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A simple construction of a fundamental solution for the extended Weinstein equation

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ABSTRACT

In this article, we study the extended Weinstein equation

$$Lu = \Delta u + \frac{k}{x_n}\frac{\partial u}{\partial x_n} + \frac{\ell}{x_n^2}u,$$

where u is a sufficiently smooth function defined in $\mathbb{R}^n$ with $x_n > 0$ and $n \geq 3$. We find a detailed construction for a fundamental solution for the operator L. The fundamental solution is represented by the associated Legendre functions Q_ν^μ.

RESUMEN

En este artículo estudiamos la ecuación de Weinstein extendida

$$Lu = \Delta u + \frac{k}{x_n}\frac{\partial u}{\partial x_n} + \frac{\ell}{x_n^2}u,$$

donde u es una función suficientemente suave definida en $\mathbb{R}^n$ con $x_n > 0$ y $n \geq 3$. Encontramos una construcción detallada para una solución fundamental del operador L. La solución fundamental está representada por las funciones de Legendre asociadas Q_ν^μ.

Keywords and Phrases: Extended Weinstein equation, fundamental solution, associated Legendre function.

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1 Introduction

In this paper, we will study the *extended Weinstein* or the *Leutwiler-Weinstein equation*

$$Lu := \Delta u + \frac{k}{x_n}\frac{\partial u}{\partial x_n} + \frac{\ell}{x_n^2}u = 0, \tag{1.1}$$

where $k, \ell \in \mathbb{R}$. The *Weinstein operator* L plays an interesting special role in the theory of partial differential equations, hyperbolic geometry and in other areas of mathematics (*cf.* Section 5). With the trivial choice of parameters $k = \ell = 0$, the Weinstein operator is the usual Euclidean *Laplacian*

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

acting on functions defined on $\mathbb{R}^n$. The solutions are called harmonic functions and the theory is well elaborated, see *e.g.* [3, 13, 14]. The next natural step is to just require the condition $\ell = 0$ to be fulfilled, in which case we are in the case presented by Alexander Weinstein, see [21] and also [4, 11]. In this case, equation (1.1) is a classic Weinstein equation and the operator L is singular on the surface $x_n = 0$. In this case, we usually look at functions that are defined in the upper half-space

$$\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0, \infty).$$

For more recent research on the Weinstein equation, see *e.g.* [5, 8]. The extended Weinstein equation (1.1) with arbitrary parameters $k, \ell \in \mathbb{R}$ was initially studied by Heinz Leutwiler in [12]. The equation has continued to be studied quite actively until these days, see *e.g.* [2].

The purpose of this article is to present the simplest possible construction (from the point of view of the authors) for the fundamental solution for the Weinstein operator L represented in (1.1). We try to present the theory in such a way that basic knowledge of partial differential equations and vector analysis are sufficient to follow the presentation, *i.e.* the so-called graduate student level. The structure of the article is as follows:

- In Section 2, we outline the required preliminaries, *i.e.* the Weinstein operator with its reduced version, and some useful notions from the theory of distributions.
- In Section 3, find the special type of "radial" solutions for (1.1).
- In Section 4, we use the "radial" solutions to define the fundamental solution and compute its proper coefficient.

2 Preliminaries

2.1 Weinstein operator

Let us look at some basic properties of the Weinstein operator L. Note that the variable x_n plays a special role in the operator. We denote elements $x = (x', x_n) \in \mathbb{R}^n_+$, where $x_n > 0$. We observe that keeping x_n fixed, the operator L admits with respect to the variable x' the same invariance properties as the Laplacian in $\mathbb{R}^{n-1}$, *i.e.* invariance under the Euclidean rigid motions (*cf.* [3]). Particularly important in what follows is the invariance with respect to translations

$$x' \mapsto x' + a' \tag{2.1}$$

for any $a' \in \mathbb{R}^{n-1}$. In the previous section, we did not discuss the fourth possible canonical special case for the Weinstein equation, namely the situation $k = 0$. In fact, this situation is significantly related to solving the extended Weinstein equation as follows. As a direct computation gives

$$L(x_n^{-\frac{k}{2}} u) = x_n^{-\frac{k}{2}} \widetilde{L}u, \tag{2.2}$$

where

$$\widetilde{L}u = \Delta u + \frac{k(2-k) + 4\ell}{4} \frac{u}{x_n^2},$$

we call the operator $\widetilde{L}$ the *reduced operator*. Subsequently, we will base our calculations largely on the reduced operator, as it is relatively close to the Laplace operator in its properties.

The reduced operator is especially useful from the point of view of the integration theory. Let U be a bounded subset of $\mathbb{R}^n_+$ with a sufficiently smooth boundary ∂U and let u and v twice differentiable real valued functions defined in an open set containing U. Hence, the usual *Green formula* for the Laplace operator is

$$\int_U (u\Delta v - v\Delta u)\, dx = \int_{\partial U} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS,$$

where the derivative with respect to the outer unit normal n is defined by

$$\frac{\partial u}{\partial n} = n \cdot \nabla u.$$

The Green formula for the reduced operator is obtained by adding and subtracting the term $\frac{k(2-k)+4\ell}{4}\frac{uv}{x_n^2}$ in the volume integral, *i.e.*

$$\int_U \left(u\widetilde{L}v - v\widetilde{L}u \right) dx = \int_{\partial U} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS. \tag{2.3}$$

2.2 Generalized functions

Generalized functions or distributions are a standard tool in modern partial differential equation theory. Their history begins in 1936, when Sergei Sobolev introduced his "*l'espace fonctionnel*" and applied them to solve a Cauchy problem of second-order partial differential equations in [17]. After this, the theory was further developed, see *e.g.*, the first larger representation of Laurent Schwartz [16]. A key work in the theory of partial differential equations is the classic book [9] by Gelfand and Shilov. In this book, distributions are examined from the point of view of solving partial differential equations, and the key tool is the connection between distributions and complex analytical functions. All the following information can be found in more detail in the literature mentioned above.

Let Ω be an open subset of $\mathbb{R}^n$ (or $\mathbb{R}^n_+$). We denote $\mathcal{D}(\Omega)$ as the space of compactly supported functions

$$C_0^\infty(\Omega) := \{\varphi \in C^\infty(\Omega) : \operatorname{supp}(\varphi) \text{ is compact and } \operatorname{supp}(\varphi) \subset \Omega\}$$

equipped with the topology of uniform convergence in compact subsets $K \subset \Omega$. Indeed, $\varphi_j \to \varphi$ in $\mathcal{D}(\Omega)$, if there exists a compact subset $K \subset \Omega$ such that $\operatorname{supp}(\varphi_j) \subset K$ for any j and all derivatives $\partial^\alpha \varphi_j \to \partial^\alpha \varphi$ uniformly, *i.e.* the convergence in the Fréchet space $C^\infty(K)$. Above, multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

is used. The preceding $\mathcal{D}(\Omega)$ is called the *test function space*. We denote by $\mathcal{D}'(\Omega)$ the space of continuous linear functionals over $\mathcal{D}(\Omega)$, and we call its elements *distributions* or *generalized functions*. If $T \in \mathcal{D}'(\Omega)$, we denote

$$T(\varphi) =: \langle T, \varphi \rangle$$

for all $\varphi \in \mathcal{D}(\Omega)$. The continuity of a functional T means, that $T(\varphi) \to 0$ for all $\varphi \to 0$ in $\mathcal{D}(\Omega)$. The convergence in $\mathcal{D}'(\Omega)$ is defined in the weak form, *i.e.* a sequence $\{T_j\}$ of distributions converges to a distribution T if and only if

$$\langle T_j, \varphi \rangle \to \langle T, \varphi \rangle, \quad \text{for} \quad j \to \infty, \tag{2.4}$$

for any $\varphi \in \mathcal{D}(\Omega)$. Important basic properties of distributions are that they have all derivatives defined by setting

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and multiplying by a smooth function $f \in C^\infty(\Omega)$ produces a distribution, *i.e.*,

$$\langle fT, \varphi \rangle = \langle T, f\varphi \rangle.$$

The above properties allow differential operators to be defined in distributional sense, *e.g.*,

$$\langle \widetilde{L}T, \varphi \rangle = \langle T, \widetilde{L}\varphi \rangle, \tag{2.5}$$

for any $T \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ when $\Omega \subset \mathbb{R}^n_+$. We also denote

$$\langle T, \varphi \rangle = \langle T(x), \varphi(x) \rangle,$$

where the x is a dummy variable (*cf.* the use of variables in integrals). Any *locally integrable* function $g \in L^1_{\text{loc}}(\Omega)$ defines a distibution via the L^2-inner product by

$$\langle g, \varphi \rangle := \langle g, \varphi \rangle_{L^2(\Omega)} = \int_\Omega g(x)\varphi(x)\, dx. \tag{2.6}$$

Remark 2.1. *The starting point for the theory of distributions can be also in measure theory. Let us elaborate on the equivalence of perspectives. If $\Omega \subset \mathbb{R}^n$ is an open set and μ a complex Borel measure on it with $\mu(K) < \infty$ for any compact $K \subset \Omega$, then*

$$T(\varphi) = \int_\Omega \varphi \, d\mu,$$

defines a distribution, where $\varphi \in \mathcal{D}(\Omega)$. If $f \in L^1_{loc}(\Omega)$, then the measure

$$\mu(E) = \int_E f(x)\, dx$$

for any Borel set $E \subset \Omega$ is a complex Borel measure with $\mu(K) < \infty$. Then the Radon-Nikodym derivative $\frac{d\mu}{dx} = f$. Hence, we can intuitively identify distributions with functions f or equivalently with measures μ.

The most important example of distributions is the Dirac delta distribution, which is defined by setting

$$\langle \delta(x-y), \varphi(x) \rangle := \varphi(y),$$

for $y \in \mathbb{R}^n$. The Dirac delta is not a distribution generated by a locally integrable function. In the distributional sense, one can see that $\delta(x-y) = 0$ for any $x \neq y$. Moreover, the Dirac delta has the obvious property

$$f(x)\delta(x-y) = f(y)\delta(x-y), \tag{2.7}$$

for $f \in C^\infty(\Omega)$, which plays a central role in this paper. If

$$Pu(x) = \sum_{k=0}^{m} \sum_{|\alpha|=k} a_\alpha(x) \partial^\alpha u(x)$$

is a linear differential operator P acting on a suitable function u, where $a_\alpha \in C^\infty(\Omega)$, we call a distribution $G(\cdot, y) \in \mathcal{D}'(\Omega)$ a *fundamental solution* at $y \in \Omega$, if it satisfies the equation

$$PG(x,y) = \delta(x-y).$$

The main motivation to find a fundamental solution is to study solutions of the equation $Pu = f$. One can see, that the solution of the problem is given by $u = G * f$, where $*$ is the convolution of a distribution and a function. See details, *e.g.* in [9].

3 Classical "radial" solutions

In this paper, our aim is to find a fundamental solution G for the Weinstein operator (1.1), *i.e.*

$$LG(x,y) = \delta(x-y)$$

where $y \in \mathbb{R}^n_+$. Our first observation is that due to the formulas (2.2) and (2.7), we obtain the following formula.

Proposition 3.1. *If* $\widetilde{L}v = \delta(x-y)$, *then* $L\left(\left(\frac{y_n}{x_n}\right)^{\frac{k}{2}} v\right) = \delta(x-y)$.

Hence, it is enough to find a fundamental solution for the reduced operator $\widetilde{L}$. A usual problem with any non-constant coefficient differential operator is that the symmetry of the operator does not match with the symmetry of the Dirac delta. We know that the Dirac delta is rotationally invariant (see [9]), *i.e.*

$$\delta(Ax) = \delta(x)$$

for any $A \in SO(n)$, but as we mentioned above, $\widetilde{L}$ is rotation invariant only around the x_n-axis, or more precisely, it is invariant under the subgroup $SO(n-1)$ in $SO(n)$ defined as the stabiliser of the x_n-axis. Hence, the x_n-direction will play a special role. Since the operator is translation invariant with respect to x', we can try to find first a fundamental solution only at the point $y = (0', y_n)$. Thus,

$$\delta(x-y) = \delta(x')\delta(x_n - y_n).$$

Consequently, the fundamental solution must be a "radial function", *i.e.*, it depends on $|x-y|$, with the special role of x_n. Hyperbolic geometry gives us an idea how to proceed. In [15], one can find the expression

$$|x-y|^2 = 2x_n y_n(\lambda(x,y) - 1), \tag{3.1}$$

where $\lambda : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [1, \infty)$ is defined by

$$\lambda(x, y) = 1 + \frac{|x-y|^2}{2x_n y_n}.$$

The reader should note that

$$\lambda \geq 1. \tag{3.2}$$

The function λ is an invariant with respect to the invariance group of the hyperbolic upper-half space, *cf.* [15]. Based on this, one can try to find a solution for the extended Weinstein equation in the form

$$u(x) = x_n^\alpha v(\lambda), \tag{3.3}$$

for a fixed $y \in \mathbb{R}^n_+$. We want to substitute this into equation (1.1). First, we compute the following technical lemma.

Lemma 3.2. *If u is of the form* (3.3) *and $y' = 0$, we have*

$$x_n^{2-\alpha} \widetilde{L} u = (\lambda^2 - 1) v''(\lambda) + \left((n - 2 + 2\alpha) \frac{x_n}{y_n} + 2(1-\alpha)\lambda \right) v'(\lambda) + \left(\alpha(\alpha - 1) + \frac{k(2-k) + 4\ell}{4} \right) v(\lambda).$$

Proof. Since $v = v(\lambda(x, y))$, we compute

$$\begin{aligned}
\frac{\partial v}{\partial x_j} &= \frac{x_j}{x_n y_n} v'(\lambda), \\
\frac{\partial v}{\partial x_n} &= \frac{x_n - y_n \lambda}{x_n y_n} v'(\lambda), \\
\frac{\partial^2 v}{\partial x_j^2} &= \frac{x_j^2 v''(\lambda) + x_n y_n v'(\lambda)}{x_n^2 y_n^2}, \\
\frac{\partial^2 v}{\partial x_n^2} &= \frac{(x_n - y_n \lambda)^2 v''(\lambda) + (2y_n^2 \lambda - x_n y_n) v'(\lambda)}{x_n^2 y_n^2},
\end{aligned}$$

for $j = 1, \ldots, n-1$. Then we compute

$$\begin{aligned}
\frac{\partial u}{\partial x_j} &= x_n^{\alpha - 1} \frac{x_j}{y_n} v'(\lambda), \\
\frac{\partial u}{\partial x_n} &= x_n^{\alpha - 1} \left(\alpha v(\lambda) + \left(\frac{x_n}{y_n} - \lambda \right) v'(\lambda) \right), \\
\frac{\partial^2 u}{\partial x_j^2} &= \frac{x_n^{\alpha - 2}}{y_n^2} \Big(x_j^2 v''(\lambda) + y_n x_n v'(\lambda) \Big), \\
\frac{\partial^2 u}{\partial x_n^2} &= \frac{x_n^{\alpha - 2}}{y_n^2} \Big((x_n - y_n \lambda)^2 v''(\lambda) + \big((2\alpha - 1) x_n y_n + (y_n^2 - \alpha y_n^2) 2\lambda \big) v'(\lambda) + \alpha(\alpha - 1) y_n^2 v(\lambda) \Big),
\end{aligned}$$

for $j = 1, \ldots, n-1$. Then we observe

$$\sum_{j=1}^{n-1} \frac{\partial^2 u}{\partial x_j} = \frac{x_n^{\alpha - 2}}{y_n^2} \Big(|x'|^2 v''(\lambda) + (n-1) y_n x_n v'(\lambda) \Big).$$

Since $y' = 0$, we have $|x'|^2 = 2x_n y_n \lambda - x_n^2 - y_n^2$ and we obtain

$$\Delta u = x_n^{\alpha-2}\left((\lambda^2-1)v''(\lambda) + \left((n-2+2\alpha)\frac{x_n}{y_n} + 2(1-\alpha)\lambda\right)v'(\lambda) + \alpha(\alpha-1)v(\lambda)\right),$$

completing the proof. □

We observe that we obtain the ordinary differential equation with respect to λ if we choose $\alpha = \frac{2-n}{2}$. Since $\alpha(\alpha-1) = \frac{1}{4}(n-2)n$, we obtain the following result.

Proposition 3.3. *The function $u(x) = x_n^{\frac{2-n}{2}} v(\lambda)$ is a solution of $\widetilde{L}u = 0$ if and only if*

$$(\lambda^2-1)v''(\lambda) + n\lambda v'(\lambda) + \frac{1}{4}(k(2-k) + (n-2)n + 4\ell)v(\lambda) = 0.$$

We denote $\beta := \frac{1}{4}(k(2-k) + (n-2)n + 4\ell)$. To solve the equation

$$(\lambda^2-1)v''(\lambda) + n\lambda v'(\lambda) + \beta v(\lambda) = 0, \tag{3.4}$$

we first observe that it is not far from the associated Legendre equation

$$(z^2-1)w''(z) + 2zw'(z) - \left(\nu(\nu+1) + \frac{\mu^2}{z^2-1}\right)w(z) = 0, \tag{3.5}$$

with parameters $\mu, \nu \in \mathbb{C}$. The associated Legendre equation has two solutions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ defined outside of singularities $z = \pm 1$, see *e.g.* [1,10]. The solutions are called *associated Legendre functions*. The solutions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are linearly independent if and only if $\mu \pm \nu \notin -\mathbb{N}$. We need to exclude this case in the future.

Assume $x \neq y$, that is, from (3.2) we obtain $\lambda = \lambda(x,y) > 1$. We look for a solution for (3.4) in the form

$$v(\lambda) = (\lambda^2-1)^\delta w(\lambda).$$

Substituting this into the equation (3.4), the equation becomes

$$(\lambda^2-1)w'' + (4\delta+n)\lambda w' + \left(2\delta + \beta + \frac{(4\delta(\delta-1)+2n\delta)\lambda^2}{\lambda^2-1}\right)w = 0. \tag{3.6}$$

To obtain the associated Legendre equation, it is required that

$$4\delta + n = 2 \iff \delta = \frac{2-n}{4}.$$

We obtain the following result.

Proposition 3.4. *The function* $v(\lambda) = (\lambda^2-1)^{\frac{2-n}{4}} w(\lambda)$ *satisfies equation* (3.4) *if and only if* w *is a solution of the associated Legendre equation*

$$(\lambda^2-1)w'' + 2\lambda w' - \left(-\frac{n(2-n)+4\beta}{4} + \frac{\frac{1}{4}(n-2)^2}{\lambda^2-1}\right) w = 0.$$

Proof. Using (3.6), we have

$$(\lambda^2-1)w'' + 2\lambda w' + \left(\frac{2-n}{2} + \beta - \frac{1}{4}(n-2)^2 \frac{\lambda^2}{\lambda^2-1}\right) w = 0.$$

On the other hand,

$$\frac{\lambda^2}{\lambda^2-1} = 1 + \frac{1}{\lambda^2-1},$$

and we obtain the result. □

Hence we obtain the solutions as follows.

Theorem 3.5. *Equation* (3.4) *has two linearly independent solutions*

$$(\lambda^2-1)^{\frac{2-n}{4}} P^{\pm\frac{n-2}{2}}_{-\frac{1}{2}\pm\frac{\sqrt{n(n-2)+1-4\beta}}{2}}(\lambda) \quad \textit{and} \quad (\lambda^2-1)^{\frac{2-n}{4}} Q^{\pm\frac{n-2}{2}}_{-\frac{1}{2}\pm\frac{\sqrt{n(n-2)+1-4\beta}}{2}}(\lambda)$$

where we can choose any $\pm$ *combination for any indices (4 possible combinations).*

Proof. To solve the reduced equation, we need to find the right parameters in the Legendre equation (3.5), that is

$$\nu(\nu+1) = -\frac{n(2-n)+4\beta}{4} \;\Leftrightarrow\; \nu = -\frac{1}{2} \pm \frac{\sqrt{n(n-2)+1-4\beta}}{2}$$

and

$$\mu^2 = \frac{1}{4}(n-2)^2 \;\Leftrightarrow\; \mu = \pm\frac{n-2}{2}.$$

Equation (3.4) admits two linearly independent solutions

$$(\lambda^2-1)^{\frac{2-n}{4}} P^{\pm\frac{n-2}{2}}_{-\frac{1}{2}\pm\frac{\sqrt{n(n-2)+1-4\beta}}{2}}(\lambda) \text{ and } (\lambda^2-1)^{\frac{2-n}{4}} Q^{\pm\frac{n-2}{2}}_{-\frac{1}{2}\pm\frac{\sqrt{n(n-2)+1-4\beta}}{2}}(\lambda).$$

Then, using the formulas 8.2.1 and 8.2.2 from [1], both functions $P^{\mu}_{-\kappa-\frac{1}{2}}$ and $Q^{\mu}_{-\kappa-\frac{1}{2}}$ can be represented by the functions $P^{\mu}_{\kappa-\frac{1}{2}}$ and $Q^{\mu}_{\kappa-\frac{1}{2}}$. Similarly, using the formulas 8.2.5 and 8.2.6, we can represent $P^{-\mu}_{\nu}$ and $Q^{-\mu}_{\nu}$ by the functions P^{μ}_{ν} and Q^{μ}_{ν}. Hence, the any $\pm$ combination gives us two linear independent solutions. □

Corollary 3.6. *Using* (2.2)*, we obtain the solutions*

$$x_n^{-\frac{k}{2}}(\lambda^2-1)^{\frac{2-n}{4}}P^{\pm\frac{n-2}{2}}_{-\frac{1}{2}\pm\frac{\sqrt{n(n-2)+1-4\beta}}{2}}(\lambda) \quad \text{and } x_n^{-\frac{k}{2}}(\lambda^2-1)^{\frac{2-n}{4}}Q^{\pm\frac{n-2}{2}}_{-\frac{1}{2}\pm\frac{\sqrt{n(n-2)+1-4\beta}}{2}}(\lambda)$$

for the Weinstein equation (1.1).

Remark 3.7. *We observe that if* $n(n-2)+1-4\beta<0$*, we obtain solutions with the functions*

$$P^{\mu}_{-\frac{1}{2}+i\theta}(\lambda) \quad \text{and} \quad Q^{\mu}_{-\frac{1}{2}+i\theta}(\lambda),$$

with some $\theta\in\mathbb{R}$*. These functions are called the Mehler functions or conical functions, see e.g. Section 8.12. in [1] or Section 8.84 in [10]. The first of the functions is real-valued, while the second is complex-valued in general. To find completely real-valued solutions, see e.g. [6].*

Remark 3.8. *The special case* $n(n-2)+1-4\beta=0$*, i.e.* $k(2-k)+4\ell=1$*, corresponds to the equation*

$$\widetilde{L}u=\Delta u+\frac{1}{4}\frac{u}{x_n^2}=0 \quad \text{or} \quad Lu=\Delta u+\frac{k}{x_n}\frac{\partial u}{\partial x_n}+\frac{\frac{1}{4}(k-1)^2}{x_n^2}u=0,$$

and the solutions are given by

$$P^{\mu}_{-\frac{1}{2}}(\lambda) \quad \text{and} \quad Q^{\mu}_{-\frac{1}{2}}(\lambda).$$

4 Finding fundamental solutions

The solutions given in Theorem 3.5 can be used as candidates for a fundamental solution. From (3.1), we infer that $x\to y$ if and only if $\lambda\to 1+$. Next, let us examine the asymptotic behavior of functions in general. In the following, we assume that the argument z of the functions is real.

Proposition 4.1. *If* $Re(\mu)>0$*, then*

$$\lim_{z\to 1}\left((z^2-1)^{\frac{\mu}{2}}P_\nu^{-\mu}(z)\right)=0.$$

Proof. For $|1-z|<2$ the associated Legendre function P_ν^μ admits the representation (see 8.1.2 in [1])

$$P_\nu^{-\mu}(z)=\frac{1}{\Gamma(1+\mu)}\left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}}{}_2F_1\left(-\nu,\nu+1;1+\mu;\frac{1-z}{2}\right),$$

where ${}_2F_1$ represents the usual hypergometric functions (see [1, 10]). Hence

$$(z^2-1)^{\frac{\mu}{2}}P_\nu^{-\mu}(z)=\frac{1}{\Gamma(1+\mu)}(z-1)^{\mu}{}_2F_1\left(-\nu,\nu+1;1+\mu;\frac{1-z}{2}\right),$$

completing the proof. ∎

Proposition 4.2. *If Re*$(\mu) > 0$ *and* $\nu + \frac{1}{2} \notin -\mathbb{N}$*, then*

$$\lim_{z\to 1+}\left((z^2-1)^{\frac{\mu}{2}} Q_\nu^\mu(z)\right) = e^{i\pi\mu}2^{\mu-1}\Gamma(\mu).$$

Proof. Using the representation 8.703 in [10], we obtain the representation

$$(z^2-1)^{-\frac{\mu}{2}} Q_\nu^\mu(z) = e^{i\pi\mu}\frac{\Gamma(\nu+\mu+1)\sqrt{\pi}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})z^{\nu+\mu+1}}\,{}_2F_1\left(\frac{\nu+\mu+2}{2},\frac{\nu+\mu+1}{2};\nu+\frac{3}{2};\frac{1}{z^2}\right).$$

Using the transformation formula 9.131.1 in [10], we have

$${}_2F_1\left(\frac{\nu+\mu+2}{2},\frac{\nu+\mu+1}{2};\nu+\frac{3}{2};\frac{1}{z^2}\right) = \frac{(z^2-1)^{-\mu}}{z^{-2\mu}}\,{}_2F_1\left(\frac{\nu-\mu+1}{2},\frac{\nu-\mu+2}{2};\nu+\frac{3}{2},\frac{1}{z^2}\right).$$

Hence

$$(z^2-1)^{\frac{\mu}{2}} Q_\nu^\mu(z) = e^{i\pi\mu}\frac{\Gamma(\nu+\mu+1)\sqrt{\pi}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})z^{\nu-\mu+1}}\,{}_2F_1\left(\frac{\nu-\mu+2}{2},\frac{\nu-\mu+1}{2};\nu+\frac{3}{2};\frac{1}{z^2}\right).$$

If $\mathrm{Re}(c-a-b) > 0$ and $c \notin -\mathbb{N}_0$, the identity 15.1.20 in [1] says

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

If $a = \frac{\nu-\mu+1}{2}$, $b = \frac{\nu-\mu+2}{2}$ and $c = \nu+\frac{3}{2}$, we have $\mathrm{Re}(c-a-b) = \mathrm{Re}(\mu) > 0$ and $\nu+\frac{3}{2} \neq 0, -1, -2, \ldots,$ *i.e.*

$${}_2F_1\left(\frac{\nu-\mu+1}{2},\frac{\nu-\mu+2}{2};\nu+\frac{3}{2};1\right) = \frac{\Gamma(\nu+\frac{3}{2})\Gamma(\mu)}{\Gamma(\frac{\nu+\mu+2}{2})\Gamma(\frac{\nu+\mu+1}{2})}.$$

Using the doubling formula for the gamma function 8.335.1 in [10], we obtain

$${}_2F_1\left(\frac{\nu-\mu+1}{2},\frac{\nu-\mu+2}{2};\nu+\frac{3}{2};1\right) = \frac{2^{\nu+\mu}\Gamma(\nu+\frac{3}{2})\Gamma(\mu)}{\sqrt{\pi}\Gamma(\nu+\mu+1)}.$$

Thus

$$\lim_{z\to 1+}\left((z^2-1)^{\frac{\mu}{2}} Q_\nu^\mu(z)\right) = e^{i\pi\mu}\frac{\Gamma(\nu+\mu+1)\sqrt{\pi}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}\frac{2^{\nu+\mu}\Gamma(\nu+\frac{3}{2})\Gamma(\mu)}{\sqrt{\pi}\Gamma(\nu+\mu+1)} = e^{i\pi\mu}2^{\mu-1}\Gamma(\mu). \qquad \square$$

We know that the homogenity of the Dirac delta distribution is $-n$ and the reduced operator is a homogeneous differential operator of degree 2. Hence, the fundamental solutions should have the homogenity $-n+2$. From (3.1), we obtain, that $(\lambda-1)^{\frac{n-2}{2}}$ has the needed homogenity. Hence the solution $(\lambda^2-1)^{-\frac{\mu}{2}}Q_\nu^\mu(\lambda)$ has the suitable homogenity. We see it by writing it in the form

$$(\lambda^2-1)^{-\frac{\mu}{2}} Q_\nu^\mu(\lambda) = \frac{(\lambda^2-1)^{\frac{\mu}{2}} Q_\nu^\mu(\lambda)}{(\lambda+1)^\mu(\lambda-1)^\mu}$$

for $\mu = \frac{n-2}{2}$. Let us define the following function with the canonical asymptotic behavior, which we can use as a candidate for a fundamental solution. This means that we fixing the constant, and proving directly that it indeed satisfies the correct equation.

Proposition 4.3. *Let* $\mu = \frac{n-2}{2}$ *and* $\mu = -\frac{1}{2} \pm \frac{\sqrt{n(n-2)+1-4\beta}}{2}$. *The function*

$$F(x,y) = \frac{f(\lambda)}{(\lambda-1)^{\mu}},$$

where

$$f(\lambda) = \frac{2e^{-\pi\mu i}}{\Gamma(\nu)} \frac{(\lambda^2-1)^{\frac{\mu}{2}} Q_{\nu}^{\mu}(\lambda)}{(\lambda+1)^{\mu}}$$

is a null solution of the reduced operator for $x \neq y$ *and*

$$\lim_{\lambda \to 1} f(\lambda) = 1.$$

Proof. Using the preceding proposition, we obtain

$$\lim_{\lambda \to 1} \left(\frac{(\lambda^2-1)^{\frac{\mu}{2}} Q_{\nu}^{\mu}(\lambda)}{(\lambda+1)^{\mu}} \right) = \frac{e^{\pi\mu i}}{2} \Gamma(\mu).$$ □

Next we need to evaluate $\widetilde{L}F(\cdot,y)$ in the distribution sense. We proceed as follows. We take a test function $\varphi \in \mathcal{D}(\mathbb{R}^n_+)$ and choose a bounded open set $U \subset \mathbb{R}^n_+$ with a sufficiently smooth boundary satisfying $\operatorname{supp}(\varphi) \subset U$, and we define $U_r(y) := U \backslash B_r(y)$ for $0 < r < R$, where $R = \inf\{|x-y| : x \in \partial U\}$. If $\chi_{U_r(y)}$ is the characteristic function of $U_r(y)$, then we define the sequence of locally integrable functions $\{F_r\}$ by $F_r := \chi_{U_r(y)} F(\cdot,y)$. Obviously, the sequence converges to the $F(\cdot,y)$ in the distributional sense (2.4). Then, using this convergence and (2.5), we obtain

$$\langle \widetilde{L}F(\cdot,y), \varphi \rangle = \langle F(\cdot,y), \widetilde{L}\varphi \rangle = \lim_{r \to 0} \langle F_r, \widetilde{L}\varphi \rangle. \tag{4.1}$$

Since F_r is locally integrable, we have using (2.6) and the Green formula (2.3),

$$\langle F_r, \widetilde{L}\varphi \rangle = \int_{U_r(y)} F(x,y)\widetilde{L}\varphi(x)\, dx = \int_{U_r(y)} \widetilde{L}F(x,y)\varphi(x)\, dx + \int_{\partial U_r(y)} \left(F\frac{\partial \varphi}{\partial n} - \varphi \frac{\partial F}{\partial n} \right) dS.$$

We observe that $\widetilde{L}F(x,y) = 0$ for $x \neq y$ and split $\partial U_r(y) = \partial U \cup (-\partial B_r(y))$, where the minus sign denotes the opposite (*i.e.* inward) orientation. Since $\operatorname{supp}(\frac{\partial \varphi}{\partial n}) \subset \operatorname{supp}(\varphi)$, we observe that the surface integral over ∂U vanishes. Hence

$$\langle F_r, \widetilde{L}\varphi \rangle = \int_{\partial B_r(y)} \left(\varphi \frac{\partial F}{\partial n} - F \frac{\partial \varphi}{\partial n} \right) dS. \tag{4.2}$$

To compute the surface integral in (4.2), we need the following technical lemma.

Lemma 4.4. *If $x \in \partial B_r(y)$ with $y = (0', y_n)$, then the normal derivative of $\lambda(x, y)$ satisfies*

$$\frac{\partial \lambda}{\partial n} = r\frac{x_n + y_n}{2x_n^2 y_n}.$$

Proof. We compute

$$\frac{\partial \lambda}{\partial x_j} = \frac{x_j}{x_n y_n}, \quad \text{for} \quad j = 1, \dots, n-1,$$
$$\frac{\partial \lambda}{\partial x_n} = \frac{-|x'|^2 + x_n^2 - y_n^2}{2x_n^2 y_n},$$

i.e.

$$\nabla \lambda = \frac{\left(x', \frac{-|x'|^2 + x_n^2 - y_n^2}{2x_n}\right)}{x_n y_n}.$$

At $x \in \partial B_r(y)$, the outward pointing unit normal is

$$n(x) = \frac{(x', x_n - y_n)}{r}.$$

Since $|x'|^2 = r^2 - (x_n - y_n)^2$, we compute

$$\frac{\partial \lambda}{\partial n} = n \cdot \nabla \lambda = r\frac{x_n + y_n}{2x_n^2 y_n}.$$ □

We also need the following asymptotics.

Remark 4.5 (Integrals over spheres). *If $f : U \to \mathbb{R}$ is a continuous function, $y \in U$ and $R > 0$ a radius such that $B_r(y) \subset U$ for all $0 < r < R$. Then there is the classical asymptotic formula of the surface integrals, depending on the singularity of the integrand. A direct consequence of the continuity of the function f is*

$$\lim_{r \to 0} \int_{\partial B_r(y)} \frac{f(x)}{r^\alpha} dS(x) = \begin{cases} 0, & \text{for} \quad 0 < \alpha < n-1, \\ \omega_{n-1} f(y), & \text{for} \quad \alpha = n-1, \\ \pm\infty, & \text{for} \quad \alpha > n-1, \end{cases} \tag{4.3}$$

where ω_{n-1} is the surface area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. These are a special case of the so-called potential type integrals, see e.g. [19].

Then we are ready to prove:

Theorem 4.6. *If $y = (0', y_n)$, then*

$$\widetilde{L}F(\cdot, y) = -\frac{n-2}{2} y_n^{n-2} \omega_{n-1} \delta(x') \delta(x_n - y_n),$$

where ω_{n-1} is the surface area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and $n \geq 3$.

Proof. Since $\lambda - 1 = \frac{r^2}{2x_n y_n}$, we have

$$F(x, y) = \frac{(2x_n y_n)^\mu f(\lambda)}{r^{n-2}},$$

where $\mu = \frac{n-2}{2}$. Hence using (4.3), we obtain

$$\lim_{r \to 0} \int_{\partial B_r(y)} F \frac{\partial \varphi}{\partial n} dS = \lim_{r \to 0} \int_{\partial B_r(y)} \frac{(2x_n y_n)^\mu f(\lambda)}{r^{n-2}} \frac{\partial \varphi}{\partial n} dS = 0.$$

Then we compute using Lemma 4.4

$$\begin{aligned} \frac{\partial F}{\partial n} &= \frac{d}{d\lambda} \left(\frac{f(\lambda)}{(\lambda - 1)^\mu} \right) \frac{\partial \lambda}{\partial n} \\ &= \left(\frac{f'(\lambda)}{(\lambda - 1)^\mu} - \frac{\mu f(\lambda)}{(\lambda - 1)^{\mu + 1}} \right) \frac{\partial \lambda}{\partial n} \\ &= \frac{1}{2} \left(\frac{(2x_n y_n)^\mu f'(\lambda)}{r^{n-3}} - \frac{\mu (2x_n y_n)^{\frac{n}{2}} f(\lambda)}{r^{n-1}} \right) \frac{x_n + y_n}{x_n^2 y_n}. \end{aligned}$$

Hence, we can compute

$$\begin{aligned} \lim_{r \to 0} \int_{\partial B_r(y)} \varphi \frac{\partial F}{\partial n} dS =& \frac{1}{2} \underbrace{\lim_{r \to 0} \int_{\partial B_r(y)} \frac{(2x_n y_n)^\mu f'(\lambda)}{r^{n-3}} \frac{x_n + y_n}{x_n^2 y_n} \varphi dS}_{=0, \text{ using } (4.3)} \\ & - \frac{1}{2} \lim_{r \to 0} \int_{\partial B_r(y)} \frac{\mu (2x_n y_n)^{\frac{n}{2}} f(\lambda)}{r^{n-1}} \frac{x_n + y_n}{x_n^2 y_n} \varphi dS \\ =& - \mu y_n^{n-2} \omega_{n-1} \varphi(y) \end{aligned}$$

again using (4.3). Hence, using (4.1) and (4.2), we conclude

$$\langle \widetilde{L}F(\cdot, y), \varphi \rangle = -\mu y_n^{n-2} \omega_{n-1} \varphi(y).$$

Using the definition of the Dirac delta distribution, we obtain the result. □

Since $\widetilde{L}$ is invariant under (2.1), we obtain a fundamental solution by the simple substitution.

Theorem 4.7. *Let $n \geq 3$. The distribution*

$$H(x,y) = \frac{h(\lambda(x,y))}{(\lambda(x,y)-1)^{\frac{n-2}{2}}}$$

where

$$h(\lambda) = -\frac{4e^{-\pi\mu i}}{(n-2)y_n^{n-2}\omega_{n-1}\Gamma(\nu)}\frac{(\lambda^2-1)^{\frac{\mu}{2}}Q_\nu^\mu(\lambda)}{(\lambda+1)^\mu}$$

is a fundamental solution for $\widetilde{L}$, i.e.

$$\widetilde{L}H(\cdot,y) = \delta(x-y),$$

for any $y \in \mathbb{R}^n_+$.

Proposition 3.1 gives the following theorem.

Theorem 4.8. *Let $n \geq 3$. The distribution*

$$G(x,y) = \frac{g(\lambda(x,y))}{(\lambda(x,y)-1)^{\frac{n-2}{2}}}$$

where

$$g(\lambda) = -\frac{1}{x_n^{\frac{k}{2}}y_n^{n-2-\frac{k}{2}}}\frac{4e^{-\pi\mu i}}{(n-2)\omega_{n-1}\Gamma(\nu)}\frac{(\lambda^2-1)^{\frac{\mu}{2}}Q_\nu^\mu(\lambda)}{(\lambda+1)^\mu}$$

is a fundamental solution for L, i.e.

$$LG(\cdot,y) = \delta(x-y),$$

for any $y \in \mathbb{R}^n_+$.

Above, the special case $n = 2$ is not considered and is left as a future research topic. The question is a natural deformation for the hyperbolic Laplace operator on the complex upper half-plane.

5 Conclusions

In this paper, we derive the fundamental solution for the operator L in detail. The reader can see that to find the fundamental solution for an operator with a non-constant coefficient is much more challenging than in the case of constant coefficients. The reader should also bear in mind how the only constant multiplication special case $k = \ell = 0$ makes calculations significantly easier. By doing the calculations presented in the paper in this case, we recover a classical derivation, based on differential equations, for the fundamental solution of the Laplace operator.

Finally, the authors would like to point out that the results of the paper may be interesting in addition to analysis in other areas of mathematics, such as analytical number theory, because the extended Weinstein equation also encompasses the famous Maaß wave equation, including the famous Maaß forms as special solutions, see *e.g.* [7, 18].

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