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Some norm inequalities for accretive Hilbert space operators

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ABSTRACT

New norm inequalities for accretive operators on Hilbert space are given. Among other inequalities, we prove that if $A, B \in \mathbb{B}(\mathbb{H})$ and B is self-adjoint and also $C_{m,M}(iAB)$ is accretive, then

$$\frac{4\sqrt{Mm}}{M+m}\|AB\| \leq \omega(AB - BA^*),$$

where M and m are positive real numbers with $M > m$ and $C_{m,M}(A) = (A^* - mI)(MI - A)$. Also, we show that if $C_{m,M}(A)$ is accretive and $(M - m) \leq k\|A\|$, then

$$\omega(AB) \leq (2 + k)\omega(A)\omega(B).$$

RESUMEN

Entregamos nuevas desigualdades para normas de operadores acretivos en espacios de Hilbert. Entre otras desigualdades, probamos que si $A, B \in \mathbb{B}(\mathbb{H})$ y B es auto-adjunto y también $C_{m,M}(iAB)$ es acretivo, entonces

$$\frac{4\sqrt{Mm}}{M+m}\|AB\| \leq \omega(AB - BA^*),$$

donde M y m son números reales positivos con $M > m$ y $C_{m,M}(A) = (A^* - mI)(MI - A)$. También mostramos que si $C_{m,M}(A)$ es acretivo y $(M - m) \leq k\|A\|$, entonces

$$\omega(AB) \leq (2 + k)\omega(A)\omega(B).$$

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1 Introduction and preliminaries

Let $\mathbb{B}(\mathbb{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathbb{H}$ with inner product $\langle \cdot, \cdot \rangle$. The numerical radius of $A \in \mathbb{B}(\mathbb{H})$ is defined by

$$\omega(A) = \sup\{ \ |\langle Ax, x\rangle| : \|x\| = 1 \ \}.$$

In [14], Yamazaki proved that for any $A \in \mathbb{B}(\mathbb{H})$

$$\omega(A) = \sup_{\theta \in \mathbb{R}} \|Re(e^{i\theta}A)\|. \tag{1.1}$$

It is well known that $\omega(\cdot)$ is a norm on $\mathbb{B}(\mathbb{H})$ which is equivalent to the usual operator norm $\|.\|$. In fact, for all $A \in \mathbb{B}(\mathbb{H})$,

$$\frac{\|A\|}{2} \leq \omega(A) \leq \|A\|. \tag{1.2}$$

The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A is normal. Several numerical radius inequalities improving the inequalities in (1.2) have been recently given in [1–3, 7, 9, 11, 12, 15, 16] and [17]. Holbrook in [6] showed that, for any $A, B \in \mathbb{B}(\mathbb{H})$,

$$\omega(AB) \leq 4\omega(A)\omega(B). \tag{1.3}$$

In the case $AB = BA$, then

$$\omega(AB) \leq 2\omega(A)\omega(B).$$

The question about the best constant k such that the inequality

$$w\,(AB) \leq k\|A\|\omega(B) \tag{1.4}$$

holds for all operators $A, B \in \mathbb{B}(\mathbb{H})$ is still open. It is shown in [4] that, for any $A, B \in \mathbb{B}(\mathbb{H})$,

$$\omega(AB \pm BA^*) \leq 2\|A\|\omega(B). \tag{1.5}$$

Let $D_A = \inf\limits_{\lambda \in \mathbb{C}} \|A - \lambda I\|$ and let R_A denote the radius of the smallest disk in the complex plane containing $\sigma(A)$ (the spectrum of A). Stampfli in [13] proved that if $A \in \mathbb{B}(\mathbb{H})$ and A is normal, then

$$D_A = R_A. \tag{1.6}$$

The following result from [10] may be stated as well: if $A, B \in \mathbb{B}(\mathbb{H})$,then

$$w\left(AB\right) \leq \omega(A)\omega(B) + D_A D_B. \tag{1.7}$$

Also, the authors in [8] proved that if $A, B \in \mathbb{B}(\mathbb{H})$ and A is self-adjointable, then

$$\omega(BA) \leq D_B\|A\|. \tag{1.8}$$

We consider the nonlinear functional $V_s : \mathbb{B}(\mathbb{H}) \longrightarrow R$, given by

$$V_s(A) = \sup_{\|x\|=1} Re\langle Ax, x\rangle.$$

Recall that, for all $A \in \mathbb{B}(\mathbb{H})$,

$$V_s(A) \leq \omega(A) \leq \|A\|. \tag{1.9}$$

We say that an operator $A : \mathbb{H} \longrightarrow \mathbb{H}$ is accretive, if $Re\langle Ax, x\rangle \geq 0$ for any $x \in \mathbb{H}$. In [3], Dragomir has shown that if M and m are positive real numbers with $M > m$ and the operator $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive, then

$$\|A\| \leq \frac{M+m}{2\sqrt{Mm}} V_s(A) \tag{1.10}$$

and

$$\|A\| \leq \frac{M+m}{2\sqrt{Mm}} \omega(A). \tag{1.11}$$

A sufficient simple condition for $C_{m,M}(A)$ to be accretive is that A is a self-adjoint operator on $\mathbb{H}$ such that $mI \leq A \leq MI$ in the usual operator order of $\mathbb{B}(\mathbb{H})$. Moreover, if $0 < m < M$, a sufficient condition for $C_{m,M}(A)$ to be accretive is that

$$\left\| A - \frac{M+m}{2} I \right\| < \frac{(M-m)}{2}.$$

The following result from [5] may be stated as well: if M and m are positive real numbers with $M > m$ and $A, B \in \mathbb{B}(\mathbb{H})$ and also $C_{m,M}(A)$ is accretive, then

$$\omega(AB - BA^*) \leq (M-m)\omega(B). \tag{1.12}$$

And also

$$\|A\| \leq \frac{M+m}{2\sqrt{Mm}} \|Re(A)\|, \tag{1.13}$$

which is a refinement of inequality (1.11).

In Section 2, we introduce some inequalities between the operator norm and the numerical radius of accretive operators on Hilbert spaces. More precisely, we establish the generalization of the inequalities (1.11) and (1.13). Also, we find a lower bound for $\omega(AB - BA^*)$.

2 Main results

We need the following lemma, to achieve our goal.

Lemma 2.1. *If* $A \in \mathbb{B}(\mathbb{H})$, *then*

$$V_s(A) \leq \|Re(A)\|.$$

Proof. Suppose that $x \in \mathbb{H}$ with $\|x\| = 1$. We have

$$Re\langle Ax, x\rangle = \frac{\langle (A + A^*)x, x\rangle}{2} \leq \frac{\|A + A^*\|}{2} \leq \|Re(A)\|.$$

Hence

$$Re\langle Ax, x\rangle \leq \|Re(A)\|.$$

Taking the supremum over $x \in \mathbb{H}$ with $\|x\| = 1$ gives

$$V_s(A) \leq \|Re(A)\|,$$

which is exactly the desired result. □

Remark 2.2. *Let* M *and* m *be positive real numbers with* $M > m$ *and* $A \in \mathbb{B}(\mathbb{H})$ *and also* $C_{m,M}(A)$ *is accretive. By* (1.10) *and Lemma 2.1 we deduce that*

$$\frac{M + m}{2\sqrt{Mm}} V_s(A) \leq \frac{M + m}{2\sqrt{Mm}} \|Re(A)\|.$$

Therefore, the inequality (1.10) *strengthens* (1.11) *and* (1.13). *Then, we continue this section and introduce some norm inequalities for products of two Hilbert space operators with inequality* (1.10).

The following result may be as well.

Theorem 2.3. *If $A, B \in \mathbb{B}(\mathbb{H})$, then*

$$V_s(AB) \leq \frac{\|B+B^*\|\omega(A)}{2} + \frac{D_A D_{B+B^*}}{2} + \frac{1}{2}\omega(AB - BA^*).$$

Proof. Clearly, $\|Re(AB)\| = \omega(Re(AB))$. Then

$$\begin{aligned}
\|Re(AB)\| &= \omega\left(\frac{AB + B^*A^*}{2}\right) \\
&= \omega\left(\frac{AB + AB^* - AB^* + B^*A^*}{2}\right) \\
&\leq \omega\left(\frac{AB + AB^*}{2}\right) + \omega\left(\frac{-AB^* + B^*A^*}{2}\right) \\
&= \frac{1}{2}\omega(A(B+B^*)) + \frac{1}{2}\omega(AB - BA^*) \\
&\leq \frac{\|B+B^*\|\omega(A)}{2} + \frac{1}{2}D_A D_{B+B^*} + \frac{1}{2}\omega(AB - BA^*). \qquad \text{(by (1.7))}
\end{aligned}$$

Hence

$$\|Re(AB)\| \leq \frac{\|B+B^*\|\omega(A)}{2} + \frac{1}{2}D_A D_{B+B^*} + \frac{1}{2}\omega(AB - BA^*) \tag{2.1}$$

and the result follows from Lemma 2.1. □

Corollary 2.4. *If $A, B \in \mathbb{B}(\mathbb{H})$, then*

$$V_s(AB) \leq \omega(B)\left(\omega(A) + D_A\right) + \frac{1}{2}\omega(AB - BA^*).$$

Proof. By Theorem 2.3,

$$V_s(AB) \leq \frac{\|B+B^*\|\omega(A)}{2} + \frac{D_A D_{B+B^*}}{2} + \frac{1}{2}\omega(AB - BA^*).$$

Since $D_{B+B^*} \leq \|B+B^*\|$, then

$$\begin{aligned}
V_s(AB) &\leq \|Re(B)\|(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*) \\
&\leq \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}B)\|(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*) \\
&\leq \omega(B)\left(\omega(A) + D_A\right) + \frac{1}{2}\omega(AB - BA^*). \qquad \text{(by (1.1))}
\end{aligned}$$

Therefore,

$$V_s(AB) \leq \omega(B)\left(\omega(A) + D_A\right) + \frac{1}{2}\omega(AB - BA^*).$$

This completes the proof. □

Corollary 2.5. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If there exist $\theta_0 \in \mathbb{R}$ such that $C_{m,M}(e^{i\theta_0}AB)$ is accretive , then*

$$\|AB\| \leq \frac{M+m}{2\sqrt{Mm}} \left(\omega(B)(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*)\right). \tag{2.2}$$

Proof. By (2.1),

$$\|Re(AB)\| \leq \frac{\|B + B^*\|\omega(A)}{2} + \frac{1}{2}D_A D_{B+B^*} + \frac{1}{2}\omega(AB - BA^*).$$

Since $D_{B+B^*} \leq \|B + B^*\|$, gives

$$\|Re(AB)\| \leq \|Re(B)\|(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*). \tag{2.3}$$

Suppose that $\theta_0 \in \mathbb{R}$. Replacing B by $e^{i\theta_0}B$ in the inequality (2.3) gives

$$\begin{aligned}
\|Re(e^{i\theta_0}AB)\| &\leq \|Re(e^{i\theta_0}B)\|(\omega(A) + D_A) + \frac{1}{2}\omega(e^{i\theta_0}(AB - BA^*)) \\
&= \|Re(e^{i\theta_0}B)\|(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*) \\
&\leq \sup_{\theta_0 \in \mathbb{R}} \|Re(e^{i\theta_0}B)\|(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*) \\
&= \omega(B)(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*). && \text{(by (1.1))}
\end{aligned}$$

Hence,

$$\|Re(e^{i\theta_0}AB)\| \leq \omega(B)(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*). \tag{2.4}$$

Since $C_{m,M}(e^{i\theta_0}AB)$ is accretive, from the inequality (1.13) we have

$$\frac{2\sqrt{Mm}}{M+m}\|AB\| \leq \|Re(e^{i\theta_0}AB)\|$$

and the result follows from (2.4). □

Remark 2.6. *The result stated in Corollary 2.5 is stronger than inequality* (1.11)*. To explain that, suppose that $C_{m,M}(B)$ is accretive. Replacing A by I in inequality* (2.2)*. Since $D_I = 0$ and $\omega(I) = \|I\| = 1$, then we have $\|B\| \leq \frac{M+m}{2\sqrt{Mm}}\omega(B)$.*

The following result may be as well.

Theorem 2.7. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}\left(\begin{bmatrix} 0 & AB \\ 0 & 0 \end{bmatrix}\right)$ is accretive, then*

$$\frac{2\sqrt{Mm}}{M+m}\|AB\| \leq \frac{\|B\|}{2}(\omega(A) + D_A) + \frac{\|AB - BA^*\|}{4}.$$

Proof. Let $A_1 = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $B_1 = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$. Since $C_{m,M}(A_1B_1) = C_{m,M}\left(\begin{bmatrix} 0 & AB \\ 0 & 0 \end{bmatrix}\right)$ is accretive, from the inequality (1.10) and Theorem 2.3 we have

$$\begin{aligned}
\frac{2\sqrt{Mm}}{M+m}\|AB\| &= \frac{2\sqrt{Mm}}{M+m}\|A_1B_1\| \\
&\leq \frac{\|B_1 + {B_1}^*\|\omega(A_1)}{2} + \frac{D_{A_1}D_{B_1+{B_1}^*}}{2} + \frac{1}{2}\omega(A_1B_1 - B_1{A_1}^*) \\
&= \frac{\|B\|\omega(A)}{2} + \frac{D_A D_{B_1+{B_1}^*}}{2} + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & AB - BA^* \\ 0 & 0 \end{bmatrix}\right) \\
&\leq \frac{\|B\|\omega(A)}{2} + \frac{D_A\|B_1\|}{2} + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & AB - BA^* \\ 0 & 0 \end{bmatrix}\right) \\
&= \frac{\|B\|\omega(A)}{2} + \frac{D_A\|B\|}{2} + \frac{\|AB - BA^*\|}{4}.
\end{aligned}$$

Consequently,

$$\frac{2\sqrt{Mm}}{M+m}\|AB\| \leq \frac{\|B\|}{2}(\omega(A) + D_A) + \frac{\|AB - BA^*\|}{4},$$

which is exactly the desired result. □

As a direct consequence of Theorem 2.7, we have:

Corollary 2.8. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}\left(\begin{bmatrix} 0 & AB \\ 0 & 0 \end{bmatrix}\right)$ is accretive and $AB = BA^*$, then*

$$\|AB\| \leq \frac{M+m}{4\sqrt{Mm}}\|B\|(\omega(A) + D_A).$$

We need the following lemma to give some applications of Theorem 2.3.

Lemma 2.9. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}(A)$ and $C_{m,M}(B)$ are accretive, then $C_{m,M}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)$ is accretive.*

Proof. Put $X = \begin{bmatrix} x \\ y \end{bmatrix}$, where $x, y \in \mathbb{H}$. First we show that if A and B are accretive, then $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is accretive. We have

$$\begin{aligned} Re(\langle TX, X\rangle) &= Re\left(\left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle\right) = Re\left(\left\langle \begin{bmatrix} Ax \\ By \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle\right) \\ &= Re(\langle Ax, x\rangle) + Re(\langle By, y\rangle). \end{aligned}$$

Since $Re(\langle Ax, x\rangle) \geq 0$ and $Re(\langle By, y\rangle) \geq 0$, then

$$Re(\langle TX, X\rangle) \geq 0 \tag{2.5}$$

and so T is accretive. On the other hand,

$$\begin{aligned} C_{m,M}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) &= \left(\begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} - \begin{bmatrix} mI & 0 \\ 0 & mI \end{bmatrix}\right)\left(\begin{bmatrix} MI & 0 \\ 0 & MI \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} A^* - mI & 0 \\ 0 & B^* - mI \end{bmatrix}\right)\left(\begin{bmatrix} MI - A & 0 \\ 0 & MI - B \end{bmatrix}\right) \\ &= \begin{bmatrix} (A^* - mI)(MI - A) & 0 \\ 0 & (B^* - mI)(MI - A) \end{bmatrix} \\ &= \begin{bmatrix} C_{m,M}(A) & 0 \\ 0 & C_{m,M}(B) \end{bmatrix}. \end{aligned}$$

Consequently,

$$C_{m,M}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \begin{bmatrix} C_{m,M}(A) & 0 \\ 0 & C_{m,M}(B) \end{bmatrix}. \tag{2.6}$$

Since $C_{m,M}(A)$ and $C_{m,M}(B)$ are accretive, the result follows from (2.5) and (2.6). □

In the following, we provide a lower bound of the $\omega(AB - BA^*)$ in terms of $\|AB\|$ for some case.

Theorem 2.10. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If B is self-adjoint and $C_{m,M}(iAB)$ is accretive, then*

$$\frac{4\sqrt{Mm}}{M+m}\|AB\| \leq \omega(AB - BA^*). \tag{2.7}$$

Proof. By Theorem 2.3,

$$V_s(AB) \leq \frac{\|B+B^*\|\omega(A)}{2} + \frac{D_A D_{B+B^*}}{2} + \frac{1}{2}\omega(AB-BA^*).$$

Replacing B by iB in the last inequality gives

$$V_s(iAB) \leq \frac{1}{2}\omega(AB-BA^*). \tag{2.8}$$

Since $C_{m,M}(iAB)$ is accretive, from the inequality (1.10) and (2.8) we have

$$\frac{2\sqrt{Mm}}{M+m}\|AB\| \leq V_s(iAB) \leq \frac{1}{2}\omega(AB-BA^*).$$

Therefore,

$$\frac{2\sqrt{Mm}}{M+m}\|AB\| \leq \frac{1}{2}\omega(AB-BA^*).$$

This completes the proof. □

Recently, some inequalities have been presented by mathematicians to find the upper bound of $\omega(AB-BA^*)$, for example inequalities (1.5) and (1.12). On the other hand, we have to use the first inequality (1.2) to find a lower bound of $\omega(AB-BA^*)$. Now, in the following we give an example to show how Theorem 2.10 improves the first inequality (1.2).

Example 2.11. *Let* $B = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$, $A = \begin{bmatrix} -1.5i & 0.2i \\ 0 & -3.2i \end{bmatrix}$, $M=3$, *and* $m=1$. *Clearly B is self-adjoint and with a simple calculation, we have*

$$\left\|iAB - \frac{M+m}{2}I\right\| = \left\|\begin{bmatrix} 0.5 & 0.1 \\ 0 & 0.4 \end{bmatrix}\right\| \simeq 0.52 \leq 1 = \frac{M-m}{2}.$$

Therefore, $C_{m,M}(iAB)$ is accretive. On the other hand,

$$\|AB\| = \left\|\begin{bmatrix} -1.5i & 0.1i \\ 0 & -1.6i \end{bmatrix}\right\| \simeq 1.62$$

and

$$\|AB-BA^*\| = \left\|\begin{bmatrix} -3i & 0.1i \\ 0.1i & -3.2i \end{bmatrix}\right\| \simeq 3.24.$$

In this case

$$\frac{\|AB-BA^*\|}{2} \simeq 1.62$$

while

$$\frac{4\sqrt{Mm}}{M+m}\|AB\| \simeq 2.80.$$

Remark 2.12. *Let M and m are positive real numbers with $M > m$ and $A \in \mathbb{B}(\mathbb{H})$ and also $C_{m,M}(A)$ is accretive. Replacing B by I and A by $-iA$ in Theorem 2.10 gives*

$$\frac{2\sqrt{Mm}}{M+m}\|A\| \leq \frac{1}{2}\omega(A+A^*) = \|Re(A)\|.$$

Therefore, the inequality (2.7) *strengthens* (1.13).

Corollary 2.13. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If B is self-adjoint and $C_{m,M}(A)$ and also $C_{m,M}(iAB)$ is accretive, then*

$$\|AB\| \leq \frac{(M^2-m^2)}{4\sqrt{Mm}}\|B\|.$$

Proof. By Theorem 2.10,

$$\frac{4\sqrt{Mm}}{M+m}\|AB\| \leq \omega(AB - BA^*).$$

From the hypothesis $C_{m,M}(A)$ is accretive and (1.12),

$$\frac{4\sqrt{Mm}}{M+m}\|AB\| \leq (M-m)\omega(B),$$

which is exactly the desired result. □

At the end of this section, we introduce some numerical radius inequalities for products of two operators.

Theorem 2.14. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}(A)$ is accretive, then*

$$\omega(AB) \leq \left(D_A + \frac{M-m}{2}\right)\omega(B).$$

Proof. Clearly, $\|Re(AB)\| = \omega(Re(AB))$. Then

$$\begin{aligned}
\|Re(AB)\| &= \omega\left(\frac{AB+B^*A^*}{2}\right) \\
&= \omega\left(\frac{AB+AB^*-AB^*+B^*A^*}{2}\right) \\
&\leq \omega\left(\frac{AB+AB^*}{2}\right) + \omega\left(\frac{-AB^*+B^*A^*}{2}\right) \\
&= \frac{1}{2}\omega(A(B+B^*)) + \frac{1}{2}\omega(AB-BA^*) \\
&\leq \frac{1}{2}D_A\|B+B^*\| + \frac{1}{2}\omega(AB-BA^*) \qquad \text{(by (1.8))}
\end{aligned}$$

$$
\begin{aligned}
&= D_A\|Re(B)\| + \frac{1}{2}\omega(AB - BA^*) \\
&\leq D_A \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}B)\| + \frac{1}{2}\omega(AB - BA^*) \\
&= D_A\omega(B) + \frac{1}{2}\omega(AB - BA^*) \\
&\leq D_A\omega(B) + \frac{M-m}{2}\omega(B). \qquad \text{(by (1.12))} \\
&= \left(D_A + \frac{M-m}{2}\right)\omega(B).
\end{aligned}
$$

Hence,

$$\|Re(AB)\| \leq \left(D_A + \frac{M-m}{2}\right)\omega(B). \tag{2.9}$$

Suppose that $\theta_0 \in \mathbb{R}$. Replacing B by $e^{i\theta_0}B$ in the inequality (2.9) gives

$$\|Re(e^{i\theta_0}AB)\| \leq \left(D_A + \frac{M-m}{2}\right)\omega(B).$$

Taking the supremum over $\theta_0 \in \mathbb{R}$ gives

$$\omega(AB) \leq \left(D_A + \frac{M-m}{2}\right)\omega(B),$$

which is exactly the desired result. □

Corollary 2.15. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}(A)$ is accretive, then*

$$\omega(AB) \leq \left(\|A\| + \frac{M-m}{2}\right)\omega(B).$$

Proof. Since $D_A \leq \|A\|$, the result follows from Theorem 2.14. □

Concerning the inequality (1.4), the following result is interesting.

Theorem 2.16. *Let k, M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}(A)$ is accretive and $(M - m) \leq k\|A\|$, then*

$$\omega(AB) \leq \left(1 + \frac{k}{2}\right)\|A\|\omega(B).$$

Proof. By Corollary 2.15,

$$\omega(AB) \leq \left(\|A\| + \frac{M-m}{2}\right)\omega(B). \tag{2.10}$$

From the hypothesis $(M - m) \leq k\|A\|$ and inequality (2.10),

$$\omega(AB) \leq \left(\|A\| + \frac{k\|A\|}{2}\right)\omega(B),$$

which is exactly the desired result. □

Corollary 2.17. *Let k, M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}(A)$ is accretive and $(M - m) \leq k\|A\|$, then*

$$\omega(AB) \leq (2 + k)\omega(A)\omega(B).$$

Proof. Since $\|A\| \leq 2\omega(A)$, the result follows from Theorem 2.16. □

Remark 2.18. *If $k < 2$, Corollary 2.16 refines the inequality* (1.3).

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Declarations

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Conflicts of interest All authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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