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Lyapunov-type inequalities for higher-order Caputo fractional differential equations with general two-point boundary conditions

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ABSTRACT

In this paper the authors present three different Lyapunov-type inequalities for a higher-order Caputo fractional differential equation with identical boundary conditions marking the inaugural instance of such an approach in the existing literature. Their findings extend and complement certain prior results in the literature.

RESUMEN

En este artículo, los autores presentan tres desigualdades de tipo Lyapunov diferentes para una ecuación diferencial fraccionaria de Caputo de alto orden con condiciones de frontera idénticas, marcando la primera vez que este enfoque aparece en la literatura existente. Sus hallazgos extienden y complementan ciertos resultados anteriores en la literatura.

Keywords and Phrases: Fractional integral, Caputo fractional derivative, boundary value problem, existence of solution, Lyapunov inequality, Green's function.

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1 Introduction

In this paper we consider the fractional differential equation

$$\left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, \quad n-1 < \gamma \leq n, \quad n \geq 3, \tag{1.1}$$

where $q : \mathbb{R} \to \mathbb{R}$ is continuous and $q(t) \neq 0$, together with the boundary conditions

$$x^{(i)}(a) = 0, \; x^{(k)}(b) = 0, \quad 0 \leq i \leq n-1 \quad \text{and} \quad i \neq k, \tag{1.2}$$

where k is a natural number between 1 and $n-1$.

Over the course of more than a century, numerous Lyapunov-type inequalities have been derived, taking into account their applications in various areas, such as eigenvalue problems, stability theory, oscillation theory, and the estimation of intervals of disconjugacy. The paper by Lyapunov [14] in 1907 is considered to be the first work in this direction. In recent decades, especially with the development of fractional differential equations, significant advancements and further generalizations of Lyapunov inequalities have been obtained. To explore some of the research that has provided some of the motivation for studying the problem (1.1)–(1.2), first note that Cabrera *et al.* [7] derived Lyapunov-like inequalities and established a lower bound for the eigenvalues of the fractional problem

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, & a < t < b, \quad \gamma \in (n-1, n], \quad n \geq 4, \\ x^{(i)}(a) = x''(b) = 0, & 0 \leq i \leq n-1, \quad i \neq 2. \end{cases}$$

It can be observed that the boundary value problem discussed in [7] is a particular case of the problem considered here, that is, of (1.1)–(1.2) with the parameter k taken to be 2. Additional notable work for $k = 2$ can found in [1, 7, 23, 24]. Compared to the problems investigated in [1, 7, 23, 24], our boundary condition (1.2) is more comprehensive and inclusive.

In [6], Bohner *et al.* applied a Vallée-Poussin theorem to obtain explicit inequality criteria for the solvability of the problem consisting of the Caputo fractional functional differential equation

$$({}^{C}D^{\gamma}_{a+}x)(t) + \sum_{i=0}^{m}(T_i x^{(i)})(t) = f(t), \quad t \in [a, b],$$

and the boundary condition (1.2), where the operator $T_i : C \to L_\infty$ with $C = C([a, b], \mathbb{R})$ can include a delay or advanced argument, an integral operator, or various linear combinations of such things. In another work, Domonshnitsky *et al.* [10] obtained such criteria for fractional functional differential equations with Riemann-Liouville derivatives again based on the Vallée-Poussin theorem. Rong and Bai [21] obtained a Lyapunov inequality for the problem

$$\begin{cases} \left({}^{\mathrm{C}}D_{a_+}^{\gamma}x\right)(t)+q(t)x(t)=0, & a<t<b, \quad 1<\gamma\leq 2,\\ x(a)=0,\ \left({}^{\mathrm{C}}D_{a}^{\beta}x\right)(b)=0, & 0<\beta\leq 1,\end{cases}$$

where $1<\gamma\leq 1+\beta$. Extensive research has been conducted on Lyapunov inequalities using different forms of fractional derivatives such as in [5,11,12,15,16,22]. For a comprehensive exploration of Lyapunov inequalities, a detailed study can be found in the recent monograph by Agarwal, Bohner and Özbekler [2].

Using estimates of the Green's function has been a common technique employed in the study of Lyapunov type inequalities. In cases where the Green's function possesses a fixed sign, estimating it becomes relatively straightforward compared to cases where the sign is unknown. Nevertheless, several researchers have successfully managed to find estimates and derive Lyapunov-type inequalities even if the sign constancy of the Green's function is not known; for example, see the recent papers [21, 22] and the book [2].

The present work is divided into six sections. Section 1 provides an introduction and background information pertaining to the problem. Preliminaries concepts are introduced in Section 2. In Section 3, we obtain a Lyapunov inequality that improves the results in [7]. In the process, we are able to obtain a new Lyapunov inequality for a third-order linear differential equation (see Corollary 3.6 below). In Section 4, we obtain a Lyapunov inequality under a restrictive condition (see (4.1)). A Lyapunov inequality for a general k with $1\leq k\leq n-2$ is discussed in Section 5. We conclude this work in Section 6 with some applications and open problems.

2 Preliminaries

The monographs [13, 18] offer a thorough examination of the basics of fractional calculus. The recent publication [22] contains the required fundamental definitions and lemmas utilized here in this study. Next, we discuss the Green's function and its sign in order to enhance our comprehension of the primary outcomes.

Lemma 2.1. *Assume that $\gamma\in(n-1,n]$, $1\leq k\leq n-1$, and $f\in L_\infty$. Then the unique solution of the fractional boundary value problem*

$$\begin{cases} ({}^{C}D_{a+}^{\gamma}x)(t)+f(t)=0, & a<t<b,\\ x^{(i)}(a)=x^{(k)}(b)=0, & 0\leq i\leq n-1 \quad \textit{and} \quad i\neq k,\end{cases} \tag{2.1}$$

is given by

$$x(t) = \int_a^b G_k(t,s) f(s) ds, \tag{2.2}$$

where $G_k(t,s)$ *is the Green's function given by*

$$G_k(t,s) = \frac{1}{\Gamma(\gamma)} \begin{cases} \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^k(b-s)^{\gamma-k-1} - (t-s)^{\gamma-1}, & a \le s \le t \le b, \\ \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^k(b-s)^{\gamma-k-1}, & a \le t \le s \le b. \end{cases} \tag{2.3}$$

Proof. Consider the equation

$$({}^C D^\gamma_{a+} x)(t) = -f(t).$$

Then, using some fundamental concepts in the fractional calculus (see [13, 18]), we see that

$$(I^\gamma_{a+}({}^C D^\gamma_{a+} x))(t) = -(I^\gamma_{a+} f)(t),$$

which, in turn, implies that there are constants $b_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, such that

$$x(t) = b_0 + b_1(t-a) + b_2(t-a)^2 + \cdots + b_{n-1}(t-a)^{n-1} - \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds,$$

for $t \in [a,b]$. From the boundary condition $x^{(i)}(a) = 0$ for $0 \le i \le n-1$ and $i \ne k$, we obtain $b_i = 0$ for $0 \le i \le n-1$, $i \ne k$. Since $x^{(k)}(a) \ne 0$, we have $b_k \ne 0$. Therefore,

$$x(t) = b_k(t-a)^k - \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds, \tag{2.4}$$

and so

$$\begin{aligned} x'(t) &= k b_k (t-a)^{k-1} - \frac{\gamma-1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-2} f(s) ds, \\ x''(t) &= k(k-1) b_k (t-a)^{k-2} - \frac{(\gamma-1)(\gamma-2)}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-3} f(s) ds, \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ x^{(k)}(b) &= k!\, b_k - \frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}{\Gamma(\gamma)} \int_a^b (b-s)^{\gamma-k-1} f(s) ds. \end{aligned}$$

Applying the boundary condition $x^{(k)}(b) = 0$ gives

$$b_k = \frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}{k!\Gamma(\gamma)} \int_a^b (b-s)^{\gamma-k-1} f(s) ds.$$

Using this value of b_k in (2.4), we obtain

$$x(t) = \frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}{k!\Gamma(\gamma)} (t-a)^k \int_a^b (b-s)^{\gamma-k-1} f(s) ds - \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds, \tag{2.5}$$

or

$$x(t) = \frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}{k!\Gamma(\gamma)}\left[\int_a^t (t-a)^k(b-s)^{\gamma-k-1}f(s)ds \right. \\ \left. + \int_t^b (t-a)^k(b-s)^{\gamma-k-1}f(s)ds\right] - \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}f(s)ds. \quad (2.6)$$

This proves the lemma. □

The following lemma provides some valuable information about the sign of the Green's function.

Lemma 2.2. *If $\gamma \in (n-1, n]$ and $\gamma > k+1$, then $G_k(t,s) > 0$ for all $t, s \in [a,b]$.*

Proof. Clearly,

$$G_k(t,s) = \frac{1}{\Gamma(\gamma)k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^k(b-s)^{\gamma-k-1} > 0,$$

for $a < t < s < b$. If $a < s \le t < b$, we obtain

$$\begin{aligned} G_k(t,s) &= \frac{1}{\Gamma(\gamma)k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^k(b-s)^{\gamma-k-1} - \frac{1}{\Gamma(\gamma)}(t-s)^{\gamma-1} \\ &\ge \frac{1}{k!\Gamma(\gamma)}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-s)^k(t-s)^{\gamma-k-1} - \frac{1}{\Gamma(\gamma)}(t-s)^{\gamma-1} \\ &= \frac{1}{k!\Gamma(\gamma)}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-s)^{\gamma-1} - \frac{1}{\Gamma(\gamma)}(t-s)^{\gamma-1} \\ &= \frac{1}{\Gamma(\gamma)}(t-s)^{\gamma-1}\left[\frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k) - 1\right] \\ &> \frac{1}{\Gamma(\gamma)}(t-s)^{\gamma-1}\left[\frac{1}{k!}k(k-1)\cdots(1) - 1\right] = 0, \end{aligned}$$

where we have used the facts that $\gamma > k+1$, $t-a \ge t-s$, and $b-s \ge t-s$, so that $(t-a)^k(b-s)^{\gamma-k-1} \ge (t-s)^k(t-s)^{\gamma-k-1}$. This completes the proof. □

3 Main results: Lyapunov type inequalities–I

We begin this section with another lemma on the properties of $G_k(t,s)$.

Lemma 3.1. *If $\gamma \in (n-1, n]$ and $\gamma > k+1$, then Green's function $G_k(t,s)$ given in (2.3) has the property that $\frac{\partial G_k(t,s)}{\partial t} > 0$ for all $t, s \in [a,b]$. Furthermore,*

$$G_k(t,s) \le G_k(b,s) \equiv \frac{1}{\Gamma(\gamma)}\left[\frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(b-a)^k(b-s)^{\gamma-k-1} - (b-s)^{\gamma-1}\right]. \quad (3.1)$$

Proof. For $a < t < s < b$, we have

$$\frac{\partial G_k(t,s)}{\partial t} = \frac{k}{k!\Gamma(\gamma)}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^{k-1}(b-s)^{\gamma-k-1} \geq 0.$$

For $a < s \leq t < b$,

$$\begin{aligned}\frac{\partial G_k(t,s)}{\partial t} &= \frac{k}{k!\Gamma(\gamma)}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^{k-1}(b-s)^{\gamma-k-1} - \frac{1}{\Gamma(\gamma)}(\gamma-1)(t-s)^{\gamma-2}\\ &\geq \frac{(\gamma-1)}{\Gamma(\gamma)}\left[\frac{k}{k!}(\gamma-2)\cdots(\gamma-k)(t-s)^{k-1}(t-s)^{\gamma-k-1} - (t-s)^{\gamma-2}\right]\\ &= \frac{(\gamma-1)}{\Gamma(\gamma)}\left[\frac{1}{(k-1)!}(\gamma-2)\cdots(\gamma-k)(t-s)^{\gamma-2} - (t-s)^{\gamma-2}\right]\\ &> \frac{(\gamma-1)}{\Gamma(\gamma)}\left[\frac{(k-1)!}{(k-1)!} - 1\right](t-s)^{\gamma-2} = 0,\end{aligned}$$

where we have used the fact that $\gamma > k+1$. Therefore, the function $G_k(t,s)$ is nondecreasing with respect to t, and this implies $G_k(t,s) \leq G_k(b,s)$ for all $t, s \in [0,1]$. This proves the lemma. □

The following theorem is the major result in this section.

Theorem 3.2. *Assume that $\gamma \in (n-1, n]$ and $\gamma > k+1$. If a nontrivial continuous solution of* (1.1)–(1.2) *exists, then*

$$\int_a^b \left[\frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(b-a)^k(b-s)^{\gamma-k-1} - (b-s)^{\gamma-1}\right]|q(s)|ds \geq \Gamma(\gamma). \tag{3.2}$$

Proof. Let $x(t)$ be a nonzero solution of (1.1)–(1.2) and let $X = C([a,b])$ be a Banach space endowed with the norm

$$\|x\| = \sup_{a\leq t\leq b}|x(t)|.$$

Then, for a solution x of (1.1)–(1.2), by Lemma 2.1,

$$x(t) = \int_a^b G_k(t,s)q(s)x(s)ds.$$

Since $q(t)$ cannot be zero,

$$|x(t)| \leq \frac{1}{\Gamma(\gamma)}\int_a^b \left[\frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(b-a)^k(b-s)^{\gamma-k-1} - (b-s)^{\gamma-1}\right]|q(s)|\,|x(s)|ds,$$

which yields (3.2). This proves the theorem. □

We have the following consequences of this result.

Corollary 3.3. *Under the conditions of Theorem 3.2, if* (1.1)–(1.2) *has a nontrivial continuous solution, then*

$$\int_a^b (b-s)^{\gamma-k-1}|q(s)|ds \geq \frac{k!\Gamma(\gamma-k)}{(b-a)^k}. \tag{3.3}$$

Corollary 3.4. *Under the conditions of Theorem 3.2, if* (1.1)–(1.2) *has a nontrivial continuous solution, then*

$$\int_a^b |q(s)|ds \geq \frac{k!\Gamma(\gamma-k)}{(b-a)^{\gamma-1}}. \tag{3.4}$$

If we set $n \equiv 3$, then $\gamma \in (2,3]$, and since $\gamma > k+1$, this means we take $k=1$. The problem (1.1)–(1.2) then reduces to

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a+}x\right)(t) + q(t)x(t) = 0, \qquad 2 < \gamma \leq 3, \\ x(a) = x''(a) = x'(b) = 0. \end{cases} \tag{3.5}$$

Fractional BVPs of the form (3.5) were studied by Qin and Bai [19, 20]. Applying Theorem 3.2, Corollary 3.3, and Corollary 3.4 to (3.5), we obtain the following corollary.

Corollary 3.5. *If* (3.5) *has a continuous nontrivial solution, then*

$$\int_a^b \left[\frac{1}{k!}(\gamma-1)(b-a)(b-s)^{\gamma-2} - (b-s)^{\gamma-1}\right]|q(s)|ds \geq \Gamma(\gamma), \tag{3.6}$$

$$\int_a^b (b-s)^{\gamma-2}|q(s)|ds \geq \frac{\Gamma(\gamma-1)}{(b-a)}, \tag{3.7}$$

and

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\gamma-1)}{(b-a)^{\gamma-1}}. \tag{3.8}$$

As discussed earlier, (3.6) implies (3.7), and (3.7) implies (3.8). In particular, applying inequality (3.8) of Corollary 3.5 to the third-order boundary value problem

$$\begin{cases} x'''(t) + q(t)x(t) = 0, \\ x(a) = x''(a) = x'(b) = 0, \end{cases} \tag{3.9}$$

we obtain the following result.

Corollary 3.6. *If* (3.9) *has a continuous nontrivial solution, then*

$$\int_a^b |q(s)|ds \geq \frac{1}{(b-a)^2}. \tag{3.10}$$

As far as our knowledge is concerned, Corollary 3.6 is new in the literature. The boundary conditions used in (3.9) are different from those of Aktaş and Çakmak [3,4] and Parhi and Panigrahi [17]. Our Corollary 3.6 can not be compared to the results in [5] because of the restrictive condition $x''(a) + x''(b) = 0$ (see the third condition of (1.7) in [5]) required there. Similarly, Corollary 3.5 can not be compared to Dhar and Kong [8, 9].

Next, suppose that $n \geq 4$. Our parameter k considered in (1.2) varies from 1 to $n-1$. In particular, if $k = 2$, we obtain the results of Cabrera, Lopez, and Sadarangani [7]. Our Green's function $G_k(t, s)$ extends the Green's functions obtained in [1, 23, 24] for $a = 0$, $b = 1$, and $k = 2$.

4 Main results: Lyapunov type inequalities–II

In this section, we derive a new Lyapunov type inequality, different from the ones presented in the previous section. We use the maximum of the Green's function $G_k(t, s)$ given in (2.3) to find a new inequality for (1.1)–(1.2) for a general k, $1 \leq k \leq n-1$, with the price being that the following restrictive inequality is imposed:

$$k! > (\gamma - 1) \cdots (\gamma - k)(\gamma - k - 1). \tag{4.1}$$

As prescribed by our boundary condition (1.2), we consider the following cases:

(A$_1$) $x(0) = x''(0) = \cdots = x^{(n-1)}(0) = 0,\ x'(1) = 0$

(A$_2$) $x(0) = x'(0) = x'''(0) = \cdots = x^{(n-1)}(0) = 0,\ x''(1) = 0$

(A$_3$) $x(0) = x'(0) = x''(0) = x''''(0) = \cdots = x^{(n-1)}(0) = 0,\ x'''(1) = 0$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

(A$_{n-1}$) $x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0,\ x^{(n-1)}(1) = 0$

Remark 4.1. *Observe that:*

(B$_1$) *For $k = 1$, that is, in the case (A$_1$), we can take $\gamma = 2.5 \in (2, 3]$. Then, condition (4.1) is satified, i.e.,*

$$1 = k! > (\gamma - 1) \cdots (\gamma - k)(\gamma - k - 1) = (2.5 - 1)(2.5 - 2) = 0.75.$$

(B$_2$) *For $k = 2$, that is, in the case (A$_2$), we can take $\gamma = 3.5 \in (3, 4]$, so that condition (4.1) becomes*

$$2 = k! > (\gamma - 1) \cdots (\gamma - k)(\gamma - k - 1) = (3.5 - 1)(3.5 - 2)(3.5 - 3) = 1.875.$$

(B$_3$) *For $k=3$, that is, in the case (A_3), we can take $\gamma = 4.4 \in (4,5]$, and (4.1) becomes*

$$6 = k! > (\gamma-1)\cdots(\gamma-k)(\gamma-k-1) = (4.4-1)(4.4-2)(4.4-3)(4.4-4) = 4.5696.$$

The following lemma gives an upper bound on $G_k(t,s)$.

Lemma 4.2. *Let $\gamma > k+1$ and assume that (4.1) is satisfied. Then*

$$G_k(t,s) \leq \frac{1}{\Gamma(\gamma)} \frac{k(b-a)^{\gamma-1}}{\gamma-k-1} \left(\frac{(\gamma-2)(\gamma-3)\cdots(\gamma-k)(\gamma-k-1)}{k!} \right)^{\frac{\gamma-1}{k}}. \tag{4.2}$$

Proof. By Lemma 3.1, we have $G_k(t,s) \leq G_k(b,s)$. Set

$$F(s) = \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(b-a)^k(b-s)^{\gamma-k-1} - (b-s)^{\gamma-1}; \tag{4.3}$$

then $G_k(b,s) = \frac{1}{\Gamma(\gamma)}F(s)$. To obtain the maximum of $F(s)$, set $F'(s)$ equal to zero to obtain

$$F'(s) = -\frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}(b-a)^k(b-s)^{\gamma-k-2} + (\gamma-1)(b-s)^{\gamma-2} = 0,$$

which is true if and only if

$$s(:= s^*) = b - \left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!} \right)^{\frac{1}{k}} (b-a). \tag{4.4}$$

Clearly, $s^* < b$. Also, if $s^* < a$, then

$$k! < (\gamma-2)\cdots(\gamma-k)(\gamma-k-1),$$

which contradicts (4.1). Hence, $s^* \geq a$.

Now,

$$\begin{aligned} F''(s) &= \frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)(\gamma-k-2)}{k!}(b-s)^{\gamma-k-3}(b-a)^k \\ &\quad - (\gamma-1)(\gamma-2)(b-s)^{\gamma-3} \\ &= (\gamma-1)(\gamma-2)(b-s)^{\gamma-k-3}\left[\frac{(\gamma-3)\cdots(\gamma-k)(\gamma-k-1)(\gamma-k-2)}{k!}(b-a)^k - (b-s)^k \right]. \end{aligned}$$

If we set

$$g(s) = \frac{(\gamma-3)\cdots(\gamma-k)(\gamma-k-1)(\gamma-k-2)}{k!}(b-a)^k - (b-s)^k,$$

then

$$\begin{aligned} g(s^*) &= \frac{(\gamma-3)\cdots(\gamma-k)(\gamma-k-1)(\gamma-k-2)}{k!}(b-a)^k - \frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}(b-a)^k \\ &= \frac{(\gamma-3)\cdots(\gamma-k)(\gamma-k-1)}{k!}(b-a)^k(\gamma-k-2-\gamma+2) \\ &= -k\frac{(\gamma-3)\cdots(\gamma-k)(\gamma-k-1)}{k!}(b-a)^k < 0. \end{aligned}$$

Therefore, $F(s)$ attains its maximum at $s = s^*$, and the maximum of $F(s)$ is given by

$$\begin{aligned} F(s) &\leq \max F(s) = F(s^*) = \\ &= \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(b-a)^k \left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right)^{\frac{\gamma-k-1}{k}} (b-a)^{\gamma-k-1} \\ &\quad - \left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right)^{\frac{\gamma-1}{k}} (b-a)^{\gamma-1} \\ &= (b-a)^{\gamma-1}\left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right)^{\frac{\gamma-k-1}{k}} \left[\frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}{k!}\right. \\ &\quad \left. - \frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right] \\ &= (b-a)^{\gamma-1}\left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right)^{\frac{\gamma-k-1}{k}} \frac{(\gamma-2)\cdots(\gamma-k)}{k!}(\gamma-1-\gamma+k+1) \\ &= k(b-a)^{\gamma-1}\left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right)^{\frac{\gamma-k-1}{k}} \frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!(\gamma-k-1)} \\ &= \frac{k(b-a)^{\gamma-1}}{(\gamma-k-1)}\left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right)^{\frac{\gamma-1}{k}}. \end{aligned}$$

Consequently, (4.2) holds, and this completes the proof of the lemma. □

Next, based on the above lemma, we present our main inequality in this section.

Theorem 4.3. *If* $\gamma > k+1$, (4.1) *is satisfied, and a nontrivial continuous solution of* (1.1)–(1.2) *exists, then*

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\gamma)(\gamma-k-1)}{k(b-a)^{\gamma-1}}\left(\frac{k!}{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}\right)^{\frac{\gamma-1}{k}}. \tag{4.5}$$

As before, we obtain the following corollaries.

Corollary 4.4. *Let* $\gamma \in (2,3)$ *and* $(\gamma-1)(\gamma-2) < 1$. *If a nontrivial continuous solution of the fractional boundary value problem* (3.5) *exists, then*

$$\int_a^b |q(t)|\mathrm{d}t \geq \frac{\Gamma(\gamma)}{(b-a)^{\gamma-1}}\frac{1}{(\gamma-2)^{\gamma-2}}.$$

Proof. This can be proved by letting $k = 1$ in (4.1) and (4.5). □

Corollary 4.5. *Let $\gamma \in (3, 4)$ and $(\gamma - 1)(\gamma - 2)(\gamma - 3) < 2!$. If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, & 3 < \gamma \leq 4, \\ x(a) = x'(a) = x'''(a) = x''(b) = 0 \end{cases} \tag{4.6}$$

exists, then

$$\int_a^b |q(t)|\mathrm{d}t \geq \frac{\Gamma(\gamma)(\gamma-3)}{2(b-a)^{\gamma-1}} \left(\frac{2!}{(\gamma-2)(\gamma-3)}\right)^{\frac{\gamma-1}{2}}.$$

Proof. This can be proved by letting $k = 2$ in (4.1) and (4.5). □

Corollary 4.6. *Let $\gamma \in (4, 5)$ and $(\gamma - 1)(\gamma - 2)(\gamma - 3)(\gamma - 4) < 3!$. If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, & 4 < \gamma \leq 5, \\ x(a) = x'(a) = x''(a) = x''''(a) = x'''(b) = 0 \end{cases} \tag{4.7}$$

exists, then

$$\int_a^b |q(t)|\mathrm{d}t \geq \frac{\Gamma(\gamma)(\gamma-4)}{3(b-a)^{\gamma-1}} \left(\frac{3!}{(\gamma-2)(\gamma-3)(\gamma-4)}\right)^{\frac{\gamma-1}{3}}.$$

Proof. This can be proved by letting $k = 3$ in (4.1) and (4.5). □

5 Main results: Lyapunov type inequalities–III

In Sections 3 and 4, we obtained two different Lyapunov-type inequalities. In this section, we obtain one more such inequality that is also different from the previous ones. Here we will have the same integrand that appeared in (3.2) in Section 3, whereas we only had q as the integrand in (4.5) in Section 4. Although the condition $\gamma > k + 1$ is required in both of these sections, the inequality (4.1) prevents us from considering many types of boundary conditions. For example, from the observations (B_2)–(B_3) and condition (4.1), we see that we cannot ask that $k < n - 2$.

In this section, we avoid condition (4.1) and find a general Lyapunov-type inequality for (1.1) together with the boundary condition (1.2), which is valid for the case $1 \leq k \leq n - 2$.

Set

$$M = \max\left\{\frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)\frac{k^k(b-a)^{\gamma-1}(\gamma-k-1)^{\gamma-k-1}}{(\gamma-1)^{\gamma-1}},\right.$$
$$\frac{k(b-a)^{\gamma-1}}{(\gamma-k-1)}\left(\frac{(\gamma-2)(\gamma-3)\cdots(\gamma-k-1)}{k!}\right)^{\frac{\gamma-1}{k}}, \tag{5.1}$$
$$\left.\frac{(b-a)^{\gamma-1}}{k!}((\gamma-1)(\gamma-2)\cdots(\gamma-k)-k!)\right\}.$$

Lemma 5.1. *Let $\gamma > k+1$. The inequality*

$$\max_{t,s\in[a,b]} G_k(t,s) \le \frac{1}{\Gamma(\gamma)}M, \tag{5.2}$$

holds, where M is defined in (5.1).

Proof. We have $\Gamma(\gamma)G_k(t,s) = \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^k(b-s)^{\gamma-k-1}$ for $a \le t \le s \le b$. Now,

$$\Gamma(\gamma)\frac{\partial G_k}{\partial t} = \frac{k}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^{k-1}(b-s)^{\gamma-k-1} \ge 0$$

implies that $G_k(t,s)$ is non decreasing with respect to t. Hence, $\Gamma(\gamma)G_k(t,s) \le G_k(s,s)\Gamma(\gamma)$. Set $\Gamma(\gamma)G_k(s,s) = g_1(s)$. Then,

$$g_1(s) = \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(s-a)^k(b-s)^{\gamma-k-1},$$

and $\dfrac{dg_1}{ds} = 0$ if and only if

$$s =: s^* = a + \frac{k(b-a)}{\gamma-1}.$$

Clearly, $a < s^* < b$, and

$$\begin{aligned}\frac{d^2g_1}{ds^2} &= \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)[k(k-1)(s-a)^{k-2}(b-s)^{\gamma-k-1}\\ &\quad - k(\gamma-k-1)(s-a)^{k-1}(b-s)^{\gamma-k-2} - k(\gamma-k-1)(s-a)^{k-1}(b-s)^{\gamma-k-2}\\ &\quad + (\gamma-k-1)(\gamma-k-2)(s-a)^k(b-s)^{\gamma-k-3}]\\ &= \frac{1}{k!}(\gamma-1)\cdots(\gamma-k)(s-a)^{k-2}(b-s)^{\gamma-k-3}[k(k-1)(b-s)^2\\ &\quad - 2k(\gamma-k-1)(s-a)(b-s) + (\gamma-k-1)(\gamma-k-2)(s-a)^2].\end{aligned} \tag{5.3}$$

Now, $s^* - a = \frac{k(b-a)}{\gamma-1}$ and

$$(b-s^*) = (b-a) - \frac{k(b-a)}{(\gamma-1)} = \frac{(b-a)(\gamma-k-1)}{(\gamma-1)}.$$

Thus, from (5.3), we have

$$\frac{d^2 g_1}{ds^2}\Big|_{s=s^*} = -\frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}{k!}\frac{k^{k-2}(b-a)^{\gamma-3}(\gamma-k-1)^{\gamma-k-2}}{(\gamma-1)^{\gamma-4}} < 0,$$

which shows that $g_1(s)$ attains its maximum at $s = s^*$. Hence,

$$\max_{a\le t\le s\le b} G_k(t,s) = \frac{1}{\Gamma(\gamma)k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)\frac{k^k(b-a)^{\gamma-1}(\gamma-k-1)^{\gamma-k-1}}{(\gamma-1)^{\gamma-1}}. \tag{5.4}$$

Next, suppose that $a \le s \le t \le b$. Since $\gamma > k+1$, $G_k(t,s)$ is nondecreasing with respect to t. Thus, for $a \le s \le t \le b$, we have

$$\max_{a\le s\le t\le b} G_k(t,s) = G_k(b,s) := \frac{1}{\Gamma(\gamma)}F(s), \tag{5.5}$$

where $F(s)$ is given in (4.3). Clearly $F'(s) = 0$ if and only if $s = s^*$, where s^* is given in (4.4). Moreover, $s^* < b$, $F(s)$ is nondecreasing for $s \le s^*$, nonincreasing for $s \ge s^*$, and attains its extreme (maximum) value at $s = s^*$.

First, suppose that $a \le s^*$. Then $F(s)$ attains its maximum at $s = s^*$, and the maximum value of $G_k(t,s)$ is given by

$$\max_{a\le s\le t\le b} G_k(t,s) = G_k(b,s^*) = \frac{1}{\Gamma(\gamma)}\frac{k(b-a)^{\gamma-1}}{(\gamma-k-1)}\left(\frac{(\gamma-2)(\gamma-3)\cdots(\gamma-k-1)}{k!}\right)^{\frac{\gamma-1}{k}}. \tag{5.6}$$

Finally, suppose that $s^* < a$. Then,

$$\begin{aligned}\max_{a\le s\le t\le b} G_k(t,s) &\le \max_{a\le s\le b} G_k(b,s) \le G_k(b,a)\\ &= \frac{(b-a)^{\gamma-1}}{k!\Gamma(\gamma)}((\gamma-1)(\gamma-2)\cdots(\gamma-k)-k!).\end{aligned} \tag{5.7}$$

Therefore, in view of (5.4), (5.6), and (5.7), the lemma is proved. □

Theorem 5.2. *Let $\gamma > k+1$. If $x(t)$ is a nonzero solution of* (1.1)–(1.2), *then*

$$\int_a^b |q(t)|\,dt > \frac{\Gamma(\gamma)}{M}. \tag{5.8}$$

6 Discussion and conclusions

In this section, we obtain Lyapunov-type inequalities for fractional differential equations of various orders and with different boundary conditions. We also compare our results with some existing ones in the literature.

6.1 The case $\gamma \in (2, 3]$

Let $\gamma \in (2, 3]$. Since $\gamma > k+1$ and $\gamma > 2$, we have $k = 1$. In this case,

$$M_1 = \max\left\{(b-a)^{\gamma-1}\left(\frac{\gamma-2}{\gamma-1}\right)^{\gamma-2},\ (b-a)^{\gamma-1}(\gamma-2)^{\gamma-2},\ (b-a)^{\gamma-1}(\gamma-2)\right\}, \tag{6.1}$$

where $M_1 = M|_{k=1}$ and M is given in (5.1). Now $2 < \gamma \leq 3$ implies $(\gamma-2)^{\gamma-2} \geq \gamma - 2$, so

$$M_1 = \left\{(b-a)^{\gamma-1}\left(\frac{\gamma-2}{\gamma-1}\right)^{\gamma-2},\ (b-a)^{\gamma-1}(\gamma-2)^{\gamma-2}\right\}. \tag{6.2}$$

We then have the following corollary.

Corollary 6.1. *Let $\gamma \in (2, 3]$. If $x(t)$ is a nonzero solution of*

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, \\ x(a) = x''(a) = x'(b) = 0, \end{cases} \tag{6.3}$$

then

$$\int_a^b |q(t)|\, dt > \frac{\Gamma(\gamma)}{M_1}. \tag{6.4}$$

Since $(b-a)^{\gamma-2} \geq \frac{1}{(\gamma-1)^{\gamma-2}}$ holds if and only if $b \geq a + \frac{1}{\gamma-1}$, we obtain the following corollary from Corollary 6.1.

Corollary 6.2. *Let $\gamma \in (2, 3]$ and $b \geq a + \frac{1}{\gamma-1}$. If $x(t)$ is a nonzero solution of (6.3), then*

$$\int_a^b |q(t)|\, dt > \frac{\Gamma(\gamma)}{(b-a)^{\gamma-1}(\gamma-2)^{\gamma-2}}. \tag{6.5}$$

Now we consider the problem (3.9). Here $n = 3$, $\gamma = 3$, and $k = 1$. In this case, Corollary 3.6 shows that if (3.9) has a nontrivial solution, then (3.10) holds. Corollary 4.4 cannot be applied because $(\gamma-1)(\gamma-2) = 2 > 1$ and so (4.1) fails. By Corollary 6.1, if x is a nonzero solution of the problem (3.9), then

$$\int_a^b |q(t)|\, dt > \frac{2}{\max\{\frac{(b-a)}{2}, (b-a)^2\}} \tag{6.6}$$

holds. If $b \geq \frac{1}{2} + a$, then $\max\{\frac{(b-a)}{2}, (b-a)^2\} = (b-a)^2$. Consequently, (6.6) yields (3.10). On, the other hand, if $b < \frac{1}{2} + a$, then $\max\{\frac{(b-a)}{2}, (b-a)^2\} = \frac{(b-a)}{2}$. In this case, (6.6) yields

$$\int_a^b |q(t)|\, dt > \frac{4}{(b-a)}. \tag{6.7}$$

6.2 The case $\gamma \in (3,4]$

Let $\gamma \in (3,4]$. Since $\gamma > k+1$ and $k \neq 0$, we consider the following two cases: $k=1$ and $k=2$. First, suppose that $k=1$; then Theorem 5.2 yields the following corollary.

Corollary 6.3. *Let $\gamma \in (3,4]$. If $x(t)$ is a nonzero solution of*

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, \\ x(a) = x''(a) = x'''(a) = x'(b) = 0, \end{cases} \tag{6.8}$$

then

$$\int_a^b |q(t)|\,dt > \frac{\Gamma(\gamma)}{M_1} \tag{6.9}$$

where M_1 is given in (6.2).

Corollary 6.4. *Let $\gamma \in (3,4]$ and $b \geq a + \frac{1}{\gamma-1}$. If $x(t)$ is a nonzero solution of* (6.8), *then (6.5) holds.*

Finally, suppose that $k=2$. Then Theorem 5.2 reduces to the following corollary.

Corollary 6.5. *Let $\gamma \in (3,4]$. If $x(t)$ is a nonzero solution of*

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, \\ x(a) = x'(a) = x'''(a) = x''(b) = 0, \end{cases} \tag{6.10}$$

then

$$\int_a^b |q(t)|\,dt > \frac{\Gamma(\gamma)}{M_2}, \tag{6.11}$$

where

$$M_2 = \left\{ \frac{2(b-a)^{\gamma-1}(\gamma-2)(\gamma-3)^{\gamma-3}}{(\gamma-1)^{\gamma-2}}, \frac{2(b-a)^{\gamma-1}}{\gamma-3}\left(\frac{(\gamma-2)(\gamma-3)}{2}\right)^{\frac{\gamma-1}{2}}, \frac{\gamma(\gamma-3)(b-a)^{\gamma-1}}{2} \right\}. \tag{6.12}$$

In this paper, we obtained Lyapunov-type inequalities for higher-order fractional differential equations of Caputo-type with general two point boundary conditions. The assumption that $\gamma > k+1$ helped us to analyze the signs of the Green's function $G_k(t,s)$ and its derivatives with the price that $k \neq n-1$. Similarly, by our assumption, we have $k \neq 0$. Therefore, it would be interesting to discover a Lyapunov-type inequality for problem (1.1) for either of the boundary conditions

$$x^{(i)}(a) = x^{(n-1)}(b) = 0, \quad 0 \leq i \leq n-2$$

or

$$x^{(i)}(a) = x(b) = 0, \quad 0 \le i \le n-1.$$

This is left to the reader.

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Conflicts of interest

The authors have no conflicts of interest to report.

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