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Extremal functions and best approximate formulas for the Hankel-type Fock space

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ABSTRACT

In this paper we recall some properties for the Hankel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. This space was introduced by Cholewinsky in 1984 and plays a background to our contribution. Especially, we examine the extremal functions for the difference operator D, and we deduce best approximate inversion formulas for the operator D on the the Hankel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$.

RESUMEN

En este artículo, resumimos algunas propiedades para el espacio de Fock the tipo Hankel $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. Este espacio fue introducido por Cholewinsky en 1984 y es un antecedente para nuestra contribución. Especialmente examinamos las funciones extremales para el operador de diferencia D y deducimos fórmulas de inversión del mejor aproximante para el operador D en el espacio de Fock de tipo Hankel $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$.

Keywords and Phrases: Analytic functions, Hankel-type Fock space, extremal functions.

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1 Introduction

The classical Fock space $\mathscr{F}(\mathbb{C}^d)$ is the Hilbert space of entire functions f on $\mathbb{C}^d$ such that

$$\|f\|^2_{\mathscr{F}(\mathbb{C}^d)} := \frac{1}{\pi^d}\int_{\mathbb{C}^d} |f(z)|^2 e^{-|z|^2}\mathrm{d}x\mathrm{d}y < \infty, \quad z = x + iy,$$

where $|z|^2 = \sum_{k=1}^d (x_k^2 + y_k^2)$ and $\mathrm{d}x\mathrm{d}y = \prod_{k=1}^d \mathrm{d}x_k\mathrm{d}y_k$.

This space was introduced by Bargmann [3], is called also Segal-Bargmann space [5] and it was the aim of many works [4, 6, 22, 28]. Recently the author of the paper studied the extremal functions for the difference and primitive operators on the Fock space $\mathscr{F}(\mathbb{C}^d)$ (see [20, 21]).

Cholewinsky [7] defined the Hankel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ associated with the poly-axially operator. The space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is the Hilbert space of entire functions f on $\mathbb{C}^d$, even with respect to the last variable, such that

$$\|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} := \left[\int_{\mathbb{C}^d} |f(z)|^2 \mathrm{d}m_\alpha(z)\right]^{1/2} < \infty,$$

where m_α is the measure defined for $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ by

$$\mathrm{d}m_\alpha(z) := \frac{1}{\pi^d}\prod_{k=1}^d \frac{|z_k|^{2\alpha_k+2}K_{\alpha_k}(|z_k|^2)}{2^{\alpha_k}\Gamma(\alpha_k+1)}\mathrm{d}z_k, \tag{1.1}$$

and K_{α_k}, $\alpha_k > -1/2$, is the Macdonald function [8].

The generalized Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is equipped with the inner product

$$\langle f, g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} := \int_{\mathbb{C}^d} f(w)\overline{g(w)}\mathrm{d}m_\alpha(w).$$

The Hankel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is also studied in [24], when the author proved an uncertainty principle of Heisenberg type for this space.

Let D be the difference operator defined for $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^{2\nu}$, by

$$Df(z) := \sum_{\nu\in\mathbb{N}^d} a_{\nu+\mathbf{1}} z^{2\nu}.$$

The main goal of the paper is to find the minimizer (denoted by $F^*_{\lambda,D}(h)$) for the extremal problem:

$$\inf_{f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\left\{\lambda\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} + \|Df - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\right\},$$

where $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ and $\lambda > 0$. We prove that the extremal function $F^*_{\lambda,D}(h)$ is given by

$$F^*_{\lambda,D}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

where $\Psi_z(w)$ is the kernel given later in Section 3.

Moreover, we establish best approximate inversion formulas for the difference operator D on the weighted Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. A pointwise approximate inversion formula for the operator D are also discussed.

Recently, the analog results are also proved, for the Fock space $\mathscr{F}(\mathbb{C}^d)$ (see [20, 21]), and for the Bessel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C})$ (see [23, 25]).

The paper is organized as follows. In Section 2 we recall some properties for the Hankel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. In Section 3 we examine the extremal functions for the difference operator D. Finally, in Section 4, we establish best approximate inversion formulas for the operator D on the Hankel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$.

Throughout this paper we shall use on $\mathbb{C}^d$ the multi-index notations.

- For all $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, $z^\nu = \prod_{k=1}^d z_k^{\nu_k}$.
- For any $\nu \in \mathbb{N}^d$, the partial ordering $\geq$ on $\mathbb{N}^d$, which is defined by

$$\nu \geq \mathbf{1} \iff \nu_j \geq 1, \quad \forall j = 1, \ldots, d, \quad \text{with } \mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^d.$$

2 Hankel-type Fock space

In this section, we recall some properties for the Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ associated with the poly-axially operator.

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$, we denote by Δ_α, the poly-axially operator [1, 9, 27] defined for $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ by

$$\Delta_\alpha := \sum_{k=1}^d \Delta_{\alpha_k, z_k}, \quad \Delta_{\alpha_k, z_k} := \frac{\partial^2}{\partial z_k^2} + \frac{2\alpha_k + 1}{z_k} \frac{\partial}{\partial z_k}.$$

This operator has important applications in both pure and applied mathematics and give rise to a generalization of multi-variable analytic structures like the Hankel transform, and the Hankel convolution [2, 15–18]. For any $w \in \mathbb{C}^d$, the system

$$\Delta_\alpha u(z) = |w|^2 u(z), \quad u(0) = 1, \quad \frac{\partial}{\partial z_k} u(z)\Big|_{z_k=0} = 0, \quad k = 1, \ldots, d,$$

admits a unique solution $I_\alpha(w,z)$, given by

$$I_\alpha(w,z) := \prod_{k=1}^{d} j_{\alpha_k}(iw_k z_k),$$

where j_{α_k} is the spherical Bessel function [26] given by

$$j_{\alpha_k}(x) := \Gamma(\alpha_k+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha_k+1)}\left(\frac{x}{2}\right)^{2n}.$$

The Bessel kernel I_α can be extended in a power series in the form

$$I_\alpha(w,z) = \sum_{\nu\in\mathbb{N}^d}\frac{w^{2\nu}z^{2\nu}}{c_\nu(\alpha)},$$

where

$$c_\nu(\alpha) = 2^{2\langle\nu\rangle}\nu!\prod_{k=1}^{d}\frac{\Gamma(\nu_k+\alpha_k+1)}{\Gamma(\alpha_k+1)} = \prod_{k=1}^{d} c_{\nu_k}(\alpha_k). \tag{2.1}$$

Here

$$c_{\nu_k}(\alpha_k) = 2^{2\nu_k}\nu_k!\frac{\Gamma(\nu_k+\alpha_k+1)}{\Gamma(\alpha_k+1)}$$

and

$$\langle\nu\rangle = \sum_{k=1}^{d}\nu_k, \quad \nu! = \prod_{k=1}^{d}\nu_k!, \quad \nu=(\nu_1,\dots,\nu_d)\in\mathbb{N}^d.$$

In the statement, and later in this work we use the following notations.

- $\mathscr{H}_*(\mathbb{C}^d)$, is the space of entire functions on $\mathbb{C}^d$ and even with respect to each variable.

- $L^2_\alpha(\mathbb{C}^d)$, is the Hilbert space of measurable functions f on $\mathbb{C}^d$, such that

$$\|f\|_{L^2_\alpha(\mathbb{C}^d)} := \left[\int_{\mathbb{C}^d}|f(z)|^2 \mathrm{d}m_\alpha(z)\right]^{1/2} < \infty,$$

  where m_α being the measure on $\mathbb{C}^d$ given by (1.1).

Cholewinsky [7] defined the Hilbert space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ as

$$\mathscr{F}_{\alpha,*}(\mathbb{C}^d) := \mathscr{H}_*(\mathbb{C}^d)\cap L^2_\alpha(\mathbb{C}^d).$$

The space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is equipped with the inner product

$$\langle f,g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} := \int_{\mathbb{C}^d} f(z)\overline{g(z)}\mathrm{d}m_\alpha(z).$$

The space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ has the reproducing kernel

$$\mathscr{K}_\alpha(w,z) = I_\alpha(w,\overline{z}), \quad w,z \in \mathbb{C}^d.$$

If $f,g \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^{2\nu}$ and $g(z) = \sum_{\nu\in\mathbb{N}^d} b_\nu z^{2\nu}$, then

$$\langle f,g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} a_\nu \overline{b_\nu} c_\nu(\alpha), \tag{2.2}$$

where $c_\nu(\alpha)$ are the constants given by (2.1).

Then, the set $\left\{\frac{z^{2\nu}}{\sqrt{c_\nu(\alpha)}}\right\}_{\nu\in\mathbb{N}^d}$ forms a Hilbertian basis for the space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$; and each $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ can be written as

$$f(z) = \sum_{\nu\in\mathbb{N}^d} \frac{\langle f, z^{2\nu}\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}}{c_\nu(\alpha)} z^{2\nu},$$

and

$$\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} \frac{\left|\langle f, z^{2\nu}\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\right|^2}{c_\nu(\alpha)}.$$

Bargmann [3] introduced the classical Fock space $\mathscr{F}(\mathbb{C}^d)$. Let $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^{2\nu}$. From [3], we have

$$\|f\|^2_{\mathscr{F}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 \nu!.$$

Using the inequality $\nu! \le c_\nu(\alpha)$, we obtain

$$\|f\|^2_{\mathscr{F}(\mathbb{C}^d)} \le \sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 c_\nu(\alpha) = \|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}.$$

Therefore

$$\mathscr{F}_{\alpha,*}(\mathbb{C}^d) \subset \mathscr{F}(\mathbb{C}^d).$$

3 Difference operator

In this section, building on the ideas of Saitoh [12–14] we examine the extremal function associated with the difference operator D. The results that are written here are a special case of [14].

Let D be the difference operator defined for $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^{2\nu}$, by

$$Df(z) := \sum_{\nu\in\mathbb{N}^d} a_{\nu+\mathbf{1}} z^{2\nu}. \tag{3.1}$$

In particular, for $f \in \mathscr{F}_{\alpha,*}(\mathbb{C})$, the difference operator [23, 25] is given

$$Df(z) := \begin{cases} \frac{1}{z^2}(f(z) - f(0)), & z \neq 0, \\ \frac{1}{2} f''(0), & z = 0. \end{cases}$$

We also define, the operators E and H for $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu z^{2\nu}$, by

$$Ef(z) := \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} \frac{c_{\nu - \mathbf{1}}(\alpha)}{c_\nu(\alpha)} a_{\nu - \mathbf{1}} z^{2\nu}, \tag{3.2}$$

and

$$Hf(z) := \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} \frac{c_{\nu - \mathbf{1}}(\alpha)}{c_\nu(\alpha)} a_\nu z^{2\nu}, \tag{3.3}$$

where $c_\nu(\alpha)$ are the constants given by (2.1).

Lemma 3.1. (i) *The operator D maps continuously from $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ into $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, and*

$$\|Df\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{2^d \sqrt{\prod_{k=1}^d (\alpha_k + 1)}} \|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}, \quad f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d).$$

(ii) *If $D^* : \mathscr{F}_{\alpha,*}(\mathbb{C}^d) \longrightarrow \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is the adjoint operator of D, then*

$$E = D^* \quad \textit{and} \quad H = D^* D.$$

Proof. (i) Let $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu z^{2\nu}$. From (3.1), we have

$$\|Df\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu \in \mathbb{N}^d} |a_{\nu + \mathbf{1}}|^2 c_\nu(\alpha) = \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} |a_\nu|^2 c_{\nu - \mathbf{1}}(\alpha).$$

Using the fact that $c_\nu(\alpha) = \left[2^{2d} \prod_{k=1}^d \nu_k(\nu_k + \alpha_k)\right] c_{\nu - \mathbf{1}}(\alpha)$, we deduce that

$$\|Df\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{2^{2d} \prod_{k=1}^d (\alpha_k + 1)} \sum_{\nu \in \mathbb{N}^d} |a_\nu|^2 c_\nu(\alpha) = \frac{1}{2^{2d} \prod_{k=1}^d (\alpha_k + 1)} \|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}.$$

(ii) If $f, g \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu z^{2\nu}$ and $g(z) = \sum_{\nu \in \mathbb{N}^d} b_\nu z^{2\nu}$, then by (2.2) and (3.1) we obtain

$$\langle Df, g \rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu \in \mathbb{N}^d} a_{\nu + \mathbf{1}} \overline{b_\nu} c_\nu(\alpha) = \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} a_\nu \overline{b_{\nu - \mathbf{1}}} c_{\nu - \mathbf{1}}(\alpha).$$

On the other hand, from (2.2) and (3.2) we have

$$\langle f, Eg \rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} a_\nu \overline{b_{\nu - \mathbf{1}}} c_{\nu - \mathbf{1}}(\alpha).$$

Then $\langle Df, g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \langle f, Eg\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$ and consequently $E = D^*$.

Finally, by relations (3.1), (3.2) and (3.3) we deduce that

$$D^*Df(z) = EDf(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{c_{\nu-\mathbf{1}}(\alpha)}{c_\nu(\alpha)} a_\nu z^{2\nu} = Hf(z).$$

The lemma is proved. □

Theorem 3.2. *For any $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ and for any $\lambda > 0$, the Tikhonov regularization problem*

$$\inf_{f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \left\{\lambda\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} + \|Df - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\right\}$$

has a unique extremal function denoted $F^*_{\lambda,D}(h)$ and is given by

$$F^*_{\lambda,D}(h)(z) = \langle h, \Psi_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

where

$$\Psi_z(w) = \sum_{\nu\in\mathbb{N}^d} \frac{(\overline{z})^{2(\nu+\mathbf{1})} w^{2\nu}}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)}, \quad w \in \mathbb{C}^d.$$

Proof. First, from [12, Theorem 2.5, Section 2], the Tikhonov regularization problem

$$\inf_{f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \left\{\lambda\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} + \|Df - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\right\}$$

has a unique extremal function denoted $F^*_{\lambda,D}(h)$ and is given by

$$F^*_{\lambda,D}(h)(z) = (\lambda I + D^*D)^{-1} D^*h(z), \quad z \in \mathbb{C}^d, \tag{3.4}$$

where I is the unit operator. We put $h(z) = \sum_{\nu\in\mathbb{N}^d} h_\nu z^{2\nu}$ and $F^*_{\lambda,D}(h)(z) = \sum_{\nu\in\mathbb{N}^d} d_\nu z^{2\nu}$. From Lemma 3.1 (ii) and (3.4) we have

$$(\lambda I + H)F^*_{\lambda,D}(h)(z) = Eh(z).$$

By relations (3.2) and (3.3) we deduce that

$$d_\nu = 0, \quad \text{if } \exists\ \nu_k = 0,$$

$$d_\nu = \frac{c_{\nu-\mathbf{1}}(\alpha) h_{\nu-\mathbf{1}}}{\lambda c_\nu(\alpha) + c_{\nu-\mathbf{1}}(\alpha)}, \quad \nu \geq \mathbf{1}.$$

Thus,

$$F^*_{\lambda,D}(h)(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{c_{\nu-\mathbf{1}}(\alpha) h_{\nu-\mathbf{1}}}{\lambda c_\nu(\alpha) + c_{\nu-\mathbf{1}}(\alpha)} z^{2\nu}. \tag{3.5}$$

Then by (2.2) and (3.5) we obtain

$$F^*_{\lambda,D}(h)(z) = \sum_{\nu\in\mathbb{N}^d} \frac{c_\nu(\alpha)h_\nu}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)} z^{2(\nu+\mathbf{1})} = \langle h, \Psi_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}, \tag{3.6}$$

where

$$\Psi_z(w) = \sum_{\nu\in\mathbb{N}^d} \frac{(\overline{z})^{2(\nu+\mathbf{1})}w^{2\nu}}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}, \quad w\in\mathbb{C}^d.$$

The theorem is proved. □

4 Approximate inversion formulas

In this section we establish the estimate properties of the extremal function $F^*_{\lambda,D}(h)(z)$, and we deduce approximate inversion formulas for the difference operator D. These formulas are the analogous of Calderón's reproducing formulas for the Fourier type transforms [10, 11, 19]. A pointwise approximate inversion formulas for the operator D are also discussed.

The extremal function $F^*_{\lambda,D}(h)$ given by (3.6) satisfies the following properties.

Lemma 4.1. *If $\lambda > 0$ and $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, then*

(i) $|F^*_{\lambda,D}(h)(z)| \le \frac{1}{2\sqrt{\lambda}}(I_\alpha(z,\overline{z}))^{1/2}\|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$,

(ii) $|DF^*_{\lambda,D}(h)(z)| \le \frac{1}{2^{d+1}\sqrt{\lambda\prod_{k=1}^d(\alpha_k+1)}}(I_\alpha(z,\overline{z}))^{1/2}\|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$,

(iii) $\|F^*_{\lambda,D}(h)\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \le \frac{1}{2\sqrt{\lambda}}\|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$.

Proof. Let $\lambda > 0$ and $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $h(z) = \sum_{\nu\in\mathbb{N}^d} h_\nu z^{2\nu}$. From (3.6) we have

$$|F^*_{\lambda,D}(h)(z)| \le \|\Psi_z\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}.$$

Using the fact that $(x+y)^2 \ge 4xy$ we obtain

$$\|\Psi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d}\left|\frac{(\overline{z})^{2(\nu+\mathbf{1})}}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}\right|^2 c_\nu(\alpha) \le \frac{1}{4\lambda}\sum_{\nu\in\mathbb{N}^d}\frac{|(\overline{z})^{2\nu}|^2}{c_\nu(\alpha)} = \frac{1}{4\lambda}I_\alpha(z,\overline{z}).$$

This gives (i).

On the other hand, from (3.1) and (3.5) we have

$$DF^*_{\lambda,D}(h)(z) = \sum_{\nu\in\mathbb{N}^d}\frac{c_\nu(\alpha)h_\nu}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}z^{2\nu} = \langle h, \Phi_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}, \tag{4.1}$$

where

$$\Phi_z(w) = \sum_{\nu\in\mathbb{N}^d} \frac{(\overline{z})^{2\nu} w^{2\nu}}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)}.$$

Then

$$|DF^*_{\lambda,D}(h)(z)| \leq \|\Phi_z\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

and

$$\|\Phi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} \left| \frac{(\overline{z})^{2\nu}}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)} \right|^2 c_\nu(\alpha) \leq \frac{1}{4\lambda} \sum_{\nu\in\mathbb{N}^d} \frac{|(\overline{z})^{2\nu}|^2}{c_{\nu+\mathbf{1}}(\alpha)}.$$

By using the fact that $c_{\nu+\mathbf{1}}(\alpha) = \left[2^{2d} \prod_{k=1}^d (\nu_k+1)(\nu_k+\alpha_k+1)\right] c_\nu(\alpha)$, we deduce that

$$\|\Phi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{2^{2(d+1)}\lambda \prod_{k=1}^d (\alpha_k+1)} \sum_{\nu\in\mathbb{N}^d} \frac{|(\overline{z})^{2\nu}|^2}{c_\nu(\alpha)} = \frac{I_\alpha(z,\overline{z})}{2^{2(d+1)}\lambda \prod_{k=1}^d (\alpha_k+1)}.$$

This gives (ii).

Finally, from (3.5) we have

$$\|F^*_{\lambda,D}(h)\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} c_\nu(\alpha) \left[\frac{c_{\nu-\mathbf{1}}(\alpha)|h_{\nu-\mathbf{1}}|}{\lambda c_\nu(\alpha) + c_{\nu-\mathbf{1}}(\alpha)} \right]^2.$$

Then we obtain

$$\|F^*_{\lambda,D}(h)\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{4\lambda} \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} c_{\nu-\mathbf{1}}(\alpha)|h_{\nu-\mathbf{1}}|^2 = \frac{1}{4\lambda}\|h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

which gives (iii) and completes the proof of the lemma. □

We establish approximate inversion formulas for the difference operator D.

Theorem 4.2. *If $\lambda > 0$ and $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, then*

(i) $\lim_{\lambda\to 0^+} \|DF^*_{\lambda,D}(h) - h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = 0,$

(ii) $\lim_{\lambda\to 0^+} \|F^*_{\lambda,D}(Dh) - h_0\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = 0,$ *where $h_0(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} h_\nu z^{2\nu}$ if $h(z) = \sum_{\nu\in\mathbb{N}^d} h_\nu z^{2\nu}$.*

Proof. Let $\lambda > 0$ and $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $h(z) = \sum_{\nu\in\mathbb{N}^d} h_\nu z^{2\nu}$. From (4.1) we have

$$DF^*_{\lambda,D}(h)(z) - h(z) = \sum_{\nu\in\mathbb{N}^d} \frac{-\lambda c_{\nu+\mathbf{1}}(\alpha) h_\nu}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)} z^{2\nu}. \tag{4.2}$$

Therefore

$$\|DF^*_{\lambda,D}(h) - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} c_\nu(\alpha) \left[\frac{\lambda c_{\nu+\mathbf{1}}(\alpha)|h_\nu|}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)} \right]^2.$$

Again, by dominated convergence theorem and the fact that

$$c_\nu(\alpha)\left[\frac{\lambda c_{\nu+\mathbf{1}}(\alpha)|h_\nu|}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}\right]^2 \le c_\nu(\alpha)|h_\nu|^2,$$

we deduce (i).

Finally, from (3.1) and (3.5) we have

$$F^*_{\lambda,D}(Dh)(z)-h_0(z)=\sum_{\nu\in\mathbb{N}^d,\ \nu\ge\mathbf{1}}\frac{-\lambda c_\nu(\alpha)h_\nu}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}z^{2\nu}. \tag{4.3}$$

So, one has

$$\|F^*_{\lambda,D}(Dh)-h_0\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}=\sum_{\nu\in\mathbb{N}^d,\ \nu\ge\mathbf{1}}c_\nu(\alpha)\left[\frac{\lambda c_\nu(\alpha)|h_\nu|}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}\right]^2.$$

Using the dominated convergence theorem and the fact that

$$c_\nu(\alpha)\left[\frac{\lambda c_\nu(\alpha)|h_\nu|}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}\right]^2 \le c_\nu(\alpha)|h_\nu|^2,$$

we deduce (ii). □

We deduce also pointwise approximate inversion formulas for the operator D.

Theorem 4.3. *If $\lambda>0$ and $h\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, then*

(i) $\lim_{\lambda\to 0^+} DF^*_{\lambda,D}(h)(z)=h(z)$,

(ii) $\lim_{\lambda\to 0^+} F^*_{\lambda,D}(Dh)(z)=h_0(z)$.

Proof. Let $h\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $h(z)=\sum_{\nu\in\mathbb{N}^d}h_\nu z^{2\nu}$. From (4.2) and (4.3), by using the dominated convergence theorem and the fact that

$$\frac{\lambda c_{\nu+\mathbf{1}}(\alpha)|h_\nu|}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}|z^{2\nu}|,\ \frac{\lambda c_\nu(\alpha)|h_\nu|}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}|z^{2\nu}|\le|h_\nu||z^{2\nu}|,$$

we obtain (i) and (ii). □

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