CUBO, A Mathematical Journal
Vol. 26, no. 2, pp. 317–326, August 2024
DOI: 10.56754/0719-0646.2602.317

Approximation and inequalities for the factorial function related to the Burnside's formula

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ABSTRACT

In this paper, we present a continued fraction approximation and some inequalities of the factorial function based on the Burnside's formula. This approximation is fast in comparison with the recently discovered asymptotic series. Finally, some numerical computations are provided for demonstrating the superiority of our approximation over the Burnside's formula and the classical Stirling's series.

RESUMEN

En este artículo, presentamos una aproximación con una fracción continua y algunas desigualdades para la función factorial basada en la fórmula de Burnside. Esta aproximación es rápida en comparación con las series asintóticas descubiertas recientemente. Finalmente, se entregan algunos cálculos numéricos para demostrar la superioridad de nuestra aproximación por sobre la fórmula de Burnside y la serie de Stirling clásica.

Keywords and Phrases: Factorial function, Stirling's formula, Burnside's formula, approximation, continued fraction.

2020 AMS Mathematics Subject Classification: 11Y60, 11A55, 41A25.

Published: 08 August, 2024
Accepted: 21 June, 2024
Received: 16 July, 2023

1 Introduction and main results

It is well known that we often need to deal with the big factorials in many situations in pure mathematics and other branches of science. To the best of our knowledge, the Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \quad n \to \infty \tag{1.1}$$

is one of the most known formulas for approximation of the factorial function. Up to now, many researchers made great efforts in the area of establishing more precise inequalities and more accurate approximation for the factorial function and its extension, called gamma function, and had a lot of inspiring results. For example, the Stirling series [1]

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \dots\right), \quad n \to \infty \tag{1.2}$$

is an extension of (1.1). Furthermore, there is a variety of approaches to Stirling's formula, ranging from elementary to advanced methods. We mention the estimations given by Schuster in [14], or the formula

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} = \sigma_n, \quad n \to \infty, \tag{1.3}$$

with $n! < \sigma_n$, due to Burnside, whose superiority over Stirling's formula was proved in [3]. There are also some approximations which are better than (1.3), Gosper's formula [7]

$$n! \sim \sqrt{2\pi\left(n + \frac{1}{6}\right)}\left(\frac{n}{e}\right)^n, \quad n \to \infty, \tag{1.4}$$

and Ramanujan's formula [13]

$$n! \sim \sqrt{2\pi}\left(\frac{n}{e}\right)^n \left(n^3 + \frac{1}{2}n^2 + \frac{1}{8}n + \frac{1}{240}\right)^{1/6}, \quad n \to \infty, \tag{1.5}$$

and Nemes's formula [12]

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n^2 - 1/10}\right)^n, \quad n \to \infty. \tag{1.6}$$

In [2], Batir obtained an asymptotic formula as follows:

$$n! \sim \sqrt{2\pi}\frac{n^{n+1}e^{-n}}{\sqrt{n - 1/6}}, \quad n \to \infty. \tag{1.7}$$

The following more accurate approximation for $n!$

$$n! \sim \sqrt{2\pi}\left(\frac{n^2+n+1/6}{e^2}\right)^{n/2+1/4}, \quad n \to \infty. \tag{1.8}$$

can be found in the literature [9].

Recently, Mortici [8] proved that for every $x \geq 0$,

$$\sqrt{2\pi e}\cdot e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+\frac{1}{2}} < \Gamma(x+1) \leq \alpha\sqrt{2\pi e}\cdot e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+\frac{1}{2}}, \tag{1.9}$$

where $\omega = \dfrac{3-\sqrt{3}}{6}$, $\alpha = 1.072042464\ldots$, and

$$\beta\sqrt{2\pi e}\cdot e^{-\zeta}\left(\frac{x+\zeta}{e}\right)^{x+\frac{1}{2}} \leq \Gamma(x+1) < \sqrt{2\pi e}\cdot e^{-\zeta}\left(\frac{x+\zeta}{e}\right)^{x+\frac{1}{2}}, \tag{1.10}$$

where $\zeta = \dfrac{3+\sqrt{3}}{6}$, $\beta = 0.988503589\ldots$

Estimates and approximations for the factorial function (and the gamma function) are a popular subject, with many papers appearing on this topic over the years. More results involving the asymptotic formulas or bounds for $n!$ or gamma function can be found in the references cited therein.

A natural question arises. It is true that the behavior of the Burnside's formula for n approaches infinity is of the form

$$n! \sim \sqrt{2\pi e}\cdot e^{-p}\left(\frac{n+p}{e}\right)^{n+q}, \tag{1.11}$$

where p, q are some constants? We propose the following sharp approximation formula as $n \to \infty$:

$$n! \sim \sqrt{2\pi e}\cdot e^{-\frac{3\pm\sqrt{3}}{6}}\left(\frac{n+\frac{3\pm\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}}. \tag{1.12}$$

These constants p, q in (1.11) given by (1.12), namely

$$p = \frac{3\pm\sqrt{3}}{6}, \quad q = \frac{1}{2}$$

are justified by the result in Theorem 1.1. Then we prove the following stronger approximation formula using continued fraction for the factorial function by the *multiple-correction method* [4–6].

Theorem 1.1. *For the factorial function, we have*

$$n! \sim \sqrt{2\pi e} \cdot e^{-p} \left(\frac{n+p}{e} \right)^{n+q} \exp \left(\frac{u_1}{n^2 + v_1 n + v_0 + \frac{s_1}{n+t_1+\frac{s_2}{n+t_2+\ddots}}} \right), \quad n \to \infty, \tag{1.13}$$

where

$$p = \frac{3 \pm \sqrt{3}}{6}, \quad q = \frac{1}{2}; \quad u_1 = \mp \frac{1}{72\sqrt{3}}, \quad v_1 = \frac{10 \pm 3\sqrt{3}}{10}, \quad v_0 = \frac{47 \pm 15\sqrt{3}}{10};$$
$$s_1 = \pm \frac{163}{21000\sqrt{3}}, \quad t_1 = \frac{815 \pm 11596\sqrt{3}}{1630};$$
$$s_2 = \frac{15531525}{106276}, \quad t_2 = \frac{19139187627 \mp 259913623163\sqrt{3}}{38278375254}; \ldots.$$

Using Theorem 1.1, we provide some inequalities for the factorial function.

Theorem 1.2. *For every $n \in \mathbb{N}$, it holds:*

$$\sqrt{2\pi e} \cdot e^{-\frac{3-\sqrt{3}}{6}} \left(\frac{n + \frac{3-\sqrt{3}}{6}}{e} \right)^{n+\frac{1}{2}} < n! < \sqrt{2\pi e} \cdot e^{-\frac{3+\sqrt{3}}{6}} \left(\frac{n + \frac{3+\sqrt{3}}{6}}{e} \right)^{n+\frac{1}{2}}. \tag{1.14}$$

To obtain Theorem 1.1, we need the following lemma which was used in [8,10,11] and is very useful for constructing asymptotic expansions.

Lemma 1.3. *If the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to zero and there exists the limit*

$$\lim_{n\to+\infty} n^s (x_n - x_{n+1}) = l \in [-\infty, +\infty] \tag{1.15}$$

with $s > 1$, then

$$\lim_{n\to+\infty} n^{s-1} x_n = \frac{l}{s-1}. \tag{1.16}$$

Lemma 1.3 was proved by Mortici in [8]. From Lemma 1.3, we can see that the speed of convergence of the sequences $(x_n)_{n\in\mathbb{N}}$ increases together with the values s satisfying (1.15).

2 Proof of Theorem 1.1

Step 0: The initial-correction.

Based on the Burnside's formula $n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}$, $n \to \infty$, we need to find the values p, q which produces the most accurate approximation of the form

$$n! \sim \sqrt{2\pi e} \cdot e^{-p}\left(\frac{n+p}{e}\right)^{n+q}, \quad n \to \infty.$$

To measure the accuracy of this approximation, a method is to define a sequence $(u_0(n))_{n\geq 1}$ by the relations

$$n! = \sqrt{2\pi e} \cdot e^{-p}\left(\frac{n+p}{e}\right)^{n+q} \exp u_0(n), \tag{2.1}$$

and to say that the approximation $n! \sim \sqrt{2\pi e} \cdot e^{-p}\left(\frac{n+p}{e}\right)^{n+q}$, $n \to \infty$ is better if $u_0(n)$ converges to zero faster.

From (2.1), we have

$$u_0(n) = \ln n! - \frac{1}{2}\ln(2\pi e) + p - (n+q)\ln(n+p) + (n+q). \tag{2.2}$$

Thus,

$$u_0(n) - u_0(n+1) = -1 - \ln(n+1) - (n+q)\ln(n+p) + (n+1+q)\ln(n+1+p). \tag{2.3}$$

Developing (2.3) into power series expansion in $1/n$, we have

$$\begin{aligned} u_0(n) - u_0(n+1) = {} & \frac{-1+2q}{2}\frac{1}{n} + \frac{2+3p^2-3q-6pq}{6}\frac{1}{n^2} \\ & + \frac{-3-6p^2-8p^3+4q+12pq+12p^2q}{12}\frac{1}{n^3} + O\left(\frac{1}{n^4}\right). \end{aligned} \tag{2.4}$$

The fastest possible sequence $(u_0(n))_{n\geq 1}$ is obtained as the first two items on the right of (2.4) vanishes, we get $p = \frac{3 \pm \sqrt{3}}{6}$, $q = \frac{1}{2}$. Thus, using Lemma 1.3, from (2.4) we have

$$u_0(n) - u_0(n+1) = \mp\frac{1}{36\sqrt{3}}\frac{1}{n^3} + O\left(\frac{1}{n^4}\right),$$

and the rate of convergence of the sequence $(u_0(n))_{n\geq 1}$ is at least n^{-2}.

Step 1: The first-correction.

Next, we define the sequence $(u_1(n))_{n\geq 1}$ by the relation

$$n! = \sqrt{2\pi e}\cdot e^{-\frac{3\pm\sqrt{3}}{6}}\left(\frac{n+\frac{3\pm\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}}\exp\left(\frac{u_1}{n^2+v_1n+v_0}\right)\exp u_1(n). \tag{2.5}$$

From (2.5), we have

$$\begin{aligned}u_1(n)-u_1(n+1) &= -1-\ln(n+1)-\left(n+\frac{1}{2}\right)\ln\left(n+\frac{3\pm\sqrt{3}}{6}\right)\\ &\quad+\left(n+\frac{3}{2}\right)\ln\left(n+1+\frac{3\pm\sqrt{3}}{6}\right)-\frac{u_1}{n^2+v_1n+v_0}+\frac{u_1}{(n+1)^2+v_1(n+1)+v_0}.\end{aligned} \tag{2.6}$$

Developing (2.6) into power series expansion in $1/n$, we have

$$\begin{aligned}u_1(n)-u_1(n+1) &= \left(\mp\frac{1}{36\sqrt{3}}-2u_1\right)\frac{1}{n^3}+\left(\frac{1}{80}\pm\frac{1}{12\sqrt{3}}+3u_1+3u_1v_1\right)\frac{1}{n^4}\\ &\quad+\left(-\frac{1}{20}\mp\frac{11}{60\sqrt{3}}-4u_1+4u_1v_0-6u_1v_1-4u_1v_1^2\right)\frac{1}{n^5}\\ &\quad+\left(\frac{599}{4536}\pm\frac{13}{36\sqrt{3}}+5u_1-10u_1v_0+10u_1v_1-10u_1v_0v_1+10u_1v_1^2+5u_1v_1^3\right)\frac{1}{n^6}+O\left(\frac{1}{n^7}\right).\end{aligned} \tag{2.7}$$

By Lemma 1.3, the fastest possible sequence $(u_1(n))_{n\geq 1}$ is obtained as the first three items on the right of (2.7) vanishes. So we can obtain

$$u_1=\mp\frac{1}{72\sqrt{3}},\quad v_1=\frac{10\pm 3\sqrt{3}}{10},\quad v_0=\frac{47\pm 15\sqrt{3}}{100},$$

and from (2.7) we have

$$u_1(n)-u_1(n+1)=\frac{163}{907200}\frac{1}{n^6}+O\left(\frac{1}{n^7}\right),$$

and the rate of convergence of the sequence $(u_1(n))_{n\geq 1}$ is at least n^{-5}.

Step 2: The second-correction.

Furthermore, we define the sequence $(u_2(n))_{n\geq 1}$ by the relation

$$n! = \sqrt{2\pi e}\cdot e^{-\frac{3\pm\sqrt{3}}{6}}\left(\frac{n+\frac{3\pm\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}}\exp\left(\frac{\mp\frac{1}{72\sqrt{3}}}{n^2+\frac{10\pm3\sqrt{3}}{10}n+\frac{47\pm15\sqrt{3}}{100}+\frac{s_1}{n+t_1}}\right)\exp u_2(n) \tag{2.8}$$

Using the same method as above, we obtain that the sequence $(u_2(n))_{n\geq 1}$ converges fastest only if $s_1 = \pm\dfrac{163}{21000\sqrt{3}}$, $t_1 = \dfrac{815 \pm 11596\sqrt{3}}{1630}$, and the rate of convergence of the sequence $(u_2(n))_{n\geq 1}$ is at least n^{-7}. We can get

$$u_2(n) - u_2(n+1) = -\frac{69029}{1877760}\frac{1}{n^8} + O\left(\frac{1}{n^9}\right).$$

Step 3: The third-correction.

Similarly, define the sequence $(u_3(n))_{n\geq 1}$ by the relation

$$\begin{aligned} n! =& \sqrt{2\pi e} \cdot e^{-\frac{3\pm\sqrt{3}}{6}} \left(\frac{n + \frac{3\pm\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}} \\ & \exp\left(\frac{\mp\frac{1}{72\sqrt{3}}}{n^2 + \frac{10\pm 3\sqrt{3}}{10}n + \frac{47\pm 15\sqrt{3}}{100} + \frac{\pm\frac{163}{21000\sqrt{3}}}{n+\frac{815\pm 11596\sqrt{3}}{1630} + \frac{s_2}{n+t_2}}}\right) \exp u_3(n). \end{aligned} \tag{2.9}$$

Using the same method as above, we obtain that the sequence $(u_3(n))_{n\geq 1}$ converges fastest only if $s_2 = \dfrac{15531525}{106276}$, $t_2 = \dfrac{19139187627 \mp 259913623163\sqrt{3}}{38278375254}$.

The new asymptotic (1.13) is obtained.

3 Proof of Theorem 1.2

The double-side inequality (1.14) may be written as follows:

$$f(n) = \ln\Gamma(n+1) - \frac{1}{2}\ln(2\pi e) + \frac{3+\sqrt{3}}{6} - \left(n+\frac{1}{2}\right)\left(\ln\left(n + \frac{3+\sqrt{3}}{6}\right) - 1\right) < 0$$

and

$$g(n) = \ln\Gamma(n+1) - \frac{1}{2}\ln(2\pi e) + \frac{3-\sqrt{3}}{6} - \left(n+\frac{1}{2}\right)\left(\ln\left(n + \frac{3-\sqrt{3}}{6}\right) - 1\right) > 0.$$

Suppose $F(n) = f(n+1) - f(n)$ and $G(n) = g(n+1) - g(n)$. For every $x > 1$, we can get

$$F''(x) = \frac{36(-1 + 4\sqrt{3} + 4\sqrt{3}n)}{(1+n)^2(3+\sqrt{3}+6n)^2(9+\sqrt{3}+6n)^2} > 0 \tag{3.1}$$

and

$$G''(x) = -\frac{36(1+4\sqrt{3}+4\sqrt{3}n)}{(1+n)^2(3-\sqrt{3}+6n)^2(9-\sqrt{3}+6n)^2} < 0. \tag{3.2}$$

It shows that $F(x)$ is strictly convex and $G(x)$ is strictly concave on $(0,\infty)$. According to Theorem 1.1, when $n\to\infty$, it holds that $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = 0$; thus $\lim_{n\to\infty} F(n) = \lim_{n\to\infty} G(n) = 0$. As a result, we can make sure that $F(x) > 0$ and $G(x) < 0$ on $(0,\infty)$. Consequently, the sequence $f(n)$ is strictly increasing and $g(n)$ is strictly decreasing while they both converge to 0. As a result, we conclude that $f(n) < 0$, and $g(n) > 0$ for every integer $n > 1$.

The proof of Theorem 1.2 is completed.

4 Numerical computations

In this section, we give Table 1 to demonstrate the superiority of our new series respectively. From what has been discussed above, we found out some new approximations as follows:

$$n! \approx \sqrt{2\pi e}\cdot e^{-\frac{3+\sqrt{3}}{6}}\left(\frac{n+\frac{3+\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}} = \beta_1(n), \tag{4.1}$$

or

$$n! \approx \sqrt{2\pi e}\cdot e^{-\frac{3-\sqrt{3}}{6}}\left(\frac{n+\frac{3-\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}} = \beta_2(n) \tag{4.2}$$

or

$$n! \approx \sqrt{2\pi e}\cdot e^{-\frac{3+\sqrt{3}}{6}}\left(\frac{n+\frac{3+\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}} \exp\left(\frac{-\frac{1}{72\sqrt{3}}}{n^2+\frac{10+3\sqrt{3}}{10}n+\frac{47+15\sqrt{3}}{100}}\right) = \beta_3(n) \tag{4.3}$$

For simplicity, we only give three approximations $\beta_1(n)$, $\beta_2(n)$, $\beta_3(n)$, more formulas can be directly obtained from Theorem 1.1.

Burnside [3] gave the formula:

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} = \beta(n). \tag{4.4}$$

The great advantage of our continued fraction approximation $\beta_3(n)$ consists in its simple form and its accuracy. From Table 1, we can see that the relative error of $\beta_3(n)$ is -1.1137×10^{-18} when calculating 500! and the relative error of $\beta(n)$ is 8.2540×10^{-4} when calculating 50!. Our formula

Table 1: Simulations for $\beta(n)$ and $\beta_i(n)$, $i = 1, 2, 3$.

n	$\frac{\beta(n)-n!}{n!}$	$\frac{\beta_1(n)-n!}{n!}$	$\frac{\beta_2(n)-n!}{n!}$	$\frac{\beta_3(n)-n!}{n!}$
50	8.2540×10^{-4}	-3.1767×10^{-6}	3.1120×10^{-6}	-8.1273×10^{-14}
500	8.3254×10^{-5}	-3.2044×10^{-8}	3.1978×10^{-8}	-1.1137×10^{-18}
1000	4.1647×10^{-5}	-8.0149×10^{-9}	8.0066×10^{-9}	-3.5367×10^{-20}
2000	2.0828×10^{-5}	-2.0042×10^{-9}	2.0032×10^{-9}	-1.1141×10^{-21}

$\beta_3(n)$ converges faster than the approximation of the Burnside's formula $\beta(n)$.

5 Acknowledgements

I wish to express my sincere gratitude and thanks to the anonymous referees, whose comments and suggestions resulted in a considerable improvement of the initial form of this paper. This work was supported by the Science and Technology Plan of Beijing Municipal Education Commission KM201910017002.

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