

Dual digraphs of finite meet-distributive and modular lattices

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ABSTRACT

We describe the digraphs that are dual representations of finite lattices satisfying conditions related to meet-distributivity and modularity. This is done using the dual digraph representation of finite lattices by Craig, Gouveia and Haviar (2015). These digraphs, known as TiRS digraphs, have their origins in the dual representations of lattices by Urquhart (1978) and Ploščica (1995). We describe two properties of finite lattices which are weakenings of (upper) semimodularity and lower semimodularity respectively, and then show how these properties have a simple description in the dual digraphs. Combined with previous work in this journal on dual digraphs of semidistributive lattices (2022), it leads to a dual representation of finite meet-distributive lattices. This provides a natural link to finite convex geometries. In addition, we present two sufficient conditions on a finite TiRS digraph for its dual lattice to be modular. We close by posing three open problems.

RESUMEN

Describimos los digrafos que son representaciones duales de reticulados finitos satisfaciendo condiciones relacionadas con encuentro-distributividad y modularidad. Esto se obtiene usando la representación digrafo dual de reticulados finitos de Craig, Gouveia y Haviar (2015). Estos digrafos, conocidos como digrafos TiRS, tienen sus orígenes en las representaciones duales de reticulados de Urquhart (1978) y Ploščica (1995). Describimos dos propiedades de reticulados finitos que son debilitamientos de la semimodularidad (superior) y semimodularidad inferior respectivamente, y luego mostramos cómo estas propiedades tienen una descripción simple en los digrafos duales. Combinado con trabajo previo sobre digrafos duales de reticulados semidistributivos (2022) en esta revista, se tiene una representación dual de reticulados encuentro-distributivos. Esto entrega una conexión natural a geometrías convexas finitas. Adicionalmente, presentamos dos condiciones suficientes en un digrafo TiRS finito para que su reticulado dual sea modular. Concluimos presentando tres problemas abiertos.

Keywords and Phrases: Semimodular lattice, lower semimodular lattice, modular lattice, TiRS digraph, meet-distributive lattice, finite convex geometry.

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1 Introduction

The first dual representation of arbitrary bounded lattices was given by Urquhart in 1978 [15]. Since then, many different authors have attempted to provide dualities and dual representations of classes of lattices that are not necessarily distributive (see the recent survey by the first author [4]).

In this paper we examine representations for finite lattices that satisfy conditions related to meet-distributivity and modularity. The dual structures of these finite lattices will be TiRS digraphs satisfying some additional conditions. It was shown by Craig, Gouveia and Haviar [6] that there is a one-to-one correspondence between the class of finite lattices and finite digraphs known as TiRS digraphs (see Definition 2.4 and Theorem 2.6). We remark that this correspondence generalises Birkhoff's one-to-one correspondence between finite distributive lattices and finite posets from the 1930s.

A goal of any representation is to use simple, familiar structures to represent the objects of interest. Finite TiRS digraphs provide a straightforward generalisation of finite posets. Moreover, digraphs are a well-studied class of mathematical structures and hence are well suited to be used as dual objects. In addition, the first-order description of TiRS digraphs can be used to study the finite ones with computational tools such as Prover9/Mace4 [11].

We introduce and study lattice-theoretic conditions which generalise both lower semimodularity and (upper) semimodularity for finite lattices and seem to be more natural and simpler than the conditions from [8]. We are also able to provide equivalent conditions to them on the dual TiRS digraph of a finite lattice. We can combine our lattice-theoretic conditions with our previous results in this journal [5] to characterise the dual digraphs of finite meet-distributive lattices, which correspond to finite convex geometries.

Currently, the only known dual characterisation of finite modular lattices is given by the theory of Formal Concept Analysis [8]. A rather complicated condition is available for the standard context dual to a finite semimodular lattice [8, Theorem 42]. We are able to provide conditions on the dual digraph of a finite lattice, which are sufficient though not necessary for modularity of the lattice.

The paper is laid out as follows. In Section 2 we provide some background definitions and results that will be needed later on in the paper. Section 3 defines two conditions which generalise, respectively, (upper) semimodularity and lower semimodularity. We focus on the generalisation of lower semimodularity—a condition we call (JM-LSM) (see Definition 3.6). We characterise the dual of (JM-LSM) on the dual digraphs of finite lattices. For completeness we state corresponding conditions and results related to upper semimodularity. In Section 4 we combine the results of Section 3 with results from a recent paper by Craig, Haviar and São João [5]. There, characterisations were given of the digraphs dual to finite join- and meet-semidistributive lattices (and hence also

finite semidistributive lattices). The combination of these dual characterisations gives us a characterisation of the dual digraphs of finite meet-distributive lattices (also known as locally distributive lattices). Furthermore, this allows us to describe a new class of structures that is in a one-to-one correspondence with finite convex geometries. In Section 5 we give two sufficient conditions on a finite TiRS digraph for the dual lattice to be modular. In Section 6 we list three open problems and indicate why the task of describing digraphs dual to finite modular lattices is challenging.

2 Preliminaries

Central to the representation of a finite lattice that we will use is the notion of a maximal-disjoint filter-ideal pair. This can, equivalently, be viewed as a maximal partial homomorphism from a lattice L into the two-element lattice.

Definition 2.1 ([15, Section 3]). *Let L be a lattice. Then $\langle F, I \rangle$ is a disjoint filter-ideal pair of L if F is a filter of L and I is an ideal of L such that $F \cap I = \emptyset$. A disjoint filter-ideal pair $\langle F, I \rangle$ is said to be a maximal disjoint filter-ideal pair (MDFIP) if there is no disjoint filter-ideal pair $\langle G, J \rangle \neq \langle F, I \rangle$ such that $F \subseteq G$ and $I \subseteq J$.*

The following fact was noted by Urquhart. It is needed for our characterisation of MDFIPs in Theorem 3.2.

Proposition 2.2 ([15, p. 52]). *Let L be a finite lattice. If $\langle F, I \rangle$ is an MDFIP of L then $\bigwedge F$ is join-irreducible and $\bigvee I$ is meet-irreducible.*

The set of join-irreducible elements of L is denoted $J(L)$ and the set of meet-irreducible elements is denoted $M(L)$.

Given a lattice L , we will add a set of arcs to the set of MDFIPs of L . The use of such digraphs for lattice representation is due to Ploščica [12]. We point out that the original work using (topologised) digraphs used so-called *maximal partial homomorphisms* (see [12, Section 1]). It is easy to show that these are in a one-to-one correspondence with MDFIPs. For a lattice L , we now present its dual digraph $G_L = (X_L, E)$ where the vertices are the MDFIPs of L . Ploščica's relation E , when transferred to the set of MDFIPs, is defined below for two MDFIPs $\langle F, I \rangle$ and $\langle G, J \rangle$:

$$(E) \quad \langle F, I \rangle E \langle G, J \rangle \iff F \cap J = \emptyset.$$

For finite lattices every filter is the up-set of a unique element and every ideal is the down-set of a unique element, so we can represent every disjoint filter-ideal pair $\langle F, I \rangle$ by an ordered pair $\langle \uparrow a, \downarrow b \rangle$ where $a = \bigwedge F$ and $b = \bigvee I$. Hence for finite lattices we have $\langle \uparrow a, \downarrow b \rangle E \langle \uparrow c, \downarrow d \rangle$ if and only if $a \not\leq d$. For a digraph $G = (V, E)$ we let $xE = \{y \in V \mid xEy\}$ and $Ex = \{y \in V \mid yEx\}$. The next lemma is easy to prove and it will be useful later on.

Lemma 2.3. Let $G_L = (X_L, E)$ be the dual digraph of a finite lattice L . If $x = \langle \uparrow a, \downarrow b \rangle$ and $y = \langle \uparrow c, \downarrow d \rangle$, then

(i) $xE \subseteq yE$ if and only if $a \leq c$;

(ii) $Ex \subseteq Ey$ if and only if $d \leq b$.

Figure 1 shows three lattices and their dual digraphs. These three examples will be important throughout this paper. To make the labelling more succinct, we have denoted by ab the MDFIP $\langle \uparrow a, \downarrow b \rangle$. We have also left out the loop on each vertex to keep the display less cluttered. Observe that the directed edge set is not a transitive relation. The labels L_4 and L_4^∂ (as well as L_3^∂ which appears later) come from the paper by Davey *et al.* [7].

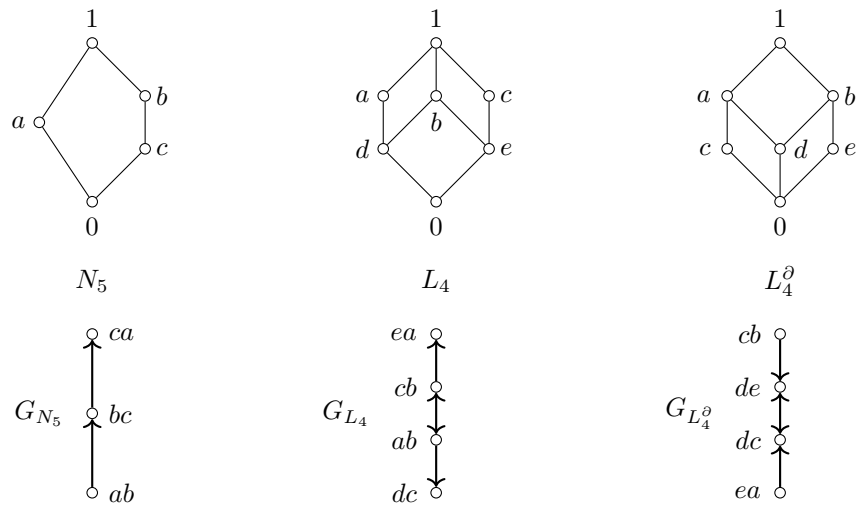


Figure 1: Finite lattices N_5 , L_4 , L_4^∂ and their dual digraphs.

The digraphs coming from lattices were described by Craig, Gouveia and Haviar [6]. The name “TiRS” comes from combining the conditions (Ti), (R), (S) below, where they are abbreviations for “transitive interval”, “reduced” and “separated” respectively.

Definition 2.4 ([6, Definition 2.2]). A *TiRS digraph* $G = (V, E)$ is a set V and a reflexive relation $E \subseteq V \times V$ such that:

(S) If $x, y \in V$ and $x \neq y$ then $xE \neq yE$ or $Ex \neq Ey$.

(R) For all $x, y \in V$, if $xE \subset yE$ then $(x, y) \notin E$, and if $Ey \subset Ex$ then $(x, y) \notin E$.

(Ti) For all $x, y \in V$, if xEy then there exists $z \in V$ such that $zE \subseteq xE$ and $Ez \subseteq Ey$.

The result below gives a description of dual digraphs of lattices with least and greatest elements.

Proposition 2.5 ([6, Proposition 2.3]). *For any bounded lattice L , its dual digraph $G_L = (X_L, E)$ is a TiRS digraph.*

We recall from [12] a fact concerning general graphs $G = (X, E)$. Let $\mathcal{Q} = (\{0, 1\}, \leq)$ denote the two-element graph. A partial map $\varphi: X \rightarrow \mathcal{Q}$ preserves the relation E if $x, y \in \text{dom } \varphi$ and xEy imply $\varphi(x) \leq \varphi(y)$. The set of maximal partial E -preserving maps (*i.e.* those that cannot be properly extended) from G to \mathcal{Q} is denoted by $\mathfrak{S}^{\text{mp}}(G, \mathcal{Q})$. We use the abbreviation MPEs for such partial maps.

For a graph $G = (X, E)$ and $\varphi, \psi \in \mathfrak{S}^{\text{mp}}(G, \mathcal{Q})$, it was shown by Ploščica [12, Lemma 1.3] that $\varphi^{-1}(1) \subseteq \psi^{-1}(1) \iff \psi^{-1}(0) \subseteq \varphi^{-1}(0)$. This implies that the reflexive and transitive binary relation \leq defined on $\mathfrak{S}^{\text{mp}}(G, \mathcal{Q})$ by $\varphi \leq \psi \iff \varphi^{-1}(1) \subseteq \psi^{-1}(1)$ is a partial order. In fact, this is a lattice order [3, Theorem 2.3]. For a graph $G = (X, E)$, denote by $\mathbb{C}(G)$ the (complete) lattice of MPEs $(\mathfrak{S}^{\text{mp}}(G, \mathcal{Q}), \leq)$.

The theorem below gives a one-to-one correspondence between finite lattices and finite TiRS digraphs. This result is essential to the work done in the rest of the current paper.

Theorem 2.6 ([6, Theorem 1.7 and p. 87]). *For any finite lattice L we have that L is isomorphic to $\mathbb{C}(G_L)$ and for any finite TiRS digraph $G = (V, E)$ we have that G is isomorphic to $G_{\mathbb{C}(G)}$.*

3 Generalising lower and upper semimodularity

For lattice elements a and b we write $a \prec b$ to denote that a is covered by b . A lattice is *upper semimodular* if whenever $a \wedge b \prec a$ then $b \prec a \vee b$. It is common to refer to such lattices as *semimodular*. A lattice is *lower semimodular* if whenever $a \prec a \vee b$ then $a \wedge b \prec b$. We use (USM) and (LSM) as abbreviations for these two conditions.

The lattices in Figure 1 provide useful examples: N_5 satisfies neither (USM) nor (LSM), L_4 satisfies (USM) but not (LSM), and L_4^∂ satisfies (LSM) but not (USM).

We will focus on lower semimodularity, rather than upper semimodularity, because of the connection between lower semimodularity and finite convex geometries (see Section 4). We note that modularity implies both semimodularity and lower semimodularity. If a lattice L has finite length and is semimodular and lower semimodular, then L is also modular (*cf.* [9, Corollary 376]). For further reading we refer to the book by Stern [14].

Figure 2 presents a number of different generalisations of distributivity and modularity (including those presented above) and the relationships between them. The ‘B’ denotes *bounded* in the sense of bounded homomorphic image of a free lattice (*cf.* [9, p. 504]). Observe that the conditions in the top left and top right, which are weakenings of (LSM) and (USM) respectively, are in fact

conditions on the standard context dual to a finite lattice. For the necessary terms and notation, we refer to the book from where Figure 2 is taken [8, p. 234].

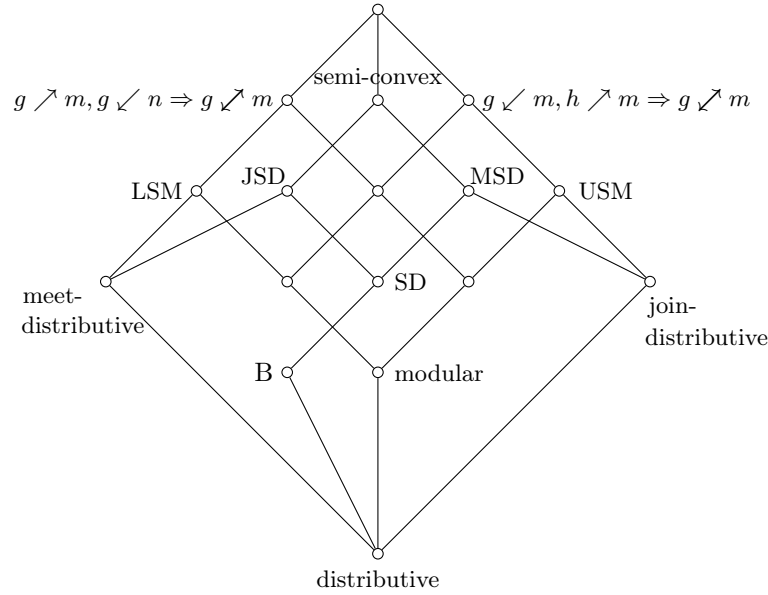


Figure 2: Relationships between generalisations of distributivity.

We begin by proving some new results about MDFIPs. These will be needed in the proofs of later results.

Lemma 3.1. *Let L be a finite lattice.*

- (i) *If $b \in M(L)$ and $b \prec a \vee b$, then $\downarrow b$ is maximal with respect to being disjoint from $\uparrow a$.*
- (ii) *If $a \in J(L)$ and $a \wedge b \prec a$, then $\uparrow a$ is maximal with respect to being disjoint from $\downarrow b$.*

Proof. Assume that $b \in M(L)$ and $b \prec a \vee b$. This implies $b < a \vee b$ and hence $a \not\leq b$ and so $\uparrow a \cap \downarrow b = \emptyset$. Suppose the ideal $\downarrow b$ were to be extended to $\downarrow c$ with $b < c$ and $\uparrow a \cap \downarrow c = \emptyset$. Since $b \in M(L)$, the element $a \vee b$ is the unique upper cover of b and so $a \vee b \in \downarrow c$. This implies $a \vee b \in \uparrow a \cap \downarrow c$, a contradiction, showing the maximality of $\downarrow b$ with respect to being disjoint from $\uparrow a$.

The proof of (ii) follows by a dual argument. □

The next theorem gives a characterisation of MDFIPs.

Theorem 3.2. *A disjoint filter-ideal pair $\langle \uparrow a, \downarrow b \rangle$ is an MDFIP if and only if it satisfies the following conditions:*

- (i) $a \in J(L)$;
- (ii) $b \in M(L)$;
- (iii) $b \prec a \vee b$;
- (iv) $a \wedge b \prec a$.

Proof. If $\langle \uparrow a, \downarrow b \rangle$ is an MDFIP, by Proposition 2.2, $a \in J(L)$ and $b \in M(L)$. We also have $b < a \vee b$, since $b = a \vee b$ would imply $a \in \downarrow b$. Suppose there exists $c \in L$ such that $b < c < a \vee b$. If $a \leq c$ then c would be an upper bound for $\{a, b\}$ and then $a \vee b \leq c$. Therefore $a \not\leq c$. This would make $\langle \uparrow a, \downarrow c \rangle$ a disjoint filter-ideal pair with $\downarrow b \subsetneq \downarrow c$, contradicting the maximality of the pair $\langle \uparrow a, \downarrow b \rangle$. A dual argument can be applied to show that $a \wedge b \prec a$.

Assume $\langle \uparrow a, \downarrow b \rangle$ satisfies (i) – (iv). Lemma 3.1 says $\downarrow b$ is maximal with respect to being disjoint from $\uparrow a$ and vice versa. Hence $\langle \uparrow a, \downarrow b \rangle$ is an MDFIP. \square

The lemmas below will be used in our later investigations.

Lemma 3.3. *Let L be a finite lattice, $a, b \in L$. The following are equivalent:*

- (i) $a \not\leq b$;
- (ii) *there exists $j \in J(L)$ such that $j \leq a$ and $j \not\leq b$;*
- (iii) *there exists $m \in M(L)$ such that $b \leq m$ and $a \not\leq m$.*

Proof. It is well-known that in a finite lattice the set $J(L)$ is join-dense. Hence $a \leq b$ is equivalent to the condition that for all $j \in J(L)$, $j \leq a$ implies $j \leq b$. This settles the equivalence of (i) and (ii). The equivalence of (i) and (iii) follows similarly from the meet-density of $M(L)$ in L . \square

For $a, b \in L$ we define the set $T_{ab} := \{m \in M(L) \mid b \leq m, a \not\leq m\}$. An important consequence of Lemma 3.3 is that T_{ab} is non-empty whenever $a \not\leq b$. This is needed for our next result.

Lemma 3.4. *Let L be a finite lattice and $a, b \in L$, $a \not\leq b$. Let d be a maximal element of T_{ab} . Then $d \prec d \vee a$.*

Proof. Firstly, we point out that T_{ab} is a non-empty finite poset and hence has a maximal element. Since $a \not\leq d$, we have $a \vee d \neq d$, and so $d < d \vee a$. Suppose there exists $c \in L$ such that $d < c < d \vee a$. As $d \vee a \not\leq c$, by Lemma 3.3 there exists $m \in M(L)$ such that $c \leq m$ but $d \vee a \not\leq m$. So $d < m$. If $a \leq m$ then $d \vee a \leq m$. It follows $a \not\leq m$ and $b \leq d < m$, so $m \in T_{ab}$. Since d was maximal in T_{ab} and $d < m$, we get a contradiction. Hence $d \prec d \vee a$. \square

From the previous lemmas one can derive the following result.

Proposition 3.5. *Let L be a finite lattice with $a \in J(L)$ and $b \in M(L)$. Then*

- (i) *there exists $m \in M(L)$ such that $\langle \uparrow a, \downarrow m \rangle$ is an MDFIP;*
- (ii) *there exists $j \in J(L)$ such that $\langle \uparrow j, \downarrow b \rangle$ is an MDFIP.*

Proof. We prove only (i), as then (ii) will follow by a dual argument. Since $a \in J(L)$, it has a unique lower cover c . Clearly $a \not\leq c$, so by Lemma 3.4, there exists a maximal element $m \in T_{ac}$ such that $m \prec m \vee a$. From Lemma 3.1(i) we know that $\downarrow m$ is maximal with respect to being disjoint from $\uparrow a$. If it were possible to extend $\uparrow a$ to $\uparrow d$ with $d < a$, then since c is the unique lower cover of a , we would get $c \in \uparrow d \cap \downarrow m$. Hence $\uparrow a$ is maximal with respect to being disjoint from $\downarrow m$. It follows that $\langle \uparrow a, \downarrow m \rangle$ is an MDFIP. \square

We now define a new condition, (JM-LSM), which will be central to the results that follow. We believe it is a more natural weakening of (LSM) than the condition given in the top left of Figure 2. The name of the condition comes from the fact that it is almost identical to the condition (LSM), but the elements involved are quantified over $J(L)$ and $M(L)$.

Definition 3.6. *A finite lattice L satisfies (JM-LSM) if for any $a \in J(L)$ and $b \in M(L)$, if $b \prec a \vee b$ then $a \wedge b \prec a$.*

Example 3.7. *Condition (JM-LSM) is a proper weakening of the condition (LSM). Indeed, the lattice in Figure 3 satisfies (JM-LSM) but not (LSM). To see this, observe that $c \prec c \vee d$ and $c \wedge d \not\prec d$, yet $d \notin J(L)$.*

We note that the lattice L_4 in Figure 1 does not satisfy (LSM), and also does not satisfy (JM-LSM): $c \in J(L)$, $a \in M(L)$ and $a \prec c \vee a$, yet $c \wedge a \not\prec c$.

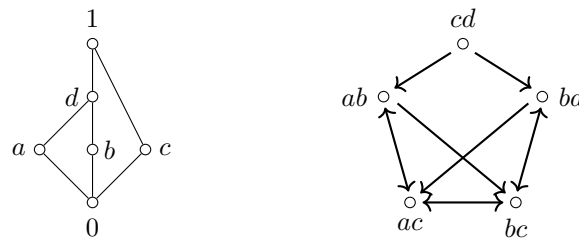


Figure 3: A finite lattice that satisfies (JM-LSM) but not (LSM). Its dual digraph (right) satisfies (LTi).

Below is a condition that we will prove is equivalent to (JM-LSM). It will assist us in proving that the digraph condition (LTi), given in Definition 3.11, can be used to characterise the dual digraphs of finite (JM-LSM) lattices.

Definition 3.8. *Condition (L-abc): Let $a \in J(L)$ and $b \in M(L)$. If $a \not\leq b$ then there exists $c \geq b$ such that $\langle \uparrow a, \downarrow c \rangle$ is an MDFIP.*

Notice that if $\langle \uparrow a, \downarrow c \rangle$ is an MDFIP, then Proposition 2.2 (cf. also Theorem 3.2) implies that for the element c in Definition 3.8 we have $c \in M(L)$. Notice also that the finite lattice L_4 in Figure 1 does not satisfy (L-abc): we have $a \in J(L)$, $c \in M(L)$ and $a \not\leq c$ and there is no $m \geq c$ such that $\langle \uparrow a, \downarrow m \rangle$ is an MDFIP.

The following theorem shows that for finite lattices the central property (JM-LSM) can be characterised exactly via the condition (L-abc).

Theorem 3.9. *A finite lattice satisfies (JM-LSM) iff it satisfies (L-abc).*

Proof. Assume (JM-LSM) and let $a \in J(L)$, $b \in M(L)$ and $a \not\leq b$. Let $T_{ab} = \{m \in M(L) \mid b \leq m \text{ \& } a \not\leq m\}$. Then T_{ab} is a non-empty finite poset. Hence it has a maximal element, say c . So $c \in M(L)$, $b \leq c$ and $\langle \uparrow a, \downarrow c \rangle$ is a disjoint filter-ideal pair. To show that $\langle \uparrow a, \downarrow c \rangle$ is an MDFIP, by Theorem 3.2 we need to show that $c \wedge a \prec a$ and $c \prec c \vee a$. By (JM-LSM) we only need to prove $c \prec c \vee a$, which follows from Lemma 3.4. We have shown that (L-abc) holds.

Now assume (L-abc). To show (JM-LSM), let $a \in J(L)$, $b \in M(L)$ and $b \prec a \vee b$. We need to prove $a \wedge b \prec a$. From $b \prec a \vee b$ we have $a \not\leq b$. By (L-abc) there exists $c \geq b$ such that $\langle \uparrow a, \downarrow c \rangle$ is an MDFIP. Hence $c \in M(L)$ and by Theorem 3.2, $c \wedge a \prec a$. We claim that $c = b$. Suppose that $c > b$. Then, since $b \in M(L)$, it has a unique upper cover b^* . As $b \prec a \vee b$, we get $b^* = a \vee b$. From $c > b$ we have $c \geq b^* = a \vee b \geq a$. This contradicts the fact that $\langle \uparrow a, \downarrow c \rangle$ is an MDFIP. Hence $c = b$. This proves $a \wedge b = c \wedge a \prec a$. \square

Remark 3.10. *We notice that if a finite lattice L satisfies (L-abc), then in the situation $a \not\leq b$ for $a \in J(L)$, $b \in M(L)$, an arbitrary maximal element of T_{ab} can be taken for the element $c \geq b$ such that $\langle \uparrow a, \downarrow c \rangle$ is an MDFIP. Indeed, if c is any maximal element of T_{ab} , then $c \in M(L)$, $a \not\leq c$, $b \leq c$ and so by the assumed condition (L-abc) there is $c' \geq c$ such that $\langle \uparrow a, \downarrow c' \rangle$ is an MDFIP. Hence $c' \in M(L)$, $a \not\leq c'$, $b \leq c'$, thus $c' \in T_{ab}$. From the maximality of c in T_{ab} we get $c = c'$ as required.*

Now we present a digraph condition dual to (JM-LSM). The condition is a strengthening of the (Ti) condition, and because of its connection to lower semimodularity, we have chosen the name (LTi). Later, in Definition 3.16, (UTi) is used for the dual condition related to upper semimodularity.

Definition 3.11. *Consider the condition below on a TiRS digraph $G = (V, E)$:*

$$(LTi) \quad uEv \implies (\exists w \in V)(wE = uE \text{ \& } Ew \subseteq Ev).$$

Note that (LTi) is not dual to (LSM) as Figure 3 shows. For each pair of E related vertices, there is some vertex making the consequent true. For example, if $u = ba$ and $v = ac$, we have $baEac$, and we

can let $w = bc$, since $baE = \{bc, ba, ac\} = bcE$, and $Ebc = \{ab, ac, ba, bc\} \subseteq \{ab, ac, ba, bc\} = Eac$. The next two results prove that it is (JM-LSM) that is dual to (LTi).

Proposition 3.12. *A finite TiRS digraph satisfies (LTi) if and only if it is the dual digraph of a lattice that satisfies (L-abc).*

Proof. Assume a finite lattice L satisfies (L-abc). To show that the dual digraph G_L satisfies (LTi), let $u = \langle \uparrow a, \downarrow m \rangle$, $v = \langle \uparrow j, \downarrow b \rangle$ be vertices of the digraph G and let uEv , whence $a \not\leq b$. Then by (L-abc) there exists $c \in M(L)$ such that $b \leq c$ and $\langle \uparrow a, \downarrow c \rangle$ is an MDFIP. If we denote $w = \langle \uparrow a, \downarrow c \rangle$ as a vertex of G , then by Lemma 2.3 we have $wE = uE$ and $Ew \subseteq Ev$ as required.

For the converse, assume that a finite TiRS digraph G satisfies (LTi). To show that its dual lattice L satisfies (L-abc), let $a \in J(L)$, $b \in M(L)$ and $a \not\leq b$. Since $a \in J(L)$ and L is finite, by Proposition 3.5(i), there exists an element $m \in M(L)$ such that $u = \langle \uparrow a, \downarrow m \rangle$ is an MDFIP. Similarly, since $b \in M(L)$, by Proposition 3.5(ii) there exists $j \in J(L)$ such that $v = \langle \uparrow j, \downarrow b \rangle$ is an MDFIP. Since $a \not\leq b$, we have uEv . Now, by (LTi), there is a vertex $w = \langle \uparrow c, \downarrow d \rangle \in V(G)$ satisfying $wE = uE$ and $Ew \subseteq Ev$. Since $wE = uE$, we get $\uparrow c = \uparrow a$, so $c = a$. Since $Ew \subseteq Ev$, Lemma 2.3(ii) tells us that $d \geq b$. This proves that d is the desired element such that $\langle \uparrow a, \downarrow d \rangle$ is an MDFIP. \square

The main theorem of this section follows directly from Theorem 3.9 and Proposition 3.12.

Theorem 3.13. *A finite TiRS digraph is the dual digraph of a finite lattice satisfying (JM-LSM) if and only if it satisfies (LTi).*

For completeness, we now state the conditions and results related to finite upper semimodular lattices and their dual digraphs.

Definition 3.14. *Let L be a finite lattice. We say that L satisfies the condition (JM-LSM) if whenever $a \in J(L)$, $b \in M(L)$, and $a \wedge b \prec a$, then $b \prec a \vee b$. We say that L satisfies (U-abc) if whenever $a \in J(L)$ and $b \in M(L)$ and $a \not\leq b$ then there exists $c \leq a$ such that $\langle \uparrow c, \downarrow b \rangle$ is an MDFIP.*

The proposition below connects the two conditions defined above.

Proposition 3.15. *A finite lattice satisfies (U-abc) iff it satisfies (JM-USM).*

Our last definition is the condition (UTi) which is, like (LTi), a strengthening of the (Ti) condition from Definition 2.4.

Definition 3.16. *Consider the condition below on a TiRS digraph $G = (V, E)$:*

$$(UTi) \quad uEv \implies (\exists w \in V)(wE \subseteq uE \text{ \& } Ev = Ew).$$

Theorem 3.17. *A finite TiRS digraph satisfies (UTi) if and only if it is the dual digraph of a finite lattice that satisfies (JM-USM).*

4 Dual digraphs of meet-distributive lattices

In this section we will combine the results from Section 3 with results about dual digraphs of finite join- and meet-semidistributive lattices from [5]. The goal is to give a description of the dual digraphs of finite meet-distributive lattices. This will give a description of a new class of structures that are in a one-to-one correspondence with the class of finite convex geometries. First, we recall some basic definitions.

A lattice L is *join-semidistributive* if it satisfies the following quasi-equation for all $a, b, c \in L$:

$$(\text{JSD}) \quad a \vee b \approx a \vee c \quad \longrightarrow \quad a \vee b \approx a \vee (b \wedge c).$$

A lattice L is *meet-semidistributive* if it satisfies the following quasi-equation for all $a, b, c \in L$:

$$(\text{MSD}) \quad a \wedge b \approx a \wedge c \quad \longrightarrow \quad a \wedge b \approx a \wedge (b \vee c).$$

A lattice is *semidistributive* if it satisfies both (JSD) and (MSD).

Considering the lattices in Figure 1 one can see that N_5 is semidistributive, L_4 is meet-semidistributive but not join-semidistributive, and L_4^∂ is join-semidistributive but not meet-semidistributive.

For a finite lattice L and $a \in L$, consider $\mu(a) = \bigwedge \{b \in L \mid b \prec a\}$. A finite lattice is *meet-distributive* (also called *locally distributive*) if for any $a \in L$, the interval $[\mu(a), a]$ is a distributive lattice (cf. [1, Section 5 - 2]). The class of finite meet distributive lattices is an important class of lattices because of their link to finite convex geometries. The following results therefore lead us to a new characterisation of finite convex geometries, which we present in Theorem 4.13 (iv) and (v).

The following equivalence is extracted from [1, Theorem 5-2.1].

Theorem 4.1. *Let L be a finite lattice. Then the following are equivalent:*

- (i) L is meet-distributive;
- (ii) L satisfies (JSD) and (LSM).

The results below use Theorem 4.1 to provide an additional characterisation of meet-distributive lattices using (JM-LSM), the condition that was central to Section 3. Later, we will use this to characterise their dual digraphs.

Theorem 4.2. *If a finite lattice L satisfies (JM-LSM) and (JSD), then it is lower semimodular.*

Proof. Let L be a finite lattice satisfying (JM-LSM) and (JSD). Let $a, b \in L$ be arbitrary such that $a \prec a \vee b$. We are going to show that $a \wedge b \prec b$. We will proceed by contradiction.

Suppose that $a \wedge b \not\prec b$. Then there exists $c \in L$ such that $a \wedge b < c < b$. Then $b \not\leq c$ and by Lemma 3.3 the set $S_{cb} = \{j \in J(L) \mid j \leq b, j \not\leq c\}$ is non-empty. Let p be a minimal element of S_{cb} .

Suppose $p \leq a$, then since $p \leq b$, we get $p \leq a \wedge b \leq c$, which is a contradiction, so $p \not\leq a$. Then by Lemma 3.3, the set $T_{pa} = \{m \in M(L) \mid a \leq m \text{ and } p \not\leq m\}$ is non-empty. Let m be a maximal element of T_{pa} . By Lemma 3.4, $m \prec m \vee p$. Since $m \in M(L)$, $p \in J(L)$, and L satisfies (JM-LSM), we obtain $m \wedge p \prec p$.

The join irreducible element p has a unique lower cover p_* ; likewise the meet irreducible element m has a unique upper cover m^* . Then $p_* \leq m$ as $p_* = m \wedge p$. Now $p \not\leq c$ and $p_* \leq c$ imply $c \wedge p = p_*$. Analogously, $p \not\leq m$ and $p \leq m^*$ imply $m \vee p = m^*$. It follows that $c \not\leq m$ as otherwise we get $c \leq m \wedge (a \vee b) = a$, whence $c \leq a \wedge b$, which contradicts $a \wedge b < c$. But $c \leq m^*$ since $m^* \geq a \vee p = a \vee b \geq b$. Here we used that since $p \leq b$, we have $a \leq a \vee p \leq a \vee b$, and since $a \prec a \vee b$, we have $a = a \vee p$ or $a \vee p = a \vee b$. But $a \neq a \vee p$ since $p \not\leq a$, so $a \vee p = a \vee b$.

Hence $m \vee c = m^*$. Combining the above,

$$m^* = m \vee p = m \vee c = m \vee (p \wedge c) = m \vee p_* = m$$

by (JSD), a contradiction. Hence c cannot exist showing that $a \wedge b \prec b$. □

Remark 4.3. *Notice in the proof we actually use a weaker form of (JSD). We will say that a lattice L is weakly join-semidistributive if it satisfies the following quasi-equation for all $a \in M(L)$, $b \in J(L)$, $c \in L$:*

$$(W\text{-JSD}) \quad a \vee b \approx a \vee c \quad \longrightarrow \quad a \vee b \approx a \vee (b \wedge c).$$

Hence in Theorem 4.2 we actually showed that (JM-LSM) and (W-JSD) implies (LSM).

We notice the lattice in Figure 3 satisfies (JM-LSM) but not (W-JSD): indeed $c \in M(L)$, $b \in J(L)$ and $c \vee b = c \vee a$ but $c \vee (b \wedge a) \neq c \vee a$.

The result below follows from Theorems 4.1 and 4.2.

Corollary 4.4. *A finite lattice is meet-distributive if and only if it satisfies both (JM-LSM) and (JSD).*

The following theorem provides a characterisation of the dual digraphs of finite join- and meet-semidistributive lattices. Its proof (see [5]) relies on the well-known κ map used in the charac-

terisation of semidistributivity. Notice that each of the conditions (i), (ii) and (iii) below is a strengthening of the (S) condition from the definition of TiRS digraphs (Definition 2.4).

Theorem 4.5 ([5, Theorem 3.6]). *Let $G = (V, E)$ be a finite TiRS digraph with $u, v \in V$. Then*

(i) *G is the dual digraph of a finite lattice satisfying (JSD) if and only if it satisfies the following condition:*

$$(dJSD) \quad \text{if } u \neq v \text{ then } Eu \neq Ev.$$

(ii) *G is the dual digraph of a finite lattice satisfying (MSD) if and only if it satisfies the following condition:*

$$(dMSD) \quad \text{if } u \neq v \text{ then } uE \neq vE.$$

(iii) *G is the dual digraph of a finite semidistributive lattice if and only if it satisfies the following condition:*

$$(dSD) \quad \text{if } u \neq v \text{ then } Eu \neq Ev \text{ and } uE \neq vE.$$

The next few results in this section link the properties discussed earlier to distributivity in lattices and transitivity in dual digraphs.

Theorem 4.6. *Let $G = (V, E)$ be a finite TiRS digraph that satisfies both (dMSD) and (LTi). Then E is transitive.*

Proof. We first claim that if a finite TiRS digraph $G = (V, E)$ satisfies both (dMSD) and (LTi), then for any vertices $u, v \in V$, uEv implies $Eu \subseteq Ev$. Indeed, uEv by (LTi) implies the existence of $w \in V$ such that $wE = uE$ and $Ew \subseteq Ev$. By the property (dMSD), $wE = uE$ means $w = u$, whence $Eu \subseteq Ev$ as required.

Now to show the transitivity of E , if uEv and vEw for some vertices $u, v, w \in V$, then by the above claim, $Eu \subseteq Ev$ and $Ev \subseteq Ew$. Hence $Eu \subseteq Ew$, which means $u \in Ew$, whence uEw as required. \square

Proposition 4.7. *If $G = (V, E)$ is TiRS digraph with transitive E , then G is a poset.*

Proof. As in a TiRS digraph $G = (V, E)$ the relation E is reflexive, it only remains to show the antisymmetry of E .

Assume for $x, y \in V$ that xEy and yEx . We firstly show that $xE \subseteq yE$: if $z \in V$ and $z \in xE$, then xEz and with yEx we get yEz by transitivity of E , hence $z \in yE$ as required. Now $xE \subset yE$ by the condition (R) from Definition 2.4 would give $(x, y) \notin E$, a contradiction. Hence $xE = yE$.

Analogously one can show that $Ey \subseteq Ex$ and since $Ey \subset Ex$ would by (R) give $(x, y) \notin E$, we have $Ey = Ex$. Using that G satisfies the separation property (S) from Definition 2.4, it follows that $x = y$ as required. \square

The result below follows from Theorem 4.6, Proposition 4.7 and Birkhoff's one-to-one correspondence between finite distributive lattices and finite posets, which was in [6] generalised into a one-to-one correspondence between the class of finite lattices and finite TiRS digraphs (*cf.* Theorem 2.6 here).

Corollary 4.8. *If a finite lattice L satisfies (MSD) and (JM-LSM), then L is distributive.*

We now return to focus on finite meet-distributive lattices, with the goal of describing a class of digraphs connected to finite convex geometries.

Using the TiRS conditions, our conditions for the dual digraphs of (JM-LSM) and (JSD), respectively, and Corollary 4.4, we get the following dual condition for meet-distributivity. Notice how (dJSD) is a strengthening of the (S) condition, and (LTi) is a strengthening of the (Ti) condition.

Theorem 4.9. *A finite digraph $G = (V, E)$ with a reflexive relation E is the dual digraph of some finite meet-distributive lattice if and only if G satisfies the following conditions:*

(dJSD) *If $x, y \in V$ and $x \neq y$ then $Ex \neq Ey$.*

(R) *For all $x, y \in V$, if $xE \subset yE$ then $(x, y) \notin E$, and if $Ey \subset Ex$ then $(x, y) \notin E$.*

(LTi) *For all $x, y \in V$, if xEy then there exists $z \in V$ such that $zE = xE$ and $Ez \subseteq Ey$.*

Proof. Let G be the dual digraph of some finite meet-distributive lattice L . Then by Theorem 2.6 the digraph G will satisfy (R). By Corollary 4.4, L satisfies (JSD) and (JM-LSM). Hence by Theorem 4.5(i), G satisfies (dJSD). Lastly, by Theorem 3.13, G will satisfy (LTi).

Conversely, assume G satisfies (dJSD), (R) and (LTi). Clearly G is a TiRS digraph, hence the dual of a finite lattice L . Theorem 4.5(i) shows that L satisfies (JSD) and Theorem 3.13 implies that L satisfies (JM-LSM). Hence by Corollary 4.4, L is meet-distributive. \square

The theorem above establishes a one-to-one correspondence between finite meet-distributive lattices and finite digraphs satisfying the conditions (dJSD), (R) and (LTi). It is a restriction of Theorem 2.6, while still generalising Birkhoff's one-to-one correspondence between finite distributive lattices and finite posets.

Definition 4.10 ([9, Definition 30]). *Let X be a set and $\phi : \wp(X) \rightarrow \wp(X)$. Then ϕ is a closure operator on X if for all $Y, Z \in \wp(X)$*

(i) $Y \subseteq \phi(Y)$;

(ii) $Y \subseteq Z$ implies $\phi(Y) \subseteq \phi(Z)$;

(iii) $\phi(\phi(Y)) = \phi(Y)$.

If X is a set and ϕ a closure operator on X then the pair $\langle X, \phi \rangle$ is called a closure system. For $Y \subseteq X$ we say that Y is closed if $\phi(Y) = Y$. The closed sets of a closure operator ϕ on X form a complete lattice, denoted by $\text{Cld}(X, \phi)$. A zero-closure system is a closure system $\langle X, \phi \rangle$ such that $\phi(\emptyset) = \emptyset$.

Now we turn our attention to convex geometries. The presentation here follows that of the book chapter by Adaricheva and Nation [1].

Definition 4.11 ([1, Definition 5-1.1]). A closure system $\langle X, \phi \rangle$ satisfies the anti-exchange property if for all $x \neq y$ and all closed sets $A \subseteq X$,

$$(AEP) \quad x \in \phi(A \cup \{y\}) \text{ and } x \notin A \text{ imply that } y \notin \phi(A \cup \{x\}).$$

Definition 4.12 ([2, Definition 1.6]). A zero-closure system that satisfies the anti-exchange property is called a convex geometry.

We now combine Theorem 4.9 with known equivalences to obtain the following characterisation of finite convex geometries. There are other equivalent conditions [1, Theorem 5-2.1] that we have not included here.

Theorem 4.13. Let L be a finite lattice. Then the following are equivalent:

- (i) L is the closure lattice $\text{Cld}(X, \phi)$ of a closure space $\langle X, \phi \rangle$ with the (AEP).
- (ii) L is a meet-distributive lattice.
- (iii) L satisfies (JSD) and (LSM).
- (iv) L satisfies (JSD) and (JM-LSM).
- (v) L is the lattice $\mathbb{C}(G)$ of a reflexive digraph G satisfying (dJSD), (R) and (LTi).

Proof. The equivalences of (i), (ii) and (iii) are known [1, Theorem 5-2.1]. The equivalence of (iii) and (iv) is the result of Corollary 4.4, and the equivalence of (iv) and (v) is Theorem 4.9. \square

5 Dual digraphs of finite modular lattices

In this section we provide two sufficient conditions for a finite TiRS digraph to be the dual digraph of a finite modular lattice.

For $i = 0, 1, 2$, let us denote by $G_i = (V_i, E_i)$ an induced subgraph of G_{N_5} (see Figure 1) with $V_i = \{x, y, z\}$ and with i of the arcs xEy and yEz missing compared to G_{N_5} . (For $i = 1$ we can,

w.l.o.g., consider the arc yEz missing.) Hence $G_0 = G_{N_5}$, G_1 has one arc and an isolated vertex, and G_2 has no arc and consists of two isolated vertices. All three digraphs are reflexive, hence they have loops at each vertex.

We introduce the following condition for the dual digraph G_L of a finite lattice L in terms of “Forbidden Induced Subgraphs”:

(FIS) G_L has neither $G_0 = G_{N_5}$ nor G_1 as an induced subgraph.

The next lemma and two propositions lead to showing that the condition (FIS) is sufficient for modularity of a finite lattice L . Note that by Lemma 3.3, for $a, b \in L$ with $a \not\leq b$, there always exist elements $\underline{a} \leq a$ and $\bar{b} \geq b$ such that $\langle \uparrow \underline{a}, \downarrow \bar{b} \rangle$ is an MDFIP. Below we write $a||b$ to indicate that $a \not\leq b$ and $b \not\leq a$.

Lemma 5.1. *Let $a, b, c, 0, 1$ be any elements of the lattice that form a sublattice isomorphic to N_5 (where $0 < a, b, c < 1$, $c < b$ and $a||b, a||c$). (See the left side of Figure 4.) Let $x = \langle \uparrow \underline{a}, \downarrow \bar{b} \rangle$, $y = \langle \uparrow \underline{b}, \downarrow \bar{c} \rangle$ and $z = \langle \uparrow \underline{c}, \downarrow \bar{a} \rangle$ be any maximal disjoint extensions of $\langle \uparrow a, \downarrow b \rangle$, $\langle \uparrow b, \downarrow c \rangle$ and $\langle \uparrow c, \downarrow a \rangle$, respectively. Then the induced subgraph $\{x, y, z\}$ of G_L is isomorphic either to $G_0 = G_{N_5}$, G_1 , or G_2 .*

Proof. First we must confirm that x, y, z are distinct MDFIPs. If $x = y$ then $\uparrow \underline{a} = \uparrow \underline{b}$ which implies $\uparrow \underline{a} \cap \downarrow \bar{b} \neq \emptyset$, i.e. x would not be an MDFIP. If $x = z$ then $\uparrow \underline{a} = \uparrow \underline{c}$ which means z would not be an MDFIP. Lastly, if $y = z$ then $\downarrow \bar{c} = \downarrow \bar{a}$ and z would not be an MDFIP.

We claim that in the induced subgraph $\{x, y, z\}$ of G_L , the arcs xEy and yEz are possible, but the induced subgraph $\{x, y, z\}$ has none of the other four possible arcs between distinct vertices: indeed, the arcs yEx , zEy , xEz and zEx are not present in G_L because clearly $b \in \uparrow \underline{b} \cap \downarrow \bar{b}$, $c \in \uparrow \underline{c} \cap \downarrow \bar{c}$, $a \in \uparrow \underline{a} \cap \downarrow \bar{a}$ and $c \in \uparrow \underline{c} \cap \downarrow \bar{b}$, respectively.

Hence $\{x, y, z\}$ is isomorphic to G_i in case i of the arcs xEy and yEz are missing in the induced subgraph $\{x, y, z\}$ for $i = 0, 1, 2$. \square

Proposition 5.2. *Let L be a finite lattice and assume that its dual digraph $G_L = (V, E)$ satisfies (FIS). Then L is lower semimodular.*

Proof. Suppose to the contrary that L does not satisfy (LSM). Then there exist elements $a, b \in L$ such that $a \prec a \vee b$ but $a \wedge b \not\prec b$. Then there exists an element $c \in L$ such that $a \wedge b < c < b$. Hence $a \vee c \leq a \vee b$. Since $a \prec a \vee b$, and $a \leq a \vee c \leq a \vee b$, we get $a \vee c = a$ or $a \vee c = a \vee b$. If $a \vee c = a$, then $c \leq a$, so $c \leq a \wedge b$, which contradicts $a \wedge b < c$. It follows that $a \vee c = a \vee b$. From $c < b$ we get $a \wedge c \leq a \wedge b$. Further, since $a \wedge b < c$ we get $a \wedge (a \vee b) = a \wedge b \leq a \wedge c$. Thus $a \wedge c = a \wedge b$.

Hence $a, c, b, a \wedge b, a \vee b$ forms a sublattice isomorphic to N_5 (see Figure 4). Let $x = \langle \uparrow \underline{a}, \downarrow \bar{b} \rangle$, $y = \langle \uparrow \underline{b}, \downarrow \bar{c} \rangle$ and $z = \langle \uparrow \underline{c}, \downarrow \bar{a} \rangle$, be arbitrary maximal disjoint extensions of $\langle \uparrow a, \downarrow b \rangle$, $\langle \uparrow b, \downarrow c \rangle$ and $\langle \uparrow c, \downarrow a \rangle$, respectively. Then by Lemma 5.1, the induced subgraph $\{x, y, z\}$ of G_L is isomorphic to $G_0 = G_{N_5}$, G_1 , or G_2 . Using the assumption (FIS), $\{x, y, z\}$ must be isomorphic to G_2 .

In particular, it follows that G_L does not have the arc yEz . Therefore $\underline{b} \leq \bar{a}$. Suppose $a = \bar{a}$. Then $\underline{b} \leq a$, so $\underline{b} \leq a \wedge b$. This gives $\underline{b} \leq c \leq \bar{c}$, which contradicts the fact that $y = \langle \uparrow \underline{b}, \downarrow \bar{c} \rangle$ is a disjoint filter-ideal pair. Hence $a < \bar{a}$. Now either $\bar{a} < a \vee b$ or $\bar{a} \parallel a \vee b$, since if $\bar{a} \geq a \vee b > c \geq \underline{c}$ then $z = \langle \uparrow \underline{c}, \downarrow \bar{a} \rangle$ could not be a disjoint filter-ideal pair.

If $a < \bar{a} < a \vee b$, this contradicts $a \prec a \vee b$, so $\bar{a} \parallel a \vee b$. If $\underline{b} > a$ then $b > \underline{b} > a$, which contradicts $a \parallel b$. If $\underline{b} \leq a$, then $\underline{b} \leq a \wedge b \leq c \leq \bar{c}$, which contradicts that $y = \langle \uparrow \underline{b}, \downarrow \bar{c} \rangle$ is a disjoint filter-ideal pair. This proves that $\underline{b} \parallel a$. Since $\underline{b} \leq b$, $a \vee \underline{b} \leq a \vee b$. If $a \vee \underline{b} = a \vee b$, then since $a < \bar{a}$ and $\underline{b} \leq \bar{a}$, we get $\bar{a} \geq a \vee \underline{b} = a \vee b$, which contradicts $\bar{a} \parallel a \vee b$. This establishes that $a \vee \underline{b} < a \vee b$ and $a < a \vee \underline{b}$ (since $\underline{b} \parallel a$), which contradicts $a \prec a \vee b$. Hence, our assumption that L does not satisfy (LSM) leads to a contradiction. \square



Figure 4: The isomorphic copies of N_5 constructed in Proposition 5.2 (left) and Proposition 5.3 (right).

Below we give the result dual to Proposition 5.2. The proof is similar to the above argument, so we omit some of the details.

Proposition 5.3. *Let L be a finite lattice and assume that its dual digraph $G_L = (V, E)$ satisfies (FIS). Then L is upper semimodular.*

Proof. Suppose L does not satisfy (USM). Then there are elements $a, b \in L$ such that $a \wedge b \prec b$ but $a \not\prec a \vee b$, i.e. there is $d \in L$ such that $a < d < a \vee b$. Analogous to the proof of Proposition 5.2, it can be shown that the elements $b, a, d, a \wedge b, a \vee b$ form a sublattice isomorphic to N_5 (see Figure 4).

Then by Lemma 5.1, arbitrary maximal disjoint extensions of $\langle \uparrow b, \downarrow d \rangle$, $\langle \uparrow d, \downarrow a \rangle$ and $\langle \uparrow a, \downarrow b \rangle$, denoted by $x = \langle \uparrow \underline{b}, \downarrow \bar{d} \rangle$, $y = \langle \uparrow \underline{d}, \downarrow \bar{a} \rangle$ and $z = \langle \uparrow \underline{a}, \downarrow \bar{b} \rangle$, respectively, form an induced subgraph $\{x, y, z\}$ of G_L that is isomorphic either to $G_0 = G_{N_5}$, G_1 , or G_2 . Using (FIS), $\{x, y, z\}$ is isomorphic to G_2 .

In particular, it follows that G_L does not have the arc xEy . Hence, $\underline{b} \leq \bar{a}$. We can then get $\underline{b} < b$ (as we got $a < \bar{a}$ in Proposition 5.2—see the left lattice in Figure 4). Now either $a \wedge b < \underline{b}$ or

$a \wedge b || \underline{b}$.

If $a \wedge b < \underline{b} < b$, this contradicts $a \wedge b \prec b$, so $\underline{b} || a \wedge b$. We can also show $\underline{b} || \bar{a}$ (as we showed $\underline{b} || a$ in Proposition 5.2).

Since $a \leq \bar{a}$, we get $a \wedge b \leq \bar{a} \wedge b$. We can again establish that $a \wedge b < \bar{a} \wedge b$ and $\bar{a} \wedge b < b$ (since $\underline{b} || \bar{a}$), which contradicts $a \wedge b \prec b$. Hence, our assumption that L does not satisfy (USM) leads to a contradiction. \square

Now we can deduce that the condition (FIS) is a sufficient condition for modularity of a finite lattice.

Theorem 5.4. (Sufficient condition for modularity) *Let L be a finite lattice with dual TiRS digraph G_L . If G_L satisfies (FIS) then L is modular.*

Proof. It follows by Propositions 5.2 and 5.3 that L satisfies both (LSM) and (USM). Since L is finite, we have that L is modular [9, Corollary 376]. \square

We notice that the dual digraph of the modular lattice M_3 has neither $G_0 = G_{N_5}$ nor G_1 as an induced subgraph (see Figure 5), hence it satisfies (FIS). The following example shows that the digraphs G_0 and G_1 cannot be dropped as forbidden induced subgraphs in the condition (FIS) for the dual digraph G_L , which guarantees the modularity of a finite lattice L .

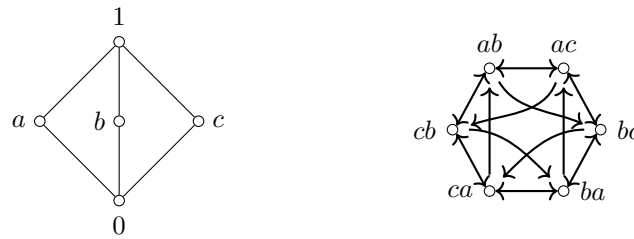


Figure 5: M_3 and its dual digraph.

Example 5.5. *The dual digraph of L_3^∂ in Figure 3 contains G_0 as an induced subgraph, but not G_1 . Hence the lattice L_3^∂ (in addition to N_5) witnesses that the digraph G_0 cannot be dropped from the condition (FIS).*

The dual digraphs of the lattices L_4 and L_4^∂ in Figure 1 do not contain G_0 as an induced subgraph but they both contain G_1 as an induced subgraph. Hence these two examples witness that the digraph G_1 cannot be dropped from the condition (FIS).

Now we are going to show that the condition (FIS) is not necessary for modularity. Indeed, it is not the case that every lattice whose dual digraph has $G_0 = G_{N_5}$ as an induced subgraph is a

non-modular lattice. The next example gives a modular lattice whose dual digraph has G_0 as an induced subgraph (but does not have G_1 as an induced subgraph).

Example 5.6. (Condition (FIS) not necessary for modularity) Figure 6 shows a modular lattice K on the left, and its dual digraph on the right. The induced subgraph isomorphic to G_0 is shown with the dotted arrows ($dcEcb$ and $cbEed$).

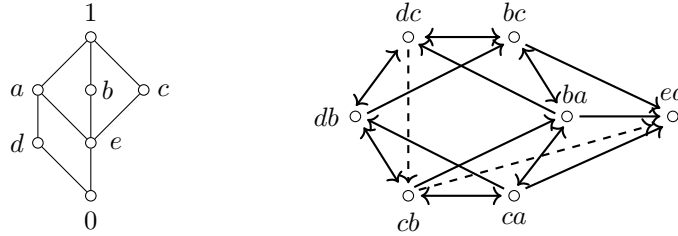


Figure 6: A finite modular lattice K whose dual digraph contains $G_0 = G_{N_5}$ as an induced subgraph.

The fact that the dual TiRS digraph $G_L = (V, E)$ of a finite modular lattice L does not contain $G_0 = G_{N_5}$ as an induced subgraph can be understood as some form of a “weak transitivity” condition for G_L . We cannot have the arcs xEy and yEz in G_L without having also the arc xEz or at least the arc zEx (provided there are no “opposite” arcs yEx and zEy in G_L):

$$\begin{aligned}
 (\text{wT0}) \quad & \text{for all vertices } x, y, z \in V, \text{ if } xEy \text{ and } yEz, \text{ but } (y, x) \notin E \text{ and} \\
 & (z, y) \notin E, \text{ then } xEz \text{ or } zEx.
 \end{aligned}$$

Similarly, the fact that the dual TiRS digraph $G_L = (V, E)$ of a finite modular lattice L does not contain the digraph G_1 as an induced subgraph can be understood as some form of a “weak transitivity” condition for G_L :

$$\begin{aligned}
 (\text{wT1}) \quad & \text{for all vertices } x, y, z \in V, \text{ if } xEy \text{ but } (y, x) \notin E \text{ and } (y, z) \notin E \\
 & \text{and } (z, y) \notin E \text{ then } xEz \text{ or } zEx.
 \end{aligned}$$

Example 5.7. It is easy to see that the dual digraph of the lattice M_3 (Figure 5) satisfies the weak transitivity conditions (wT0) and (wT1). The lattices L_4 and L_4^∂ in Figure 1, and L_3^∂ in Figure 3 are non-modular lattices. The weak transitivity condition (wT0) is not satisfied in the dual digraph of L_3^∂ . In the dual digraphs of the lattices L_4 and L_4^∂ we see the failures of (wT1).

We notice that the weak transitivity conditions (wT0) and (wT1) are essentially expressing on the digraph side that the digraph G_L does not contain respectively the graphs G_0 and G_1 as induced subgraphs.

Hence the sufficiency of the quasi-equations (wT0) and (wT1) on the dual TiRS digraphs G_L for the modularity of L comes as no surprise:

Corollary 5.8 (Sufficient condition for modularity by “weak transitivity”). *Let L be a finite lattice with dual TiRS digraph $G_L = (V, E)$. If G_L satisfies the weak transitivity conditions (wT0) and (wT1), then L is modular.*

Proof. Let the weak transitivity conditions (wT0) and (wT1) be satisfied in G_L . Suppose for contradiction that the lattice L is not modular. Then by Theorem 5.4, for some $i \in \{0, 1\}$ the digraph G_L contains the digraph G_i as an induced subgraph on certain vertices $x, y, z \in V$. It follows that the weak transitivity condition (wTi) is not satisfied. \square

6 Conclusions and future work

In Section 3 we defined two lattice conditions which generalise lower semimodularity and (upper) semimodularity respectively. We were motivated by Figure 2, taken from Ganter and Wille’s book [8] (see also the PhD thesis of Reppe [13, Chapter 3.7]). There, weakenings of (LSM) and (USM) are given using complicated conditions on standard contexts. Our lattice-theoretic conditions on finite lattices that are weakenings of (LSM) and (USM), which we call (JM-LSM) and (JM-USM), seem to be simpler than the mentioned conditions in Figure 2 and they are easily seen to be generalisations of (LSM) and (USM). The top left and top right conditions in Figure 2 were shown to be equivalent to (JM-LSM) and (JM-USM) by Kadima [10, Theorem 4.9].

In Section 4 we used the results of Section 3 to obtain a new characterisation of meet-distributive lattices in Theorem 4.1. Combining this with previous results [5], we obtained a characterisation of the dual digraphs of finite meet-distributive lattices. Theorem 4.13 shows that we have identified a new class of structures that is in a one-to-one correspondence with finite convex geometries.

In Remark 4.3 we gave a condition, (W-JSD), which is a weakening of join-semidistributivity. The lattice M_3 satisfies (LSM) but not (W-JSD) and hence shows that (LSM) is not equivalent to (JM-LSM) and (W-JSD). This leads us to ask the following question.

Problem 6.1. *Is there another weakening of (JSD) such that when it is combined with (JM-LSM), this will be equivalent to (LSM)?*

Theorem 4.9 gave three conditions ((dJSD), (R) and (LTi)) on reflexive digraphs, which characterise the dual digraphs of finite meet-distributive lattices. This leads to the posing of the following open problem.

Problem 6.2. *Can the conditions (dJSD), (R) and (LTi) be combined to give fewer, and possibly simpler, conditions?*

In Section 5 we introduced the condition (FIS) on dual digraphs and showed that it implies both lower and upper semimodularity of a finite lattice. Hence (FIS) was shown to be a sufficient condition for modularity of a finite lattice (Theorem 5.4). We also formulated a sufficient condition for modularity in different terms in Corollary 5.8. The condition (FIS) was shown not to be necessary for modularity of a finite lattice and hence we raise the following open question.

Problem 6.3. *Is it possible to find forbidden induced subgraphs that characterise the dual digraphs of finite modular lattices in an analogous way to how N_5 characterises modularity?*

The task of representing structures (in our case digraphs) dual to finite modular lattices has proved to be very challenging. We note that in the setting of formal contexts dual to finite lattices, a condition dual to semimodularity has been obtained (*cf.* item (4) of [8, Theorem 42]). We have attempted to translate this condition to TiRS digraphs and the result was a complicated and opaque condition. We do not believe that the translation of this condition and its dual will yield a useful characterisation of the TiRS digraphs dual to finite modular lattices.

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