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# $L_p$-boundedness of the Laplace transform

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**ABSTRACT**

In this paper, we discuss about the boundedness of the Laplace transform $\mathcal{L} : L_p([0,\infty)) \to L_p(A)$ ($p \geq 1$) for the cases $A = [0,\infty)$, $A = [1,\infty)$ and $A = [0,1]$. We also provide examples for the cases where $\mathcal{L}$ is unbounded.

**RESUMEN**

En este artículo, discutimos sobre la acotación de la transformada de Laplace $\mathcal{L} : L_p([0,\infty)) \to L_p(A)$ ($p \geq 1$) para los casos $A = [0,\infty)$, $A = [1,\infty)$ y $A = [0,1]$. También entregamos ejemplos para los casos donde $\mathcal{L}$ es no acotada.

**Keywords and Phrases:** Laplace transform, Integral transform, $L_p$-strong boundedness.

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# 1 Introduction

The Laplace transform $\mathcal{L}$ is a well-known classical linear integral operator defined for every appropriate function $f$ on $[0,\infty)$ by

$$\mathcal{L}f(t) = \int_0^\infty e^{-st} f(s)\, ds, \quad t \in (0,\infty).$$

Laplace transform is widely used for solving ordinary and partial differential equations. Hence it is a useful tool not only for mathematicians but also for physicists and engineers. It is also useful in Probability Theory (see [1], [8] and [10]).

Searching among the literature, we found that the study of the boundedness of the Laplace transform for some unknown reason has been neglected. In this regard, we could only find the references [3] and [6,7], in which the authors stated some results about the boundedness of the Laplace transform. In [3], the optimal rearrangement-invariant space on either side of $\mathcal{L} : X \to Y$ is characterized when the other space is given. In [6], the authors studied both the Laplace transform and a more general class of operators (also in weighted $L_p$ spaces), and in [7], they provided for them a spectral representation in $L_2$. For more on the Laplace transform and its optimal domain of definition, the interested reader is invited to check [2, 9, 11] and the references therein.

In such a sense, in a self contained presentation, we study the boundedness of the Laplace transform on Lebesgue $L_p$-spaces. Our main goal is to show that:

(1) $\mathcal{L} : L_p([0,\infty)) \to L_p([0,\infty))$ is bounded only if $p = 2$.

(2) $\mathcal{L} : L_p([0,\infty)) \to L_p([1,\infty))$ is bounded only if $p > 2$.

(3) $\mathcal{L} : L_p([0,\infty)) \to L_p([0,1])$ is bounded only if $1 < p < 2$.

# 2 Main results

We would like to discuss now about the boundedness of the Laplace transform $\mathcal{L}$. For example, for $f \in L_1([0,\infty))$, it holds that

$$|\mathcal{L}f(t)| \le \int_0^\infty |f(s)||e^{-st}|\, ds \le \int_0^\infty |f(s)|\, ds = \|f\|_{L_1([0,\infty))} < \infty.$$

This means that $\mathcal{L}(f)$ exists and it is bounded for all $t \ge 0$. By taking the supremum over $t \in [0,\infty)$, we obtain

$$\|\mathcal{L}(f)\|_{L_\infty([0,\infty))} \le \|f\|_{L_1([0,\infty))},$$

which means that

$$\mathcal{L} : L_1([0,\infty)) \longrightarrow L_\infty([0,\infty)),$$

is a bounded operator.

For our next result we will use the so called *Minkowski integral inequality*, stated below. Details and proof of this inequality may be found in [4].

**Theorem 2.1** (Minkowski integral inequality)**.** *Let $(X,\mathscr{A},\mu)$ and $(Y,\mathscr{B},\nu)$ be $\sigma$-finite measure spaces. Suppose that $f$ is $\mathscr{A}\times\mathscr{B}$-measurable function and $f(\cdot,y)\in L_p(\mu)$ for all $y\in Y$. Then for $1\le p\le\infty$ we have*

$$\left(\int\limits_X \left|\int\limits_Y f(x,y)\,d\nu\right|^p d\mu\right)^{\frac{1}{p}} \le \int\limits_Y \left(\int\limits_X |f(x,y)|^p\,d\mu\right)^{\frac{1}{p}} d\nu. \tag{2.1}$$

The next result is an exercise in the 1958 book of Dunford and Schwartz [5]. It states that

$$\mathcal{L} : L_2([0,\infty)) \to L_2([0,\infty)),$$

is a bounded operator. For the sake of completeness, we provide its proof.

**Theorem 2.2.** *Let $f\in L_2([0,\infty))$. Then*

$$\|\mathcal{L}f\|_{L_2([0,\infty))} \le \sqrt{\pi}\|f\|_{L_2([0,\infty))}.$$

*Proof.* Let $f\in L_2([0,\infty))$ and

$$\mathcal{L}f(t) = \int_0^\infty f(s)e^{-st}\,ds. \tag{2.2}$$

Now, making the change of variables $u=st$, (2.2) becomes

$$\mathcal{L}f(t) = \int_0^\infty e^{-u} f\left(\frac{u}{t}\right)\frac{dt}{t}.$$

By means of the Minkowski integral inequality (Theorem 2.1), one has

$$\begin{aligned}
\|\mathcal{L}\|_{L_2([0,\infty))} &= \left(\int_0^\infty |\mathcal{L}f(t)|^2\,dt\right)^{\frac{1}{2}} = \left(\int_0^\infty \left|\int_0^\infty e^{-u} f\left(\frac{u}{t}\right)\frac{du}{t}\right|^2 dt\right)^{\frac{1}{2}} \\
&\le \int_0^\infty \left(\int_0^\infty \left|e^{-u} f\left(\frac{u}{t}\right)t^{-1}\right|^2 dt\right)^{\frac{1}{2}} du = \int_0^\infty e^{-u}\left(\int_0^\infty \left|f\left(\frac{u}{t}\right)\right|^2 \frac{dt}{t^2}\right)^{\frac{1}{2}} du \\
&= \int_0^\infty u^{-\frac{1}{2}}e^{-u}\left(\int_0^\infty |f(\omega)|^2\,d\omega\right)^{\frac{1}{2}} du = \left(\int_0^\infty u^{\frac{1}{2}-1}e^{-u}\,du\right)\|f\|_{L_2([0,\infty))} \\
&= \Gamma\left(\frac{1}{2}\right)\|f\|_{L_2([0,\infty))}.
\end{aligned}$$

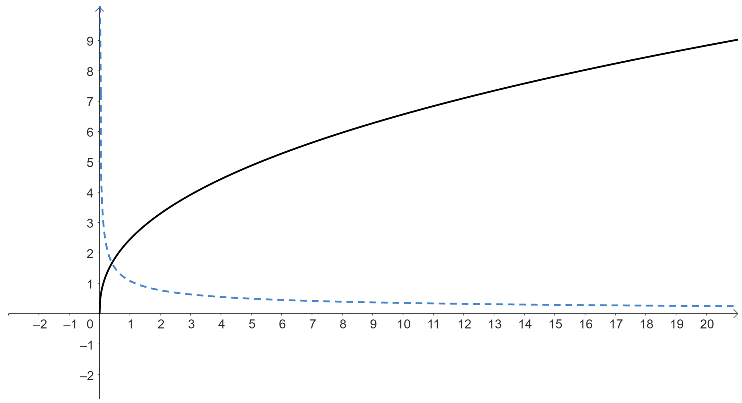


Figure 1: The graph of $F(a) = \frac{\|\mathcal{L}(f_a)\|_{L_p([0,\infty))}}{\|f_a\|_{L_p([0,\infty))}}$ for $p = 1.5$ (solid) and $p = 5$ (dashed).

It is a well known fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, so we finally arrive to

$$\|\mathcal{L}f\|_{L_2([0,\infty))} \leq \sqrt{\pi}\|f\|_{L_2([0,\infty))}. \qquad \square$$

**Remark 2.3.** *A routine calculation shows that, for $p > 1$, if $f_a(t) = e^{-at}$ where $a > 0$, we have*

$$\|f_a\|_{L_p([0,\infty))} = \left(\frac{1}{ap}\right)^{1/p}, \quad \|\mathcal{L}(f_a)\|_{L_p([0,\infty))} = \left(\frac{a^{1-p}}{p-1}\right)^{1/p}.$$

*Hence*

$$\frac{\|\mathcal{L}(f_a)\|_{L_p([0,\infty))}}{\|f_a\|_{L_p([0,\infty))}} = \left(\frac{p}{p-1}\right)^{1/p} a^{\frac{2}{p}-1} \to \infty,$$

*as $a \to \infty$ for $1 < p < 2$, and as $a \to 0^+$ for $p > 2$ (see e.g. Figure 1 below). This shows that*

$$\mathcal{L} : L_p([0,\infty)) \to L_p([0,\infty))$$

*is not a bounded operator for $p \neq 2$.*[1]

Our next result states that

$$\mathcal{L} : L_p([0,\infty)) \to L_p([1,\infty)),$$

is a bounded operator for $p > 2$.

[1]All plots in the present article were made using the software DESMOS.

**Theorem 2.4.** *Let $f \in L_p([0,\infty))$ with $2 < p < \infty$, then*

$$\|\mathcal{L}(f)\|_{L_p([1,\infty))} \leq C_p \|f\|_{L_p([0,\infty))}.$$

*Proof.* Let $f \in L_p$ with $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder's inequality one has

$$\begin{aligned}
\|\mathcal{L}(f)\|^p_{L_p([1,\infty))} &= \int_1^\infty |\mathcal{L}f(x)|^p \, dx = \int_1^\infty \left( \int_0^\infty e^{-xy} f(y)\, dy \right)^p dx \\
&\leq \int_1^\infty \left( \int_0^\infty |f(y)|^p \, dy \right) \left( \int_0^\infty e^{-qxy}\, dy \right)^{p/q} dy = \int_1^\infty \left( - \frac{e^{-qxy}}{qx} \Big|_0^\infty \right)^{p/q} \|f\|^p_{L_p([0,\infty))} \, dx \\
&= \left(\frac{1}{q}\right)^{p/q} \left( \int_1^\infty x^{-p/q}\, dx \right) \|f\|^p_{L_p([0,\infty))} = \left(\frac{1}{q}\right)^{p/q} \frac{1}{(2-p)x^{p-2}} \Big|_1^\infty \|f\|^p_{L_p([0,\infty))} \\
&= \left(\frac{1}{q}\right)^{p/q} \frac{1}{p-2} \|f\|^p_{L_p([0,\infty))}.
\end{aligned}$$

Finally,

$$\|\mathcal{L}(f)\|_{L_p(1,\infty)} \leq \left(\frac{1}{q}\right)^{1/q} \left(\frac{1}{p-2}\right)^{1/p} \|f\|_{L_p(0,\infty)},$$

hence

$$\|\mathcal{L}(f)\|_{L_p(1,\infty)} \leq \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \left(\frac{1}{p-2}\right)^{1/p} \|f\|_{L_p(0,\infty)}.$$

□

**Remark 2.5.** *Theorem 2.4 does not hold for $1 < p < 2$. Let us check this. As in the previous remark, for $f_a(t) = e^{-at}$ with $a > 0$, we have $\|f_a\|_{L_p([0,\infty))} = \left(\frac{1}{ap}\right)^{1/p}$, and also*

$$\|\mathcal{L}(f_a)\|_{L_p(1,\infty)} = \left(\frac{1}{p-1}\right)^{1/p} \left((1+a)^{1-p}\right)^{1/p} = \left(\frac{1}{p-1}\right)^{1/p} (1+a)^{1/p-1}.$$

*Hence*

$$\frac{\|\mathcal{L}(f_a)\|_{L_p(1,\infty)}}{\|f_a\|_{L_p(0,\infty)}} = \frac{\left(\frac{1}{p-1}\right)^{1/p} (1+a)^{1/p-1}}{\left(\frac{1}{ap}\right)^{1/p}} = \left(\frac{p}{p-1}\right)^{1/p} \cdot \frac{(a+a^2)^{1/p}}{1+a} \to \infty$$

*as $a \to \infty$ and $1 < p < 2$ (see, for example, Figure 2 below). So,*

$$\mathcal{L} : L_p([0,\infty)) \to L_p([1,\infty)),$$

*is not a bounded operator for $1 < p < 2$.*

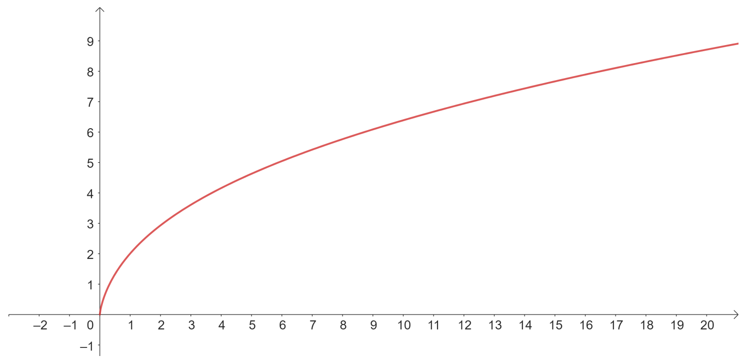


Figure 2: The graph of $G(a) = \frac{\|\mathcal{L}(f_a)\|_{L_p(1,\infty)}}{\|f_a\|_{L_p(0,\infty)}}$ for $p = 1.4$.

In our last result, we will show that

$$\mathcal{L} : L_p([0,\infty)) \to L_p([0,1]),$$

is a bounded operator for $1 < p < 2$.

**Theorem 2.6.** *Let $f \in L_p([0,\infty))$ with $1 < p < 2$. Then*

$$\|\mathcal{L}(f)\|_{L_p([0,1])} \leq C_p \|f\|_{L_p([0,\infty))}.$$

*Proof.* Let $q$ denote the conjugate exponent of $p$, *i.e.* $1/p + 1/q = 1$. Assuming $1 < p < 2$, then $q > 2$ and also $1 - p/q > 0$. Now,

$$\begin{aligned}
\|\mathcal{L}f\|^p_{L_p([0,1])} &= \int_0^1 |\mathcal{L}f(t)|^p \, dt = \int_0^1 \left( \int_0^\infty e^{-st} f(s) \, ds \right)^p dt \\
&\leq \int_0^1 \left( \int_0^\infty |f(s)|^p \, ds \right) \left( \int_0^\infty e^{-sqt} \, ds \right)^{p/q} dt = \int_0^1 \left( - \left. \frac{e^{-sqt}}{qt} \right|_0^\infty \right)^{p/q} dt \cdot \|f\|^p_{L_p([0,\infty))} \\
&= \int_0^1 \left( \frac{1}{qt} \right)^{p/q} dt \cdot \|f\|^p_{L_p([0,\infty))} = \left( \frac{1}{q} \right)^{p/q} \int_0^1 t^{-p/q} \, dt \cdot \|f\|^p_{L_p([0,\infty))} \\
&= \left( \frac{1}{q} \right)^{p/q} \frac{1}{1 - p/q} \cdot \|f\|^p_{L_p([0,\infty))} = \left( \frac{p-1}{p} \right)^{p-1} \frac{1}{2-p} \cdot \|f\|^p_{L_p([0,\infty))},
\end{aligned}$$

where we used Hölder's inequality in the third line. Finally, we conclude that

$$\|\mathcal{L}f\|_{L_p([0,1])} \leq C_p \|f\|_{L_p([0,\infty))},$$

where $C_p = \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \left(\frac{1}{2-p}\right)^{\frac{1}{p}}$. □

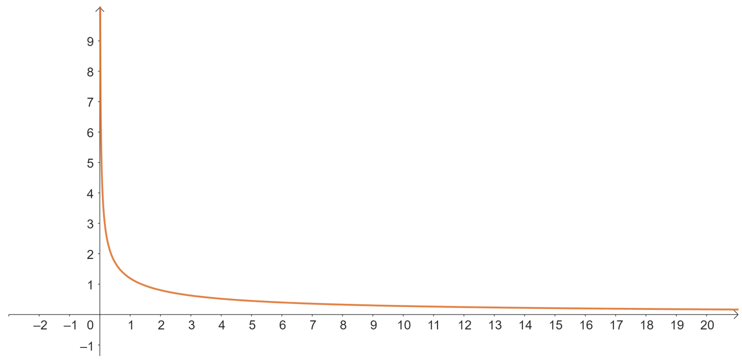


Figure 3: The graph of $H(a) = \frac{\|\mathcal{L}(f_a)\|_{L_p[0,1]}}{\|f_a\|_{L_p(0,\infty)}}$ for $p = 3.9$.

**Remark 2.7.** *Theorem 2.6 does not hold for $p > 2$. Again, for $f_a(t) = e^{-at}$ with $a > 0$, we have $\|f_a\|_{L_p([0,\infty))} = \left(\frac{1}{ap}\right)^{1/p}$, and also*

$$\|\mathcal{L}(f_a)\|_{L_p([0,1])} = \left(\frac{a^{1-p} - (1+a)^{1-p}}{p-1}\right)^{1/p}.$$

*Hence*

$$\frac{\|\mathcal{L}(f_a)\|_{L_p([0,1])}}{\|f_a\|_{L_p([0,\infty))}} = \left(\frac{p}{p-1}\right)^{1/p} \left(a^{2-p} - a(1+a)^{1-p}\right)^{1/p} \to \infty,$$

*as $a \to 0^+$ and $p > 2$ (see e.g. Figure 3). So,*

$$\mathcal{L} : L_p([0,\infty)) \to L_p([0,1]),$$

*is not a bounded operator for $p > 2$.*


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