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Characterizations of kites as graceful graphs

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ABSTRACT

We introduce and study an infinite family of graceful graphs, which we call kites. The kites are graphs where a path is joined with a graph "forming" a kite. We study and characterize three classes of the kites: kites formed by cycles known to be graceful, fan kites and lantern kites. Beside showing in a transparent way that all these graphs are graceful, we provide characterizations of these graphs among all simple graphs via three tools: via Sheppard's labelling sequences introduced in the 1970s and via labelling relations and graph chessboards. The latter are relatively new tools for the study of graceful graphs introduced by Haviar and Ivaška in 2015. The labelling relations are closely related to Sheppard's labelling sequences while the graph chessboards provide a nice visualization of the graceful labellings.

RESUMEN

Introducimos y estudiamos una familia infinita de grafos agraciados que llamamos cometas. Las cometas son grafos en los cuales un camino está unido con un grafo "formando" una cometa. Estudiamos y caracterizamos tres clases de cometas: cometas formadas por ciclos conocidas por ser agraciadas, cometas abanicos y cometas linternas. Además de mostrar de manera transparente que todos estos grafos son agraciados, entregamos caracterizaciones de estos grafos entre todos los grafos simples a través de tres herramientas: a través de sucesiones de etiquetados de Sheppard introducidos en los 1970s y vía relaciones de etiquetados y tableros de ajedrez de grafos. Los últimos son herramientas relativamente nuevas en el estudio de grafos agraciados introducidos por Haviar e Ivaška en 2015. Las relaciones de etiquetados están estrechamente relacionadas con las sucesiones de etiquetados de Sheppard mientras que los tableros de ajedrez de grafos entregan una visualización agradable para los etiquetados agraciados.

Keywords and Phrases: Graph, graceful labelling, graph chessboard, labelling sequence, labelling relation.

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1 Introduction

The Graceful Tree Conjecture stated by Rosa in the mid 1960s says that every tree can be gracefully labelled. The conjecture is one of the most attractive open problems in Graph Theory. It has led to a great interest in the study of gracefulness of simple graphs. Yet not much is known about the structure of graceful graphs after almost sixty years.

A *graceful labelling* of a graph of size m is a vertex labelling by numbers from the set $\{0, 1, \ldots, m\}$ such that no two vertices share the same label, each edge is assigned the label, which is the absolute value of the difference of the vertex labels, and the edge labels cover all values of the set $\{1, 2, \ldots, m\}$. If a graph is gracefully labelled, we say it is a *graceful graph*.

The Graceful Tree Conjecture was stated by Rosa in [7] and [8]. The best source of information on attacks of the conjecture and on the study of labellings of graphs is the electronic book *A Dynamic Survey of Graph Labeling* by Gallian [1].

In this paper we introduce and study an infinite family of graceful graphs, which we call kites. The kites $K_n(G)$ are graphs where a path P_n is joined with a graph G "forming" a kite. In our work the graph G can be a cycle C_m known to be graceful (*i.e.* $m = 0 \pmod 4$ or $m = 3 \pmod 4$), a fan graph F_m or a lantern L_m. These kites $K_n(G)$ have been studied in the second author's M.Sc. thesis [5].

Characterizations of the kites are presented using the tools of labelling sequences, labelling relations and graph chessboards. Labelling sequences were introduced in 1976 by Sheppard [9]. The labelling relations and graph chessboards as new tools for the study of graceful graphs were introduced and applied in 2015 by Haviar and Ivaška [3]. We also refer to recent papers [6] and [2], in which another classes of graceful graphs were studied by these tools.

The basic terms and facts needed in this paper are presented in Section 2. This includes the concepts of graph chessboards, labelling sequences and labelling relations. In Section 3 we describe graceful labellings of the kites formed by graceful cycles and we present their characterizations by the mentioned concepts. In Section 4 we introduce and similarly characterize other two classes of the kites: fan kites and lantern kites.

2 Preliminaries

In this section we recall necessary basic terms concerning the graph labellings as well as the concepts of labelling sequences, labelling relations and simple chessboards. These definitions are taken primarily from [8] and [3].

Throughout this paper we consider only finite *simple graphs*, that is, finite unoriented graphs without loops and multiple edges. The following concept was called *valuation* by Rosa in his seminal paper [8].

Definition 2.1 ([3,8])**.** *A vertex labelling f of a simple graph $G = (V, E)$ is a one-to-one mapping of its vertex set V into the set of non-negative integers assigning so-called vertex labels to the vertices of G.*

In this paper by a *labelling* we mean a *vertex labelling*. The number $|f(u)-f(v)|$, where $f(u), f(v)$ are the labels of the vertices u, v respectively, will be called the *induced label of the edge uv* in the labelling f.

Definition 2.2 ([3, Definition 1.2.3])**.** *Let $G = (V, E)$ be a graph of size m and let $f : V \to \mathbb{N}$ be its labelling. Then f is called a graceful labelling if*

(1) $f(V) \subseteq \{0, 1, \dots, m\}$, *and*

(2) $f(E) = \{1, 2, \dots, m\}$.

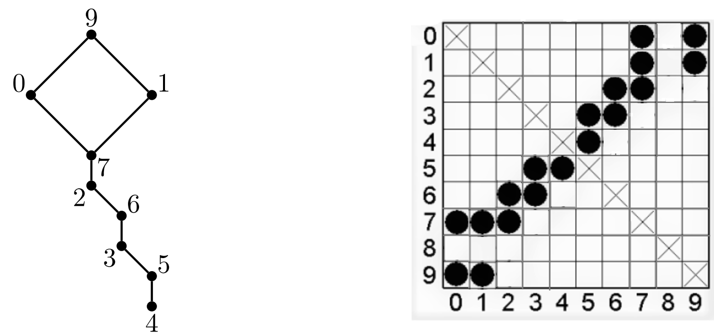
A *simple chessboard* is a square table with n rows and n columns, and dots which represent the edges of a graph are placed in the cells of the table. Every edge uv corresponds to the dots with coordinates $[u, v]$ and $[v, u]$ (the dot with coordinates $[i, j]$ means the dot in the i-th row and the j-th column of the table, where $i, j \in \{1, \dots, n\}$). Let the r-th diagonal be the set of all cells with the coordinates $[i, j]$ where $i - j = r$ and $i \geq j$. The 0-th diagonal (also called the *main* diagonal) has no dots, because we consider simple graphs, the other diagonals are called *associate*. Simple chessboard is called *graceful* if there is exactly one dot on each of its associate diagonals.

The simple chessboard is a useful visualization of a graph because via it one can easily see some properties of the graph such as its size, degrees of vertices, gracefulness, etc. In Figure 1 we see the simple chessboard of a graph of size 9 (the kite $K_6(C_4)$ formed by the cycle C_4).

To represent gracefully labelled graphs, we will use other two tools: *labelling sequences* and *labelling relations*. A concept of the labelling sequence was introduced by Sheppard in [9]. He proved that there is a unique correspondence between gracefully labelled graphs and labelling sequences. Later in [3] Haviar and Ivaška proved a correspondence between labelling sequences and labelling relations. Let us now define these concepts.

Definition 2.3 ([3,9])**.** *For a positive integer m, the sequence $(j_1, j_2, \dots, j_m)$ of integers, denoted (j_i), is a labelling sequence if $0 \leq j_i \leq m - i$ for all $i \in \{1, 2, \dots, m\}$.*

The correspondence between gracefully labelled graphs and labelling sequences is described in the following theorem.



1	2	3	4	5	6	7	8	9
4	3	3	2	2	1	0	1	0
5	5	6	6	7	7	7	9	9

Figure 1: Representations of the kite $K_6(C_4)$

Theorem 2.4 ([3, 9])**.** *There exists a one-to-one correspondence between graphs of size m having a graceful labelling f and between labelling sequences (j_i) of m terms. The correspondence is given by $j_i = \min\{f(u), f(v)\}$, $i \in \{1, 2, \ldots, m\}$, where u, v are the end-vertices of the edge labelled i.*

Definition 2.5 ([3, Definition 3.5.1])**.** *Let $L = (j_1, j_2, \ldots, j_m)$ be a labelling sequence. Then the relation $A(L) = \{[j_i, j_i + i] \mid i \in \{1, 2, \ldots, m\}\}$ will be called the labelling relation assigned to the labelling sequence L.*

In [3] also a *labelling table* was assigned to a graceful graph of size m, which displays its labelling sequence and the labelling relation together. The labelling table consists of a header and two rows. The header just lists the numbers $1, 2, \ldots, m$. The first row of the labelling table consists of the labelling sequence j_i as defined in Definition 2.3. The numbers in the second row are sums of the numbers from the header and the numbers of the first row (the members of the labelling sequence). The pairs from the first and second rows in each column are the elements of the labelling relation.

In Figure 1 we see the labelling table of the kite $K_6(C_4)$. In the first row of the table we see the labelling sequence $(4, 3, 3, 2, 2, 1, 0, 1, 0)$ of this graceful graph. The pairs from the first and second rows in each column of the labelling table form the labelling relation representing the edges of this graph. For example the pair $[4, 5]$ represents the last edge of the path.

3 Characterizations of kites formed by graceful cycles

It is well-known (see [8], [3] or [1]) that a cycle C_m is graceful if and only if $m = 0 \pmod 4$ or $m = 3 \pmod 4$. Therefore our first two studied classes of kites are those formed by graceful cycles C_m.

3.1 Kites formed by cycles C_m for $m = 0 \pmod 4$

In this subsection we present our characterization of the kites $K_n(C_m)$ formed by cycles C_m for $m = 0 \pmod 4$, where $m \geq 4$ and $n \geq 1$. The special case are the quadrangular kites.

By a *quadrangular kite* we mean a graph obtained by joining the cycle C_4 to the end-point of the path P_n with $n \geq 1$. We denote it $K_n(C_4)$. The size of this graph is $s = n + 3$, where $n - 1$ is the length of the path P_n.

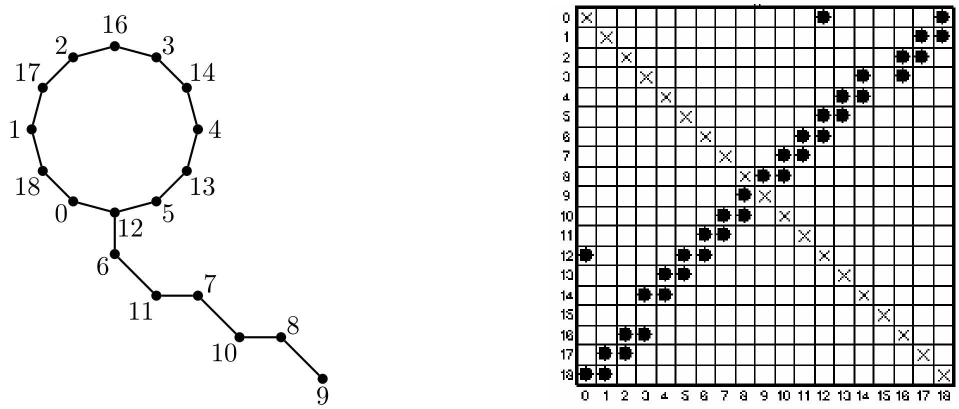
Example 3.1. *We again consider the quadrangular kite $K_6(C_4)$ presented in Figure 1. Its labelling sequence (LS, for short) $(4, 3, 3, 2, 2, 1, 0, 1, 0)$ consists of two groups: the first is $(4, 3, 3, 2, 2)$, which is the LS of the path P_6 and the second is $(1, 0, 1, 0)$, which is the LS of the cycle C_4. These two groups are clearly seen in the graph chessboard as the dots forming the "stairway" representing the path pattern and the dots forming the "square block" representing the cycle C_4.*

Definition 3.2. *Let $s = n+3$ for some $n \geq 1$. By a QK-graph chessboard (QK standing for "quadrangular kite") of size s we mean a simple chessboard such as in Figure 1 described in the previous example. Its dots start in the lower left corner with two dots in the column 0 and two dots in the column 1, which together create the "square block". The remaining dots form a "stairway" attached to the square block (the "stairway" starts with the dot with coordinates $[s - 2, 2]$).*

The characterization of the quadrangular kites via their chessboards, labelling sequences and labelling relations is a special case of Theorem 3.4, which will be presented with a full proof. It follows from it that a graph G of size $s = n + 3$ for some $n \geq 1$ is the quadrangular kite $K_n(C_4)$ if and only if G has a graceful labelling and a QK-graph chessboard of size s.

Consider now the kite $K_7(C_{12})$ and its representations in Figure 2. In the first row of the labelling table is the LS $(8, 8, 7, 7, 6, 6, 5, 5, 4, 4, 3, 0, 3, 2, 2, 1, 1, 0)$.

Definition 3.3. *Let $s = m + n - 1$ for $m = 0 \pmod 4$ and $m \geq 4$, $n \geq 1$. By a C0K-graph chessboard (C0K standing for "kite formed by cycle C_m for $m = 0 \pmod 4$") of size s we mean a simple chessboard such as in Figure 2 whose dots can be divided into three groups: the first group of dots form a "stairway" starting in the lower left corner (with the dot with coordinates $[s, 0]$), the second group consists of a single dot with coordinates $[\frac{m}{2} + (n - 1), 0]$ and the third group is again a "stairway". (It starts with the dot with coordinates $[\lfloor \frac{s-i}{2} \rfloor + i, \lfloor \frac{s-i}{2} \rfloor]$ for $i = \frac{m}{2} + (n - 2)$.)*



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
8	8	7	7	6	6	5	5	4	4	3	0	3	2	2	1	1	0
9	10	10	11	11	12	12	13	13	14	14	12	16	16	17	17	18	18

Figure 2: Representations of the kite $K_7(C_{12})$

The following theorem characterizes the kites $K_n(C_m)$ for $m = 0 \pmod 4$ via their graph chessboards, labelling sequences and labelling relations.

Theorem 3.4. *Let G be a graph of size $s = m + n - 1$ for $m = 0 \pmod 4$ and $m \geq 4$, $n \geq 1$. Then the following are equivalent:*

(1) G is the kite $K_n(C_m)$.

(2) G has a graceful labelling and a C0K-graph chessboard of size s.

(3) There exists a labelling sequence $L = (j_1, j_2, \dots, j_s)$ of G such that

$$j_i = \begin{cases} \left\lfloor \frac{s-i}{2} \right\rfloor, & \text{if } i < \frac{m}{2} + (n-1); \\ 0, & \text{if } i = \frac{m}{2} + (n-1); \\ \left\lfloor \frac{s-i+1}{2} \right\rfloor, & \text{if } i > \frac{m}{2} + (n-1). \end{cases} \tag{LSC0K}$$

(4) There exists a labelling sequence L of G with the labelling relation

$$A(L) = \left\{ \left[\left\lfloor \frac{s-i}{2} \right\rfloor, \left\lfloor \frac{s-i}{2} \right\rfloor + i \right] \middle| i < \frac{m}{2} + (n-1) \right\} \cup \left\{ \left[0, \frac{m}{2} + (n-1) \right] \right\} \cup \left\{ \left[\left\lfloor \frac{s-i+1}{2} \right\rfloor, \left\lfloor \frac{s-i+1}{2} \right\rfloor + i \right] \middle| i > \frac{m}{2} + (n-1) \right\}.$$

Proof. (1) ⇒ (2)**:** Let G be the kite $K_n(C_m)$ for $m = 0 \pmod 4$. Let us label its vertices as follows: we label the vertex joining the cycle C_m with the path P_n (let us call it the "joining vertex") by number $s - \frac{m}{2}$, and we label every second vertex from the joining vertex in the clockwise direction by numbers $s, s-1, s-2, \ldots,$ but we skip the number $\frac{3}{4}m+n-1$. The remaining vertices of the cycle C_m will be labelled in the clockwise direction from the joining vertex by numbers $0, 1, 2, \ldots, \frac{m}{2}-1$. Next we label the path P_n. We start from the joining vertex labelled by $s-\frac{m}{2}$ and we label every second vertex from it by numbers $s-(\frac{m}{2}+1), s-(\frac{m}{2}+2), \ldots$ The remaining vertices of the path P_n will be labelled by numbers $\frac{m}{2}, \frac{m}{2}+1, \ldots, \lfloor \frac{s}{2} \rfloor$. The labelling is illustrated in Figure 3.

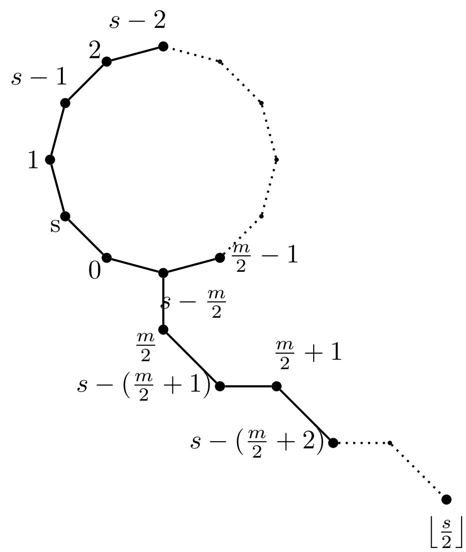


Figure 3: Vertex labelling of the kite $K_n(C_m)$ for $m = 0 \pmod 4$

Gracefulness of the described vertex labelling of G will be shown by using visualization via the corresponding chessboard of G. The dots in the $(\frac{m}{2}) \times (\frac{m}{2}+1)$ left lower rectangle of the chessboard represent the cycle C_m of the kite. (Specifically, the columns 0 to $\frac{m}{2}-1$ and the rows s to $s-\frac{m}{2}$.) The remaining dots represent the path P_n and form a "stairway". So we obtain a C0K graph chessboard. This yields that the labelling is graceful because each diagonal of the chessboard has exactly one dot.

(2) ⇒ (3)**:** Let G be a gracefully labelled graph with a C0K graph chessboard. We show that the corresponding labelling sequence (LS) satisfies (LSC0K). The "stairway" in the direction from the main diagonal corresponds in the LS to numbers $\lfloor \frac{s-i}{2} \rfloor$ for $i < \frac{m}{2}+(n-1)$. The "single" dot represents in the LS the number 0. The remaining "stairway" corresponds in the LS to numbers $\lfloor \frac{s-i+1}{2} \rfloor$ for $i > \frac{m}{2}+(n-1)$. So the formula (LSC0K) holds.

(3) ⇒ (4)**:** Let (LSC0K) hold. We show that the LS L has the labelling relation $A(L)$ as described in (4). The numbers $\lfloor \frac{s-i}{2} \rfloor$ in the LS correspond in $A(L)$ to the pairs $\left[\lfloor \frac{s-i}{2} \rfloor, \lfloor \frac{s-i}{2} \rfloor + i\right]$ for $i < \frac{m}{2}+(n-1)$. The number 0 clearly corresponds to the pair $[0, \frac{m}{2}+(n-1)]$. The numbers $\lfloor \frac{s-i+1}{2} \rfloor$ in the LS correspond in $A(L)$ to the pairs $\left[\lfloor \frac{s-i+1}{2} \rfloor, \lfloor \frac{s-i+1}{2} \rfloor + i\right]$ for $i > \frac{m}{2}+(n-1)$.

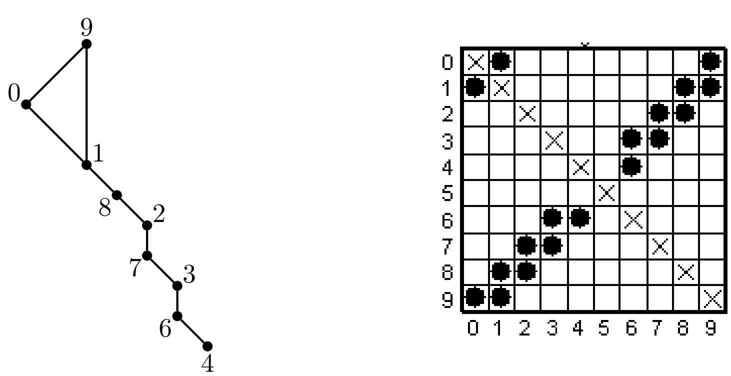
The first coordinates of these pairs are members of the LS and the second coordinates are obtained by adding numbers i to the members of the LS. So the formula $A(L)$ from (4) holds.

(4) $\Rightarrow$ (1)**:** Let (4) hold, *i.e.* there exists a LS L of the graph G with the labelling relation $A(L)$ as in (4). We show that the pairs in $A(L)$ represent the edges of the kite $K_n(C_m)$. The pairs $\left[\left\lfloor\frac{s-i}{2}\right\rfloor, \left\lfloor\frac{s-i}{2}\right\rfloor + i\right]$ for $i < n$ clearly represent the path P_n. The pairs $\left[\left\lfloor\frac{s-i}{2}\right\rfloor, \left\lfloor\frac{s-i}{2}\right\rfloor + i\right]$ for $i = n, n+1, \ldots, \frac{m}{2}+n-2$ represent a part of the cycle C_m starting from the joining vertex and going in the anticlockwise direction. The pair $[0, \frac{m}{2}+(n-1)]$ represents the edge of the cycle C_m with the joining vertex $\frac{m}{2}+(n-1)$ and the vertex 0. The pairs $\left[\left\lfloor\frac{s-i+1}{2}\right\rfloor, \left\lfloor\frac{s-i+1}{2}\right\rfloor + i\right]$ for $i > \frac{m}{2}+(n-1)$ represent the remaining edges of the cycle C_m. Hence, G is the kite $K_n(C_m)$. □

3.2 Kites formed by cycles C_m for $m = 3 \pmod 4$

By the *triangular kite* $K_n(C_3)$ we mean a graph obtained by joining the cycle C_3 to end-point of the path P_n with $n \geq 1$. The size of the triangular kite $K_n(C_3)$ is $s = n+2$.

In this subsection we present our characterization of the kites $K_n(C_m)$ formed by cycle C_m for $m = 3 \pmod 4$ and n sufficiently big, more precisely $n \geq \left\lfloor\frac{m}{2}\right\rfloor$. This will cover all the triangular kites $K_n(C_3)$. For general $m \geq 3$ with $m = 3 \pmod 4$ and $n \geq \left\lfloor\frac{m}{2}\right\rfloor$ we distinguish two subclasses of the kites $K_n(C_m)$ according to the order of their path P_n: n is even and n is odd. Both cases are similar, but they differ in details.



1	2	3	4	5	6	7	8	9
0	4	3	3	2	2	1	1	0
1	6	6	7	7	8	8	9	9

Figure 4: Representations of the triangular kite $K_7(C_3)$

Example 3.5. *In Figure 4 we see the triangular kite $K_7(C_3)$ obtained by joining the cycle C_3 to the path P_7. Its graceful labelling is depicted in the graph diagram. We also see the corresponding graph chessboard and the labelling relation. The labelling sequence (LS) is $(0,4,3,3,2,2,1,1,0)$.*

We can easily recognize the LS of the path P_7, which is $(4,3,3,2,2,1)$, and the LS of the cycle C_3, which is $(0,1,0)$.

Definition 3.6. *Let $s = n+2$ for some $n \geq 1$. By a TK-graph chessboard (TK standing for the "triangular kite") we mean a simple chessboard such as in Figure 4. It has the "single" dot with coordinates $[1,0]$ and the remaining dots form a "stairway" in the chessboard starting in the lower left corner (with the dot with coordinates $[s,0]$).*

The characterization of the triangular kites $K_n(C_3)$ by the simple chessboards, the labelling sequences and the labelling relations is a special case of the coming Theorem 3.9 (for even n) and Theorem 3.12 (for odd n), where $n \geq \lfloor \frac{m}{2} \rfloor$. It follows from these theorems that a graph G of size $s = n+2$ for some $n \geq 1$ is the triangular kite $K_n(C_3)$ if and only if G has a graceful labelling and a TK-graph chessboard of size s.

Now we are going to describe the kites $K_n(C_m)$ formed by cycle C_m for $m = 3 \pmod 4$ and general parameter $m \geq 3$ where we assume that n is sufficiently big, more precisely $n \geq \lfloor \frac{m}{2} \rfloor$. We will start with the subcase where n is even.

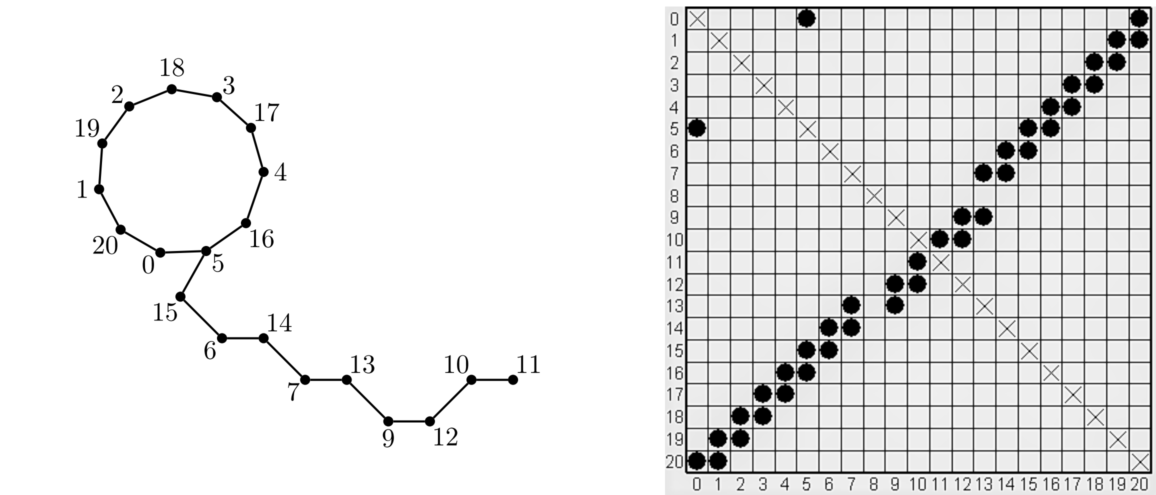
Example 3.7. *In Figure 5 we see a gracefully labelled graph diagram of the kite $K_{10}(C_{11})$ and its corresponding simple chessboard.*

The chessboard can be divided into three parts: the first part is a "stairway" (from column 0 to column 7), the second part is the "single dot" with coordinates $[5,0]$ and the third part is a "stairway" starting with two vertical *dots.*

The labelling sequence (LS) is $(10,10,9,9,0,7,7,6,6,5,5,4,4,3,3,2,2,1,1,0)$. We can divide it into four parts. The first part $(10,10,9,9)$ is the LS of one part of the path, then $(7,7,6,6,5)$ represents the remaining part of the path. The third part is the number 0 in the place $\lfloor \frac{m}{2} \rfloor = 5$ and the last part $(5,4,4,3,3,2,2,1,1,0)$ represents the cycle C_{11}.

Definition 3.8. *Let $s = m+n-1$ for $m = 3 \pmod 4$ and $n \geq \lfloor \frac{m}{2} \rfloor$, where $m \geq 3$, $n \geq 1$ and n is even. By an even C3K-graph chessboard of size s we mean the simple chessboard such as in Figure 5, which has three parts: the first part is a "stairway" (from column 0 to column $\lceil \frac{1}{2}(s - \lfloor \frac{m}{2} \rfloor) \rceil - 1$), the second part is the "single" dot with coordinates $[\lfloor \frac{m}{2} \rfloor, 0]$ and the third part is again a "stairway" starting with two vertical dots (starting in the column $\lceil \frac{1}{2}(s - \lfloor \frac{m}{2} \rfloor) \rceil + 1$).*

Now we are ready for the characterization of the kites formed by cycle C_m for $m = 3 \pmod 4$ and $m \geq 3$ with $n \geq \lfloor \frac{m}{2} \rfloor$ in the case of even n.



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
10	10	9	9	0	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0
11	12	12	13	5	13	14	14	15	15	16	16	17	17	18	18	19	19	20	20

Figure 5: Representations of the kite $K_{10}(C_{11})$ with path P_n for even n

Theorem 3.9. *Let G be a graph of size $s = m + n - 1$ for $m = 3 \pmod 4$ and $n \geq \lfloor \frac{m}{2} \rfloor$, where $m \geq 3$, $n \geq 1$ and n is even. Then the following are equivalent:*

(1) G is the kite $K_n(C_m)$.

(2) G has a graceful labelling and an even C3K-graph chessboard of size s.

(3) There exists a labelling sequence $L = (j_1, j_2, \dots, j_s)$ of G such that

$$j_i = \begin{cases} \left\lceil \frac{s-i+1}{2} \right\rceil, & \textit{if } i < \left\lfloor \frac{m}{2} \right\rfloor; \\ 0, & \textit{if } i = \left\lfloor \frac{m}{2} \right\rfloor; \\ \left\lceil \frac{s-i}{2} \right\rceil, & \textit{if } i > \left\lfloor \frac{m}{2} \right\rfloor. \end{cases} \tag{LSC3K-even}$$

(4) There exists a labelling sequence L of G with the labelling relation

$$A(L) = \left\{ \left[\left\lceil \frac{s-i+1}{2} \right\rceil, \left\lceil \frac{s-i+1}{2} \right\rceil + i \right] \;\middle|\; i < \left\lfloor \frac{m}{2} \right\rfloor \right\} \cup \left\{ \left[0, \left\lfloor \frac{m}{2} \right\rfloor \right] \right\} \cup$$
$$\left\{ \left[\left\lceil \frac{s-i}{2} \right\rceil, \left\lceil \frac{s-i}{2} \right\rceil + i \right] \;\middle|\; i > \left\lfloor \frac{m}{2} \right\rfloor \right\}.$$

Proof. (1) ⇒ (2)**:** Let G be the kite $K_n(C_m)$. We label it such that we start labelling the cycle C_m: we label the vertex joining the cycle with the path (the "joining vertex") by number $\lfloor \frac{m}{2} \rfloor$ and we follow in the clockwise direction by numbers $0, s, 1, s-1, \dots, s - \lfloor \frac{m}{2} \rfloor + 1$. Next we label the path P_n. We start from number $\left\lfloor \frac{m}{2} \right\rfloor$ and label every second vertex by numbers $\left\lfloor \frac{m}{2} \right\rfloor + 1, \left\lfloor \frac{m}{2} \right\rfloor + 2, \ldots$, but we skip the number $\left\lceil \frac{1}{2}(s - \left\lfloor \frac{m}{2} \right\rfloor) \right\rceil$. In the remaining part of the path we start with the vertex next to the joining vertex and we label every second vertex by numbers $s - \left\lfloor \frac{m}{2} \right\rfloor, s - (\left\lfloor \frac{m}{2} \right\rfloor + 1), \ldots$ To show that this labelling is graceful, we use the corresponding simple chessboard of G (see Figure 5). The cycle C_m of the kite is in the corresponding chessboard represented by the "stairway" in columns 0 to $\left\lfloor \frac{m}{2} \right\rfloor$ and by the "single" dot with coordinates $\left[\left\lfloor \frac{m}{2} \right\rfloor, 0\right]$. The path P_n is in the chessboard represented by part of the "stairway" starting with the upper dot in column $\left\lfloor \frac{m}{2} \right\rfloor$ and by another "stairway" starting with two vertical dots, but the column $\left\lceil \frac{1}{2}(s - \left\lfloor \frac{m}{2} \right\rfloor) \right\rceil$ is without any dots. So we obtain an even C3K-graph chessboard, which means that our labelling is graceful, because each diagonal of the simple chessboard has exactly one dot.

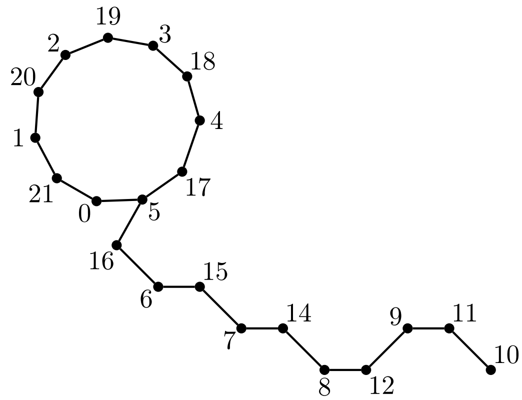
(2) $\Rightarrow$ (3)**:** Assume we have a graceful labelling of the graph G with an even C3K-graph chessboard. We show that the corresponding LS satisfies the formula (LSC3K-even). The "stairway" in the direction from the main diagonal represents in the corresponding LS numbers $\left\lceil \frac{s-i+1}{2} \right\rceil$ for $i < \left\lfloor \frac{m}{2} \right\rfloor$. The "single" dot represents the number 0. The remaining "stairway" represents numbers $\left\lceil \frac{s-i}{2} \right\rceil$ for $i > \left\lfloor \frac{m}{2} \right\rfloor$. So the formula (LSC3K-even) holds.

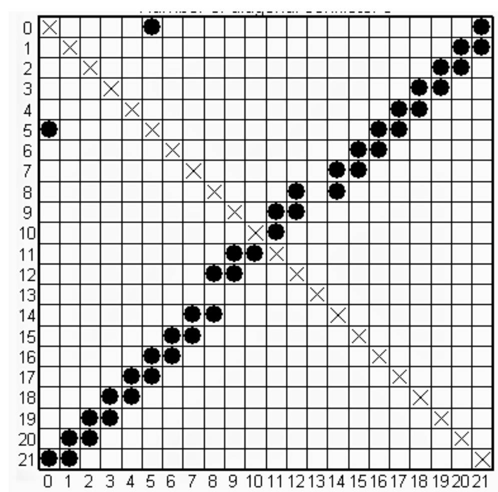
(3) $\Rightarrow$ (4)**:** Assume now that (3) holds, *i.e.* there exists a LS $L = (j_1, j_2, \ldots, j_s)$ that satisfies the formula (LSC3K-even). We show that this LS has the labelling relation $A(L)$ as described in (4). The numbers $\left\lceil \frac{s-i+1}{2} \right\rceil$ obviously correspond in $A(L)$ to the pairs $\left[\left\lceil \frac{s-i+1}{2} \right\rceil, \left\lceil \frac{s-i+1}{2} \right\rceil + i\ \right]$ for $i < \left\lfloor \frac{m}{2} \right\rfloor$. The number 0 clearly corresponds to the pair $[0, \left\lfloor \frac{m}{2} \right\rfloor]$. The numbers $\left\lceil \frac{s-i}{2} \right\rceil$ correspond in $A(L)$ to the pairs $\left[\left\lceil \frac{s-i}{2} \right\rceil, \left\lceil \frac{s-i}{2} \right\rceil + i\ \right]$ for $i > \left\lfloor \frac{m}{2} \right\rfloor$. The first coordinates of these pairs are members of the LS and the second coordinates arise by adding the number i to them.

(4) $\Rightarrow$ (1)**:** Let (4) hold, *i.e.* there exists a LS L with the labelling relation $A(L)$ as described in (4). We show that the pairs in $A(L)$ represent the edges of the kite $K_n(C_m)$. The pairs $\left[\left\lceil \frac{s-i+1}{2} \right\rceil, \left\lceil \frac{s-i+1}{2} \right\rceil + i\ \right]$ for $i < \left\lfloor \frac{m}{2} \right\rfloor$ represent the ending part of a path. The pairs $\left[\left\lceil \frac{s-i}{2} \right\rceil, \left\lceil \frac{s-i}{2} \right\rceil + i\ \right]$ for $i = \left\lfloor \frac{m}{2} \right\rfloor + 1, \ldots, s - m + 1$ represent the remaining edges of the path P_n. The pairs $\left[\left\lceil \frac{s-i}{2} \right\rceil, \left\lceil \frac{s-i}{2} \right\rceil + i\ \right]$ for $i = s - m + 2, \ldots, s$ represent the edges of the cycle C_m starting from the joining vertex in the anticlockwise direction and skipping the last edge of the cycle ending in the joining vertex. This last edge of the cycle is represented by the pair $[0, \left\lfloor \frac{m}{2} \right\rfloor]$ in $A(L)$. Hence G is the kite $K_n(C_m)$. □

The description for the subcase with odd n is similar and it will follow now.

Example 3.10. *In Figure 6 we see the kite $K_{11}(C_{11})$, so it differs from Example 3.7 only by the length of the path, which is now an odd number. We see a gracefully labelled graph diagram of this kite and its corresponding simple chessboard. Again, the simple chessboard can be divided into*





1	2	3	4	5	6	7	8	9	10	11
10	9	9	8	0	8	7	7	6	6	5
11	11	12	12	5	14	14	15	15	16	16

12	13	14	15	16	17	18	19	20	21
5	4	4	3	3	2	2	1	1	0
17	17	18	18	19	19	20	20	21	21

Figure 6: Representations of the kite $K_{11}(C_{11})$

three parts: the first part is a "stairway", the second part is the "single dot" with coordinates $[5, 0]$ *and the third part is again a "stairway" starting now with two* horizontal *dots.*

The labelling sequence (LS) is $(10, 9, 9, 8, 0, 8, 7, 7, 6, 6, 5, 5, 4, 4, 3, 3, 2, 2, 1, 1, 0)$*. We can divide it into four parts: the first part* $(10, 9, 9, 8)$ *is the LS of a part of the path* P_{11}*, then* $(8, 7, 7, 6, 6, 5)$ *is the LS of the remaining part of the path* P_{11}*. The number* 0 *and the last part of the LS, which is* $(5, 4, 4, 3, 3, 2, 2, 1, 1, 0)$*, together represent the cycle* C_{11}*.*

We now define an odd C3K-graph chessboard.

Definition 3.11. *Let* $s = m + n - 1$ *for* $m = 3 \pmod 4$ *and* $n \geq \lfloor \frac{m}{2} \rfloor$*, where* $m \geq 3$*,* $n \geq 1$ *and* n *is odd. By an odd C3K-graph chessboard of size* s *we mean the simple graph chessboard such as in Figure 6, which has three parts: the first part is a "stairway" starting from the left lower corner, the second part is the "single" dot with coordinates* $\left[\lfloor \frac{m}{2} \rfloor, 0\right]$ *and the third part is a 'stairway" starting with two* horizontal *dots.*

The proof of the following characterization of the kites $K_n(C_m)$ formed by cycle C_m for $m = 3 \pmod 4$ in this subcase with odd n, which is sufficiently big, is analogous to the proof of Theorem 3.9 and we leave it for the reader.

Theorem 3.12. *Let G be a graph of size $s = m + n - 1$ for $m = 3 \pmod 4$ and $n \geq \lfloor \frac{m}{2} \rfloor$, where $m \geq 3$, $n \geq 1$ and n is odd. Then the following are equivalent:*

(1) G is the kite $K_n(C_m)$.

(2) G has a graceful labelling and an odd C3K-graph chessboard of size s.

(3) There exists a labelling sequence $L = (j_1, j_2, \dots, j_s)$ of G such that

$$j_i = \begin{cases} \lfloor \frac{s-i}{2} \rfloor, & \text{if } i < \lfloor \frac{m}{2} \rfloor; \\ 0, & \text{if } i = \lfloor \frac{m}{2} \rfloor; \\ \lfloor \frac{s-i+1}{2} \rfloor, & \text{if } i > \lfloor \frac{m}{2} \rfloor. \end{cases} \tag{LSC3K-odd}$$

(4) There exists a labelling sequence L of G with the labelling relation

$$A(L) = \left\{ \left[\left\lfloor \frac{s-i}{2} \right\rfloor, \left\lfloor \frac{s-i}{2} \right\rfloor + i \right] \;\middle|\; i < \left\lfloor \frac{m}{2} \right\rfloor \right\} \cup \left\{ \left[0, \left\lfloor \frac{m}{2} \right\rfloor \right] \right\} \cup$$
$$\left\{ \left[\left\lfloor \frac{s-i+1}{2} \right\rfloor, \left\lfloor \frac{s-i+1}{2} \right\rfloor + i \right] \;\middle|\; i > \left\lfloor \frac{m}{2} \right\rfloor \right\}.$$

4 Characterizations of fan kites and lantern kites

In this section we describe graceful labellings of other two classes of the kites: fan kites and lantern kites. We present their characterizations again by the graph chessboards, labelling sequences and labelling relations.

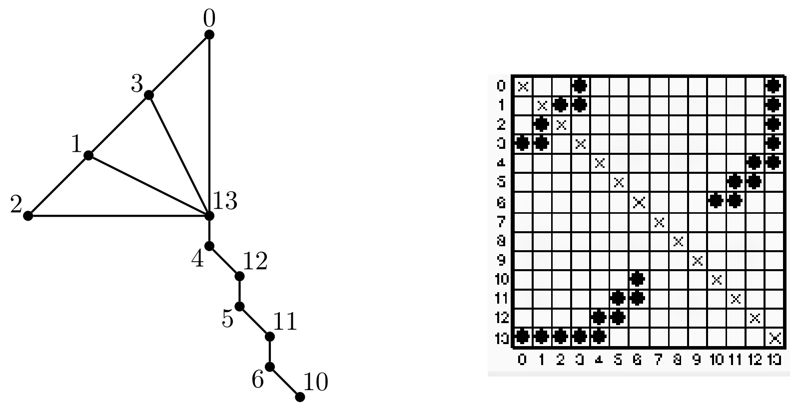
4.1 Fan kites

By a *fan kite* we mean a fan-graph kite, which is a graph obtained by joining the fan-graph F_m (see Definition 4.1) with end-point of the path P_n (see Figure 7). We will denote it by $K_n(F_m)$. The fan kite $K_n(F_m)$ is the graph of size $s = 2m + n - 2$, where $2m - 1$ is the size of the fan graph F_m and $n - 1$ is the length of the path P_n.

Definition 4.1 ([3, Section 4.4.7], [6, Section 4.1])**.** *Let $m \geq 2$. The fan graph F_m is a join of the path P_m and a single vertex K_1.*

Clearly, the fan graph F_m has order $m + 1$ and size $2m - 1$.

Example 4.2. *The fan kite $K_7(F_4)$ is obtained by joining the fan-graph F_4 with the path P_7. In Figure 7 we see its gracefully labelled graph diagram and its corresponding graph chessboard. In the chessboard we can recognize a "fan-graph pattern" in the columns from 0 to 3 and a "path pattern".*



1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0	6	6	5	5	4	4	3	2	1	0
2	3	3	10	11	11	12	12	13	13	13	13	13

Figure 7: Representations of the fan kite $K_7(F_4)$

The labelling sequence (LS) is $(1,1,0,6,6,5,5,4,4,3,2,1,0)$. *It consists of the LS* $(1,1,0,3,2,1,0)$ *of the fan-graph* F_4 *and the LS* $(6,6,5,5,4,4)$ *of the path* P_7. *The pairs from the second and third rows of the table form the labelling relation.*

Definition 4.3. *Let* $s = 2m+n-2$ *for* $m \geq 2$, $n \geq 1$. *By an FK-graph chessboard of size* s *we mean a simple chessboard such as in Figure 7. It starts with* m *dots in the last row and continues with dots creating a "path pattern". In the upper left corner of the simple chessboard there are* $m-1$ *dots forming a "stairway".*

Now we present the result where we show gracefulness of the fan kites. We present their characterization by the graph chessboards, labelling sequences and labelling relations.

Theorem 4.4. *Let* G *be a graph of size* $s = 2m+n-2$ *for* $m \geq 2$, $n \geq 1$. *Then the following are equivalent:*

(1) G *is the fan kite* $K_n(F_m)$.

(2) G *has a graceful labelling and an FK-graph chessboard of size* s.

(3) There exists a labelling sequence $L = (j_1, j_2, \ldots, j_s)$ *of* G *such that*

$$j_i = \begin{cases} \left\lceil \frac{m-1-i}{2} \right\rceil, & \text{if } i \leq m-1; \\ \left\lceil \frac{s-i+m-1}{2} \right\rceil, & \text{if } m \leq i < s-m+1; \\ s-i, & \text{if } i \geq s-m+1. \end{cases} \tag{LSFK}$$

(4) There exists a labelling sequence L of G with the labelling relation

$$A(L) = \left\{ \left[\left\lceil \frac{m-1-i}{2} \right\rceil , \left\lceil \frac{m-1-i}{2} \right\rceil + i \right] \;\middle|\; i \leq m-1 \right\} \cup$$
$$\left\{ \left[\left\lceil \frac{s-i+m-1}{2} \right\rceil , \left\lceil \frac{s-i+m-1}{2} \right\rceil + i \right] \;\middle|\; m \leq i < s-m+1 \right\} \cup$$
$$\{ [\, s-i, s \,] \mid i \geq s-m+1 \}.$$

Proof. (1) $\Rightarrow$ (2)**:** Let G be the fan kite $K_n(F_m)$ of size s. It contains two paths: the path P_m as the path of the fan-graph F_m, and the main path P_n of the kite. We label the vertex connecting the fan-graph F_m with the path P_n (the "joining vertex") by number s. The joining vertex is adjacent to every vertex of the path P_m of the fan-graph F_m. We label P_m by numbers $0, m-1, 1, m-2, 2, \ldots \lceil \frac{m-1}{2} \rceil - 1, \lceil \frac{m-1}{2} \rceil$. Now we label the path P_n: we start from the joining vertex and we continue gradually with numbers $m, s-1, m+1, \ldots$ We show that the labelling of $K_n(F_m)$ is graceful by using its corresponding chessboard. The dots in the left upper corner represent the path P_m of the fan graph F_m, the dots in the last row represent edges connecting the joining vertex with the vertices of the path P_m. The remaining dots represent the path P_n. There is exactly one dot on each diagonal, so the labelling is graceful.

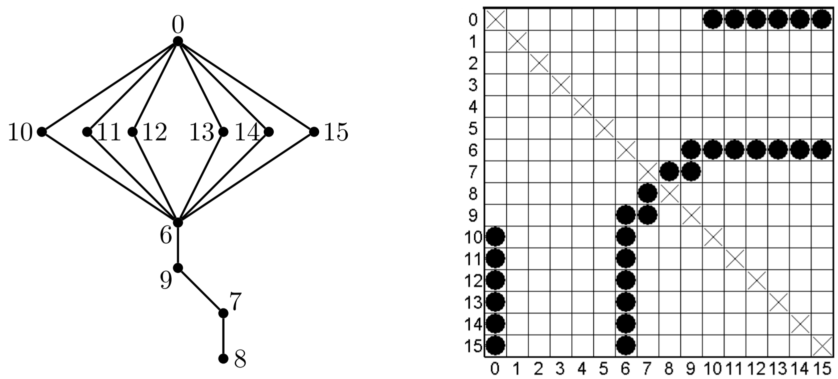
(2) $\Rightarrow$ (3)**:** Let G have a graceful labelling with an FK-graph chessboard. The dots in the left upper corner of the graph chessboard correspond in the labelling sequence (LS) to numbers $\lceil \frac{m-1-i}{2} \rceil$ for $i \leq m-1$. The "path pattern" in the bottom right of the graph chessboard corresponds in the LS to numbers $\lceil \frac{s-i+m-1}{2} \rceil$ for $m \leq i < s-m+1$. The m dots in the last row correspond in the LS to numbers $s-i$ for $i \geq s-m+1$.

(3) $\Rightarrow$ (4)**:** Let (3) hold, we show that this LS L has the labelling relation $A(L)$ as in (4). Every member of the LS L creates in the labelling relation the first coordinate. The second coordinate in each of the pairs in $A(L)$ is obtained by adding the number i to the first coordinate. So $A(L)$ satisfies (4).

(4) $\Rightarrow$ (1)**:** Let (4) hold. We show that the pairs in $A(L)$ represent the edges of the graph $K_n(F_m)$. The pairs $\left[\lceil \frac{m-1-i}{2} \rceil, \lceil \frac{m-1-i}{2} \rceil + i\right]$ for $i \leq m-1$ correspond to the path P_m. The pairs $\left[\lceil \frac{s-i+m-1}{2} \rceil, \lceil \frac{s-i+m-1}{2} \rceil + i\right]$ for $m \leq i < s-m+1$ correspond to the path P_n. The pairs $[s-i, s]$ for $i \geq s-m+1$ correspond to the edges connecting the vertex labelled s with the path P_m. So G is the fan kite $K_n(F_m)$. □

4.2 Lantern kites

By a *lantern kite* we mean a graph obtained by joining a "lantern" to the end-point of a path. By a lantern we mean a complete bipartite graph $K_{2,m}$, but we will denote it simply by L_m. We denote the lantern kite obtained by joining the lantern L_m to the end-point of the path P_n by $K_n(L_m)$ and we assume that $m \geq 2$, $n \geq 1$. The size of the graph is $s = 2m + n - 1$ where $2m$ is the size of lantern L_m and $n - 1$ is the length of the path P_n. We note that the lantern kite $K_n(L_2)$ is just the quadrangular kite $K_n(C_4)$.



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
7	7	6	6	6	6	6	6	6	0	0	0	0	0	0
8	9	9	10	11	12	13	14	15	10	11	12	13	14	15

Figure 8: Representations of the lantern kite $K_4(L_6)$

Example 4.5. *The lantern kite $K_4(L_6)$ is obtained by joining the vertex labelled by $m = 6$ of the lantern L_6 to the path P_4. In Figure 8 we see a gracefully labelled graph diagram of the kite $K_4(L_6)$ and its corresponding graph chessboard. The second row of the labelling table is the labelling sequence (LS) $(7, 7, 6, 6, 6, 6, 6, 6, 6, 0, 0, 0, 0, 0, 0)$. It consists of two parts: one is the LS $(7, 7, 6)$ of the path P_4, and the second is the LS $(6, 6, 6, 6, 6, 6, 0, 0, 0, 0, 0, 0)$ of the lantern L_6.*

Definition 4.6. *Let $s = 2m + n - 1$ for $m \geq 2$, $n \geq 1$. By an LK-graph chessboard of size s we mean a simple chessboard as in Figure 8. It has m dots in the column 0 and m dots in the column m, and the pattern continues from the mth column with dots forming a "path pattern".*

Now we characterize via our tools the lantern kites $K_n(L_m)$:

Theorem 4.7. *Let G be a graph of size $s = 2m + n - 1$ for $m \geq 2$, $n \geq 1$. Then the following are equivalent:*

(1) G is the lantern kite $K_n(L_m)$.

(2) G has a graceful labelling and an LK-graph chessboard of size s.

(3) There exists a labelling sequence $L = (j_1, j_2, \ldots, j_s)$ *of* G *such that*

$$j_i = \begin{cases} \left\lceil \frac{s-i}{2} \right\rceil, & \textit{if } i \leq s - 2m; \\ m, & \textit{if } s - 2m < i \leq s - m; \\ 0, & \textit{if } s - m < i \leq s. \end{cases} \tag{LSLK}$$

(4) There exists a labelling sequence L *of* G *with the labelling relation*

$$A(L) = \left\{ \left[\left\lceil \frac{s-i}{2} \right\rceil, \left\lceil \frac{s-i}{2} \right\rceil + i \right] \;\middle|\; i \leq s - 2m \right\} \cup \\ \{ [m, m+i] \mid s - 2m < i \leq s - m \} \cup \{ [0, i] \mid s - m < i \leq s \}.$$

Proof. (1) $\Rightarrow$ (2)**:** Let G be the lantern kite $K_n(L_m)$. We label the vertex joining the path P_n with the lantern L_m (the "joining vertex") by m and the vertex on the top of the lantern by 0. We label the vertices in the middle of the lantern (adjacent to the vertices 0 and m) by $s-m+1, s-m+2, \ldots, s$. We finally label the vertices of the path P_n from the joining vertex by numbers $s-m, m+1, s-(m+1), m+2, \ldots, \left\lceil \frac{s}{2} \right\rceil$. To show that this labelling is graceful we use the corresponding graph chessboard. The edges connecting the vertex 0 with the "middle" vertices of the lantern are represented in the column 0 of the graph chessboard. The edges connecting the joining vertex labelled m with the "middle" vertices of the lantern are represented in the column m of the graph chessboard. The path P_n is represented by a "path pattern". We obtain an LK-graph chessboard, where each diagonal has exactly one dot, so the labelling is graceful.

(2) $\Rightarrow$ (3)**:** Let G be a gracefully labelled graph with an LK-graph chessboard. The dots in the column 0 of the graph chessboard correspond in the labelling sequence (LS) to the number 0. The dots in column m of the graph chessboard correspond in the LS to the number m. The dots forming the "path pattern" correspond in the LS to numbers $\left\lceil \frac{s-i}{2} \right\rceil$. So the formula (LSLK) holds.

(3) $\Rightarrow$ (4)**:** Let (3) hold. We verify that the labelling relation $A(L)$ of the LS L satisfying (LSLK) consists of the pairs as described in (4). The numbers $\left\lceil \frac{s-i}{2} \right\rceil$ from the LS L correspond in $A(L)$ to the pairs $\left[\left\lceil \frac{s-i}{2} \right\rceil, \left\lceil \frac{s-i}{2} \right\rceil + i\right]$ for $i \leq s - 2m$. The numbers m from L corresponds in $A(L)$ to the pairs $[m, m+i]$ for $s - 2m < i \leq s - m$. Finally, the numbers 0 from L corresponds in $A(L)$ to the pairs $[0, i]$ for $s - m < i \leq s$. So $A(L)$ is exactly as in (4).

(4) $\Rightarrow$ (1)**:** Let (4) hold. We show that the pairs in $A(L)$ represent the edges of $K_n(L_m)$. The pairs $\left[\left\lceil \frac{s-i}{2} \right\rceil, \left\lceil \frac{s-i}{2} \right\rceil + i\right]$ for $i \leq s - 2m$ represent the edges of the path P_n. The pairs $[m, m+i]$ for $s-2m < i \leq s-m$ represent the edges from the lantern L_m connecting the joining vertex with the "middle" vertices. Finally, the pairs $[0, i]$ for $s - m < i \leq s$ represent the edges from

the lantern L_m connecting the top vertex with the "middle" vertices. So G is the lantern kite $K_n(L_m)$. □

5 Conclusion and further research directions

We introduced and studied classes of graceful graphs, which we call kites. We described kites formed by cycles known to be graceful, fan kites and lantern kites. We showed in a transparent way that the studied graphs are graceful and we provided characterizations of these graphs among all simple graphs via Sheppard's labelling sequences, labelling relations and graph chessboards. The labelling relations are closely related to Sheppard's labelling sequences while the graph chessboards provide a nice visualization of the graceful labellings.

In particular, in Section 3 we firstly presented the characterization of the kites $K_n(C_m)$ formed by cycles C_m for $m = 0 \pmod 4$, where $m \geq 4$ and $n \geq 1$. It follows from it as a special case that a graph G of size $s = n+3$ for some $n \geq 1$ is the quadrangular kite $K_n(C_4)$ if and only if G has a graceful labelling and a QK-graph chessboard of size s. Then in Section 3 we presented the characterization of the kites $K_n(C_m)$ formed by cycle C_m for $m = 3 \pmod 4$ and $n \geq \lfloor \frac{m}{2} \rfloor$. We distinguished two subclasses of the kites $K_n(C_m)$ with n even and n odd. Both cases are rather similar, yet they differ in details. Our theorems also cover all triangular kites $K_n(C_3)$. It follows from them as a special case that a graph G of size $s = n+2$ for some $n \geq 1$ is the triangular kite $K_n(C_3)$ if and only if G has a graceful labelling and a TK-graph chessboard of size s.

In Section 4 we described graceful labellings of other two classes of the kites: fan kites and lantern kites. We showed that a graph G of size $s = 2m + n - 2$ for $m \geq 2$, $n \geq 1$ is the fan kite $K_n(F_m)$ if and only if G has a graceful labelling and an FK-graph chessboard of size s. We finally proved that a graph G of size $s = 2m + n - 1$ for $m \geq 2$, $n \geq 1$ is the lantern kite $K_n(L_m)$ if and only if G has a graceful labelling and an LK-graph chessboard of size s. For both fan kites and lantern kites we also gave characterizations of these graphs among all simple graphs via Sheppard's labelling sequences and the labelling relations.

Before we present possible further research directions, we notice that the gracefulness of certain similar graphs was studied in 1980 by Koh, Rogers, Teo, and Yap [4] and in 1984 by Truszczyński [10]. In [4] the authors call them *tadpoles*, but the journal with the paper has not been accessible to us, and so we do not know which tadpoles exactly were studied there. Yet we are sure they could not be described by the graph chessboards and labelling relations as the concepts invented much later.

M. Truszczyński in [10] refers to his graphs as *dragons* and denotes by $D_k(m)$ a dragon with the cycle C_k and the path P_{m+1}. He gives a proof that all dragons are graceful for $k \geq 3$ and $m \geq 1$. His proof uses a method that is laborious, technical, has lots of sub-cases and is hardly visualizable. We proved here in Section 3 two cases, $k = 0 \pmod 4$ and $k = 3 \pmod 4$, the latter with sufficiently big path, but we use visualization, from which gracefulness of the graph is clearly seen. In addition, we characterized these kites formed by graceful cycles by the simple chessboards and we gave formulas for Sheppard's labelling sequences and the labelling relations. The aim of our approach has been to study interesting kites, find their graceful labellings and characterize them by the simple chessboards, labelling sequences and labelling relations. Finally, the author of the paper [10] studied gracefulness of the so-called unicyclic graphs, *i.e.* those with one cycle and connected to anything possible (the path P_m was only one of the possibilities, he also connects them *e.g.* to stars). He has expressed his belief that all unicyclic graphs are graceful. So our and his approach overlap a bit, but both approaches have different intentions.

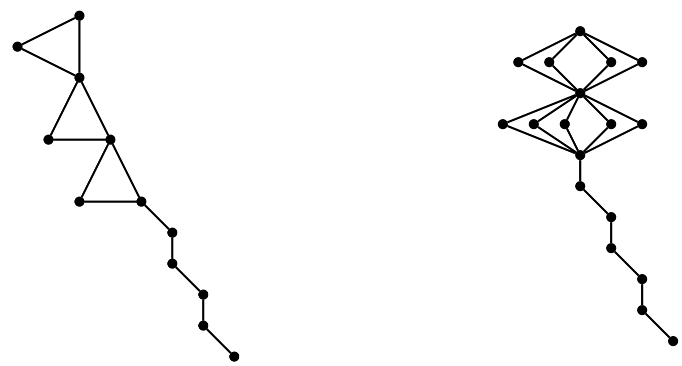


Figure 9: Further interesting kites

The first possible further research direction that we propose here is to characterize some tadpoles from the paper [4] and the remaining kites formed by the cycles C_m for $m = 1, 2 \pmod 4$ from the paper [10] via the simple chessboards, labelling sequences and labelling relations similarly as here. The second possible research direction is to take some further classes of gracefully labelled graphs (like fan graphs here) from Chapter 4 of [3] and describe them in the analogous manner like here. Another possible research direction is to consider "chain kites" in the way that the chain is a collection of C_3 graphs ("triangular chain kites", see the left graph in Figure 9) or a collection of any C_m graphs for a fixed $m \geq 4$. (We notice that the case $m = 4$, so-called "quadrangular chain kites", were studied and described in [5].) Or one could consider the chain kites in the way that the chain is a collection of C_m graphs of different sizes. Also interesting could be the chains as collections of the lanterns L_m of different sizes (see the right graph in Figure 9).

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