410.6535*.

[22] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, “A new definition of fractional derivative,” *J. Comput. Appl. Math.*, vol. 264, pp. 65–70, 2014, doi: 10.1016/j.cam.2014.01.002.

[23] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, ser. North-Holland Mathematics Studies. Elsevier Science B.V., Amsterdam, 2006, vol. 204.

[24] T. A. Komleva and A. V. Plotnikov, “Differential inclusions with the Hukuhara derivative,” *Nelīnīĭnī Koliv.*, vol. 10, no. 2, pp. 229–246, 2007, doi: 10.1007/s11072-007-0017-x.

[25] T. A. Komleva, A. V. Plotnikov, L. I. Plotnikova, and N. V. Skripnik, “Conditions for the existence of basic solutions of linear multivalued differential equations,” *Ukrainian Math. J.*, vol. 73, no. 5, pp. 758–783, 2021.

[26] T. A. Komleva, A. V. Plotnikov, and N. V. Skripnik, “Differential equations with multivalued solutions,” *Ukraïn. Mat. Zh.*, vol. 60, no. 10, pp. 1326–1337, 2008, doi: 10.1007/s11253-009-0150-z.

[27] T. O. Komleva, A. V. Plotnikov, and N. V. Skripnik, “Existence of solutions of linear set-valued integral equations and their properties,” *J. Math. Sci. (N.Y.)*, vol. 277, no. 2, pp. 268–280, 2023, doi: 10.1007/s10958-023-06831-1.

[28] T. A. Komleva, L. I. Plotnikova, N. V. Skripnik, and A. V. Plotnikov, “Some remarks on linear set-valued differential equations,” *Stud. Univ. Babeş-Bolyai Math.*, vol. 65, no. 3, pp. 411–427, 2020, doi: 10.24193/subbmath.2020.3.09.

[29] V. Lakshmikantham, T. G. Bhaskar, and J. Vasundhara Devi, *Theory of set differential equations in metric spaces.* Cambridge Scientific Publishers, Cambridge, 2006.

[30] V. Lakshmikantham and R. N. Mohapatra, *Theory of fuzzy differential equations and inclusions*, ser. Series in Mathematical Analysis and Applications. Taylor & Francis Group, London, 2003, vol. 6, doi: 10.1201/9780203011386.

[31] V. Lakshmikantham, S. Leela, and J. Vasundhara Devi, *Theory of fractional dynamic systems.* Cambridge Scientific Publ., 2009.

[32] A. Lasota and A. Strauss, “Asymptotic behavior for differential equations which cannot be locally linearized,” *J. Differential Equations*, vol. 10, pp. 152–172, 1971, doi: 10.1016/0022-0396(71)90103-3.

[33] M. Martelli and A. Vignoli, “On differentiabiliy of multi-valued maps,” *Boll. Un. Mat. Ital. (4)*, vol. 10, pp. 701–712, 1974.

[34] A. A. Martynyuk, “A fractional-like Hukuhara derivative and its properties,” *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki*, no. 4, pp. 10–16, 2019, doi: 10.15407/dopovidi2019.04.010.

[35] A. A. Martynyuk, G. T. Stamov, and I. M. Stamova, “Fractional-like Hukuhara derivatives in the theory of set-valued differential equations,” *Chaos, Solitons & Fractals*, vol. 131, 2020, Art. ID 109487, doi: 10.1016/j.chaos.2019.109487.

[36] A. A. Martynyuk, *Qualitative analysis of set-valued differential equations.* Birkhäuser/Springer, Cham, 2019, doi: 10.1007/978-3-030-07644-3.

[37] N. A. Perestyuk, V. A. Plotnikov, A. M. Samoilenko, and N. V. Skripnik, *Differential equations with impulse effects*, ser. De Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, 2011, vol. 40, multivalued right-hand sides with discontinuities.

[38] M. Piszczek, "On a multivalued second order differential problem with Hukuhara derivative," *Opuscula Math.*, vol. 28, no. 2, pp. 151–161, 2008.

[39] A. V. Plotnikov, "Differentiation of multivalued mappings. The T-derivative," *Ukraïn. Mat. Zh.*, vol. 52, no. 8, pp. 1119–1126, 2000, doi: 10.1023/A:1010361206391.

[40] A. V. Plotnikov and N. V. Skripnik, "Conditions for the existence of local solutions of set-valued differential equations with generalized derivative," *Ukrainian Math. J.*, vol. 65, no. 10, pp. 1498–1513, 2014, doi: 10.1007/s11253-014-0875-1.

[41] A. V. Plotnikov and A. V. Tumbrukaki, "Integrodifferential equations with multivalued solutions," *Ukraïn. Mat. Zh.*, vol. 52, no. 3, pp. 359–367, 2000, doi: 10.1007/BF02513136.

[42] A. V. Plotnikov, T. A. Komleva, and I. V. Molchanyuk, "Existence and uniqueness theorem for set-valued Volterra-Hammerstein integral equations," *Asian-Eur. J. Math.*, vol. 11, no. 3, 2018, Art. ID 1850036, doi: 10.1142/S1793557118500365.

[43] A. V. Plotnikov, T. A. Komleva, and L. I. Plotnikova, "Averaging of a system of set-valued differential equations with the hukuhara derivative," *Journal of Uncertain Systems*, vol. 13, no. 1, pp. 3–13, 2019.

[44] A. V. Plotnikov and N. Skripnik, "An existence and uniqueness theorem to the Cauchy problem for generalised set differential equations," *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, vol. 20, no. 4, pp. 433–445, 2013.

[45] A. V. Plotnikov and N. V. Skripnik, "Set-valued differential equations with generalized derivative," *J. Adv. Res. Pure Math.*, vol. 3, no. 1, pp. 144–160, 2011, doi: 10.5373/jarpm.475.062210.

[46] A. Plotnikov and N. Skripnik, *Differential equations with clear and fuzzy multivalued right-hand sides.* Asymptotics Methods. AstroPrint, Odessa, 2009.

[47] A. Plotnikov and N. Skripnik, "Existence and uniqueness theorems for generalized set differential equations," *Int. J. Control Sc. Eng*, vol. 2, no. 1, pp. 1–6, 2012.

[48] V. Plotnikov, A. Plotnikov, and A. Vityuk, *Differential equations with a multivalued right-hand side: Asymptotic methods.* Odessa: AstroPrint, 1999.

[49] N. V. Plotnikova, "Systems of linear differential equations with\pi-derivative and linear differential inclusions," *Sbornik: Mathematics*, vol. 196, no. 11, pp. 1677–1691, 2005, doi: 10.1070/SM2005v196n11ABEH003726.

[50] I. Podlubny, *Fractional differential equations*, ser. Mathematics in Science and Engineering. Academic Press, Inc., San Diego, CA, 1999, vol. 198.

[51] E. S. Polovinkin, *Set-valued analysis and differential inclusions.* Moscow: Fizmatlit, 2014, vol. 524.

[52] H. Rådström, "An embedding theorem for spaces of convex sets," *Proc. Amer. Math. Soc.*, vol. 3, pp. 165–169, 1952, doi: 10.2307/2032477.

[53] A. Tolstonogov, *Differential inclusions in a Banach space*, ser. Mathematics and its Applications. Kluwer Academic Publishers, Dordrecht, 2000, vol. 524, doi: 10.1007/978-94-015-9490-5.

[54] Y. N. Tyurin, "Mathematical formulation of a simplified production planning model," *Economy and mat. methods*, no. 3, pp. 391–409, 1965.

[55] A. N. Vityuk, "Fractional differentiation of multivalued mappings," *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki*, no. 10, pp. 75–79, 2003.

[56] A. Vityuk, "Differential equations of fractional order with set-valued solutions," *Visn. Odes. Derzh. Univ., Ser. Fiz.-Mat. Nauky*, vol. 8, no. 2, pp. 108–112, 2003.

[57] P. Wang and J. Bi, "The stability of set-valued differential equations with different initial time in the sense of fractional-like Hukuhara derivatives," *Fractal and Fractional*, vol. 7, no. 1, p. 20, 2022, doi: 10.3390/fractalfract7010020.