CUBO, A Mathematical Journal
Vol. 26, no. 3, pp. 387–406, December 2024
DOI: 10.56754/0719-0646.2603.387

New values of the Julia Robinson number

Carlos Muñoz Sandoval[1]

[1] *Departamento de Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Concepción, Chile.*
cmunoz2016@udec.cl

ABSTRACT

We extend results of Vidaux and Videla concerning the set of Julia Robinson numbers.

RESUMEN

Extendemos los resultados de Vidaux y Videla respecto del conjunto de números de Julia Robinson.

Keywords and Phrases: Decidability, definability, 2-towers, totally real towers, iterates of quadratic polynomials.

2020 AMS Mathematics Subject Classification: 03B25, 11U05, 11R80.

Published: 07 October, 2024
Accepted: 03 September, 2024
Received: 28 June, 2023

1 Introduction

Given a ring R of totally real algebraic integers and $t \in \mathbb{R} \cup \{\infty\}$, consider the set

$$R_t = \{x \in R \colon 0 \ll x \ll t\},$$

where $x \ll y$ means that all the conjugates of $y - x$ are positive, the interval (or singleton $\{\infty\}$)

$$\{t \in \mathbb{R} \cup \{\infty\} \colon \# R_t = \infty\}$$

and the so-called *Julia Robinson number*

$$\mathrm{JR}(R) = \inf\{t \in \mathbb{R} \cup \{\infty\} \colon \# R_t = \infty\}.$$

When the interval is closed or $\{\infty\}$, we say that R has the JR property. Notice that $\mathrm{JR}(R) \geq 4$ by a result of Kronecker (see [3]). Using the above definition, J. Robinson proved in [6] a result that can be formulated as

Theorem 1.1. *Let R be a ring of totally real algebraic integers. If R has the JR property, then it is possible to define $\mathbb{N}$ in R, and hence, R has undecidable first-order theory.*

Originally Robinson only considered R when it was the ring of integers of a totally real field, but it is not difficult to see that the proof of this theorem can be adapted to apply to any subring of the ring of integers of a totally real field (see [1, Theorem 1.2.2 and Lemma 1.2.3] for more details).

In the same work, J. Robinson proved that the ring of integers of the field $\mathbb{Q}^{\mathrm{tr}}$ of all totally real algebraic numbers (whose conjugates are all real numbers) has the JR property with JR number equal to 4, and the ring of integers of $K = \mathbb{Q}(\sqrt{p} \colon p \text{ prime})$ also has the JR property with JR number equal to ∞. In the case of the ring of integers of a totally real number field K has JR number equal to ∞ and hence, has undecidable theory. In [5] J. Robinson proved that every ring of integers of a number field (not necessarily totally real) has undecidable theory.

So, all known examples at that time had JR numbers equal to 4 or ∞ and the natural question, asked by J. Robinson in [6], was

> Does the JR property hold for every ring of integers of any totally real algebraic field?

Motivated by the attempt to find rings that do not satisfy one or the other of these two properties of the JR number, Vidaux and Videla constructed in [7] infinitely many rings $\mathcal{O}$ depending on two parameters (ν, x_0) for which the JR number of $\mathcal{O}$ is a minimum but is not 4 or ∞, and also also show that for infinitely many values of (ν, x_0) the JR number is not a minimum, but satisfies another topological property called *isolation property* defined as:

R has the isolation property if and only if R does not have the JR property and there exists $M > 0$ such that for every $\varepsilon > 0$, if $\varepsilon < M$ then the set $R_{\mathrm{JR}(R)+M} \smallsetminus R_{\mathrm{JR}(R)+\varepsilon}$ is finite.

In case that R has the *isolation property* then the natural numbers are definable in R, so in particular the theory of the ring R is undecidable (see [7] for details).

In [2] P. Gillibert and G. Ranieri built infinite rings with JR number strictly between 4 and infinity, which are the ring of integers of their field of fractions, however, the JR number of each of these rings is a minimum, also leaving J. Robinson's question open.

The objective of this article is to obtain new Julia Robinson numbers, having either the JR property or the isolation property, and hence produce new examples of totally real undecidable rings — see [4] for recent results on this *spectrum* problem.

Given (non-zero) natural numbers ν, λ and x_0, put $x_n = \sqrt{\nu + \lambda x_{n-1}}$ for every $n \geq 1$, and consider the ring $\mathcal{O}$ equal to the union of all the $\mathbb{Z}[x_n]$. Vidaux and Videla [7], and Castillo [1], study the definability of $\mathbb{N}$ in $\mathcal{O}$ when $\lambda = 1$. Generalizing their results, we have the following:

In section 2 we will start studying properties of the sequence (x_n) and we will give necessary and sufficient conditions for the ring $\mathcal{O}$ to be totally real (which is necessary to be able to apply Julia Robinson's techniques).

Theorem 2.7. $\mathcal{O}$ is totally real if and only if either $\nu > x_0^2 - \lambda x_0$ and $\nu \geq 2\lambda^2$ or $\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_0 < \nu^2 - \lambda^2 \nu$.

Later, we will give sufficient conditions for the tower $(K_n)_{n\geq 0}$, of the fraction fields of $\mathcal{O}_n = \mathbb{Z}[x_n]$ is a 2-tower, that is, such that $[K_{n+1} \colon K_n] = 2$ for all $n \geq 0$ (the latter is necessary to apply the argument given by Vidaux and Videla in [7]). More precisely, we will show that the tower grows when $\nu + \lambda x_0$ is congruent to 2 or 3 modulo 4, and λ is congruent to 1 or 3 modulo 4 (Proposition 2.13).

In section 3 we will study the increasing case, giving rise to our main result (in the following theorem, the case $\lambda = 1$ is done in [7] and [1]):

Theorem 1.2. *Assume $\nu > x_0^2 - \lambda x_0$ and $\nu \geq 2\lambda^2$. Assume that for every $n \geq 0$ we have $[K_{n+1} \colon K_n] = 2$. If $\lambda = 1$ and $\nu \neq 3$, then $\mathcal{O}$ has JR number equal to $\lceil \alpha \rceil + \alpha$ and satisfies the JR property. If $\lambda \geq 2$, $\nu \geq 2\lambda^2 + 2$, and $x_0 \geq \frac{\lambda}{4}$, then $\mathcal{O}$ has JR number equal to $\lceil \alpha \rceil + \alpha$ and satisfies the JR property.*

This theorem gives us new values of JR numbers, *e.g.* for the parameters $\lambda = 3$, $\nu = 20$ and $x_0 = 2$, the JR number is equal to 13.217 approximately, but with $\lambda = 1$ this number is not obtained.

In section 4 we present two new theorems: the first of them is a direct adaptation of [7, Proposition 3.4, Proposition 3.5 and Proposition 3.6]:

Theorem 1.3. *Assume $\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_0 < \nu^2 - \lambda^2\nu$. Assume that for every $n \geq 0$ we have $[K_{n+1}\colon K_n] = 2$. Assume that $\nu - \lambda x_1 \geq 1$ and $x_1 < \lfloor\alpha\rfloor + 1$. The JR number of $\mathcal{O}$ is $\lfloor\alpha\rfloor + \alpha + 1$ and satisfies the isolation property. Moreover, there are infinitely many rings $\mathcal{O}$ that satisfy these hypotheses.*

The following theorem solves the problem for infinitely many values of the parameters ν and x_0 when $\lambda = 3$, removing the hypothesis $\nu - \lambda x_1 \geq 1$. The proof of this theorem can be easily adapted to $\lambda = 2, 4, 5, \ldots,$ as long as λ is not too large, nevertheless, despite the fact that the number of cases to be considered seems to decrease as λ grows, we were not able to find a pattern that would allow as to write a proof for arbitrary λ.

Theorem 1.4. *Assume $\nu < x_0^2 - 3x_0$ and $27x_0 < \nu^2 - 9\nu$. Assume that for every $n \geq 0$ we have $[K_{n+1}\colon K_n] = 2$. If $x_1 < \lfloor\alpha\rfloor + 1$ and $\nu \neq 19$, then $\mathcal{O}$ has JR number equal to $\lfloor\alpha\rfloor + \alpha + 1$ and satisfies the isolation property. Moreover, there are infinitely many rings $\mathcal{O}$ that satisfy these hypotheses.*

This article is a contribution to two long term projects:

1) Does the ring of integers of any 2-tower over a number field have undecidable theory?

2) Study the topology of the set of JR numbers on the interval $[4, +\infty)$ — *e.g.* is it a dense set?

2 Basic properties of the tower

We define the sequence (x_n) whose general term is $x_n = \sqrt{\nu + \lambda x_{n-1}}$ and

- ν and x_0 are non-negative integers and not zero simultaneously,
- $\lambda > 0$ is a rational integer, and
- $\nu \neq x_0^2 - \lambda x_0$ (in order to avoid $x_1 = x_0$).

We define the following rings and their field of fractions:

$$\begin{aligned} \mathcal{O}_0 &= \mathbb{Z} & K_0 &= \mathbb{Q} \\ \mathcal{O}_n &= \mathcal{O}_{n-1}[x_n] & K_n &= K_{n-1}[x_n] \\ \mathcal{O} &= \bigcup_{n\geq 0} \mathcal{O}_n & K &= \bigcup_{n\geq 0} K_n \end{aligned}$$

Let us begin by stating the following lemma, whose proof is essentially the same as those given in [7, Lemma 2.2, 2.3 and 2.14].

Lemma 2.1. (1) *The sequence* (x_n) *is strictly increasing if and only if* $\nu > x_0^2 - \lambda x_0$ *or is strictly decreasing if and only if* $\nu < x_0^2 - \lambda x_0$.

(2) *The sequence* (x_n) *converges to the limit* $\alpha = \frac{\lambda+\sqrt{\lambda^2+4\nu}}{2}$.

(3) *If* $\mathcal{O}$ *is totally real, then the JR number of* $\mathcal{O}$ *is finite, in particular the extension of* K *over* $\mathbb{Q}$ *is infinite.*

Lemma 2.2. *There exists an integer* $n_0 \geq 0$ *such that for every* $n \geq 0$, *we have* $n \leq n_0$ *if and only if* x_n *is a rational integer.*

Proof. If $x_n \notin \mathbb{Z}$ for some $n \geq 0$, then $x_n \notin \mathbb{Q}$ since x_n is an algebraic integer. Hence, $\lambda x_n \notin \mathbb{Q}$ for every $\lambda \geq 1$. So, $x_{n+1} = \sqrt{\nu + \lambda x_n} \notin \mathbb{Z}$. Since (x_n) is bounded, the sequence takes finitely many integer values. We choose n_0 to be the largest index n such that x_n is a rational integer. □

2.1 The totally real condition

As was indicated in [7], Julia Robinson's criterion is only applicable for rings of totally real algebraic integers. In this section we will give a sufficient and necessary condition for the ring $\mathcal{O}$ to be totally real.

Lemma 2.3. *We have* $\nu \geq 2\lambda^2$ *if and only if* $\nu \geq \lambda\alpha$.

Proof. Observe that $\nu \geq \lambda\alpha$ if and only if

$$\nu \geq \lambda\left(\frac{\lambda+\sqrt{\lambda^2+4\nu}}{2}\right) \geq \frac{\lambda^2}{2},$$

which implies $2\nu \geq \lambda^2$. Therefore, we have

$$\begin{aligned} \nu \geq 2\lambda^2 &\iff 4\nu^2 \geq 8\lambda^2\nu \iff 4\nu^2 - 4\lambda^2\nu + \lambda^4 \geq \lambda^4 + 4\lambda^2\nu \\ &\iff (2\nu-\lambda^2)^2 \geq \lambda^2(\lambda^2+4\nu) \iff 2\nu - \lambda^2 \geq \lambda\sqrt{\lambda^2+4\nu} \\ &\iff \nu \geq \lambda\alpha. \end{aligned}$$

□

Lemma 2.4. *If* $\mathcal{O}$ *is totally real and* $\nu > x_0^2 - \lambda x_0$, *then* $\nu \geq 2\lambda^2$.

Proof. Since K_{n+1} has degree 2 over K_n for infinitely many n by Lemma 2.1, we have a subsequence of (x_n), namely (x_{n_k}), such that $\sqrt{\nu - \lambda x_{n_k}}$ is a conjugate of x_{n_k+1}. In particular, $\nu \geq \lambda x_{n_k}$ for every $k \geq 1$ since the ring $\mathcal{O}$ is totally real. From this, and the fact that x_n converges to α, we can deduce $\nu \geq \lambda\alpha$. We can conclude using Lemma 2.3. □

Lemma 2.5. *If $\mathcal{O}$ is totally real, then we have $\lambda^3 x_{n_0} < \nu^2 - \lambda^2\nu$, where n_0 comes from Lemma 2.2.*

Proof. We write $n_1 = n_0 + 1$. By the definition of n_0, we have $x_{n_1} \notin K_{n_0}$ and therefore K_{n_1} is a quadratic extension over K_{n_0}. Thus $\sqrt{\nu - \lambda x_{n_1}}$ is a conjugate of x_{n_1+1}. Since $\mathcal{O}$ is totally real, $\sqrt{\nu - \lambda x_{n_1}}$ will be a real number, which is not zero because λx_{n_1} is an irrational number and ν is a rational integer. So we have $\nu > \lambda x_{n_1} = \lambda\sqrt{\nu + \lambda x_{n_0}}$ if and only if $\lambda^3 x_{n_0} < \nu^2 - \lambda^2\nu$. □

Remark 2.6. *Let $x \in \mathcal{O}$. We use the notation $\overline{|x|}$ for the largest absolute value of conjugates of x over $\mathbb{Q}$.*

The following theorem gives us a characterization of when our ring $\mathcal{O}$ is totally real and therefore, will allow us to use Julia Robinson's methods.

Theorem 2.7. *The ring $\mathcal{O}$ is totally real if and only if*

(1) *either $\nu > x_0^2 - \lambda x_0$ and $\nu \geq 2\lambda^2$, or*

(2) *$\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_{n_0} < \nu^2 - \lambda^2\nu$.*

If $\mathcal{O}$ is totally real, then $\overline{|x_n|} = x_n$ for each $n \geq 0$.

Proof. Let us start proving that $\overline{|x_n|} = x_n$ for each $n \geq 0$ if $\mathcal{O}$ is totally real. We will show this by induction over n. The case $n = 0$ is trivial. Assume $\overline{|x_n|} = x_n$ for some n. We have

$$x_{n+1} = \sqrt{\nu + \lambda x_n} \geq \pm\sqrt{\nu + \lambda x_n^\sigma}$$

for every embedding σ and since the only possible conjugates of x_{n+1} are of the form $\pm\sqrt{\nu + \lambda x_n^\sigma}$ for some embedding σ, we are done. For the rest of the proof, the implication from left to right is an immediate consequence of Lemma 2.4 and Lemma 2.5. We show the other implication by induction on n. Let $n_1 = n_0 + 1$. If $n \leq n_0$, then $\mathcal{O}_n = \mathbb{Z}$ which is totally real and hence $\overline{|x_n|} = x_n$. For n_1 we have $x_{n_1} \notin \mathbb{Z}$ and hence its conjugates are of the form $\pm\sqrt{\nu + \lambda x_{n_0}}$. Therefore, $\mathcal{O}_{n_1} = \mathbb{Z}[x_{n_1}]$ is totally real and $\overline{|x_{n_1}|} = x_{n_1}$. Suppose that for some $n \geq n_1$, $\mathcal{O}_n$ is totally real and $\overline{|x_n|} = x_n$. Note that the conjugates of x_{n+1} are of the form $\pm\sqrt{\nu + \lambda x_n^\sigma}$ for some embedding σ. Since $\overline{|x_n|} = x_n$, we have $\overline{|x_{n+1}|} = x_{n+1}$ and it will be enough to prove that $\nu \geq \lambda x_n$ for each $n \geq n_1$. We can separate the proof into cases where the sequence (x_n) is increasing or decreasing:

- If $\nu > x_0^2 - \lambda x_0$ and $\nu \geq 2\lambda^2$, then (x_n) is strictly increasing by Lemma 2.1 and hence $\lambda x_n < \lambda\alpha \leq \nu$ by Lemma 2.3.
- If $\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_{n_0} < \nu^2 - \lambda^2\nu$, then (x_n) is strictly decreasing by Lemma 2.1 and $\lambda x_{n_1} < \nu$. Hence, $\lambda x_n \leq \lambda x_{n_1} < \nu$ for each $n \geq n_1$. □

We can assume, without loss of generality, that $n_0 = 0$, since if $n_0 > 0$, then we can define a new sequence $y_n = x_{n+n_0}$, and the rings $\mathcal{O}$ corresponding to (x_n) and (y_n) are the same.

Assumption 2.8. *The number x_1 is a non-rational integer*

Lemma 2.9. *In the decreasing case, we have $\nu \geq 3$ and $x_0 \geq 3$.*

Proof. This is an immediate consequence of the inequalities $\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_0 < \nu^2 - \lambda^2 \nu$, and the fact that λ is at least 1. □

Lemma 2.10. *Assume that (x_n) is increasing. If $\nu \geq 2\lambda^2 + 2$, then $x_n \geq 2$ for each $n \geq 1$.*

Proof. Since the sequence (x_n) is increasing, we have

$$x_n \geq x_1 = \sqrt{\nu + \lambda x_0} \geq \sqrt{2\lambda^2 + 2} \geq 2.$$

for each $n \geq 1$. □

Lemma 2.11. *We have $\alpha \geq 2$.*

Proof. If (x_n) is decreasing, then by Lemma 2.9 we have $\nu \geq 3$, and if (x_n) is increasing, then $\nu \geq 2\lambda^2 \geq 2$. In all cases, we have $\nu \geq 2$. Hence, we have

$$2\alpha = \lambda + \sqrt{\lambda^2 + 4\nu} \geq 4$$

because $\lambda \geq 1$ and $\nu \geq 2$. □

2.2 Conditions for the tower to increase at each step

For the induction arguments to work in the next sections, we will need the tower (K_n) to increase at each step. In this subsection, we will provide sufficient conditions for that.

Let $f(t) = \dfrac{t^2 - \nu}{\lambda}$ be a function of the real variable t. We define for each $n \geq 1$

$$P_n = \lambda^{2^n - 1} f^{\circ n}(t) - \lambda^{2^n - 1} x_0,$$

where $f^{\circ n}$ stands for the composition of f with itself n times.

Lemma 2.12. *The polynomial P_n is monic for each $n \geq 1$.*

Proof. We prove it by induction on n. If $n = 1$, then $P_1 = \lambda f(t) - \lambda x_0 = t^2 - \nu - \lambda x_0$ is monic. Suppose that for some $n \geq 2$ the polynomial P_n is monic. We have

$$\begin{aligned} P_{n+1}(t) &= \lambda^{2^{n+1}-1} f^{\circ(n+1)}(t) - \lambda^{2^{n+1}-1} x_0 = \lambda^{2^{n+1}-1}\left(\frac{(f^{\circ n}(t))^2 - \nu}{\lambda}\right) - \lambda^{2^{n+1}-1} x_0 \\ &= \lambda^{2^{n+1}-2}(f^{\circ n}(t))^2 - \lambda^{2^{n+1}-2}\nu - \lambda^{2^{n+1}-1}x_0 = \left(\lambda^{2^n-1} f^{\circ n}(t)\right)^2 - \lambda^{2^{n+1}-2}\nu - \lambda^{2^{n+1}-1}x_0 \\ &= \left(P_n(t) + \lambda^{2^n-1}x_0\right)^2 - \lambda^{2^{n+1}-2}\nu - \lambda^{2^{n+1}-1}x_0, \end{aligned}$$

and since P_n is monic by hypothesis, P_{n+1} is monic too. □

Proposition 2.13. *If $\nu + \lambda x_0$ is congruent to 2 or 3 modulo 4 and λ is congruent to 1 or 3 modulo 4, then for each $n \geq 1$, we have $[K_{n+1} \colon K_n] = 2$.*

Proof. From the definition of f we have $f^{\circ n}(x_n) = x_0$ for each $n \geq 1$. Therefore, x_n is a root of P_n. Also note that, by Lemma 2.12, P_n is monic for each $n \geq 1$. Given $a, b \in \mathbb{Z}$, we have

$$P_1(t+a) = (t+a)^2 - \nu - \lambda x_0 = t^2 + 2at + a^2 - (\nu + \lambda x_0), \tag{2.1}$$

and

$$\begin{aligned} P_2(t+b) &= \lambda^3 f^{\circ 2}(t+b) - \lambda^3 x_0 \\ &= t^4 + 4bt^3 + 2(3b^2 - \nu)t^2 + 4(b^3 - b\nu)t + (b^4 - 2b^2\nu + \nu^2 - \lambda^2(\nu + \lambda x_0)). \end{aligned} \tag{2.2}$$

Also, for each $n \geq 1$, we have

$$\begin{aligned} P_{n+2}(t) &= \lambda^{2^{n+2}-1}(f^{\circ(n+2)}(t) - x_0) = \lambda^{2^{n+2}-1}(f^{\circ 2}(f^{\circ n}(t)) - x_0) \\ &= \lambda^{2^{n+2}-1}\left(f^{\circ 2}\left(\frac{P_n(t)}{\lambda^{2^n-1}} + x_0\right) - x_0\right) \\ &= \lambda^{2^{n+2}-1}\left(\left(\frac{P_2\left(\frac{P_n(t)}{\lambda^{2^n-1}} + x_0\right)}{\lambda^3} + x_0\right) - x_0\right) = \lambda^{4(2^n-1)} P_2\left(\frac{P_n(t)}{\lambda^{2^n-1}} + x_0\right) \\ &= P_n^4(t) + 4\lambda^{2^n-1}x_0 P_n^3(t) + 2\lambda^{2(2^n-1)}(3x_0^2 - \nu)P_n^2(t) + 4\lambda^{3(2^n-1)}(x_0^3 - x_0\nu)P_n(t) \\ &\quad + \lambda^{4(2^n-1)}\left(x_0^4 - 2x_0^2\nu + \nu^2 - \lambda^2(\nu + \lambda x_0)\right). \end{aligned} \tag{2.3}$$

We prove by induction on n that the polynomial P_n is irreducible. If $n = 1$, then using Equation (2.1) we choose $a = 0$ if $\nu + \lambda x_0$ is congruent to 2 modulo 4, and $a = 1$ if $\nu + \lambda x_0$ is congruent to 3 modulo 4. In both cases $P_1(t+a)$ is an Eisenstein polynomial for 2. If $n = 2$, then using Equation (2.2), we have that $P_2(t+x_0)$ is an Eisenstein polynomial for 2, because $x_0^4 - 2x_0^2\nu + \nu^2 - \lambda^2(\nu + \lambda x_0)$

is congruent to 2 modulo 4 when $\nu + \lambda x_0$ is congruent to 2 or 3 modulo 4 and λ is congruent to 1 or 3 modulo 4 (we leave the verification to the reader). Note that λ^2 is congruent to 1 modulo 4 by hypothesis. Therefore, the constant term of $P_{n+2}(t)$, seen as a polynomial in $P_n(t)$, is congruent to 2 modulo 4. So, using Equation (2.3), if $P_n(t+c)$ is an Eisenstein polynomial for 2 for some $c \in \mathbb{Z}$, then $P_{n+2}(t+c)$ is an Eisenstein polynomial for 2 too. Thus, we can prove the irreducibility of P_n by induction on n, separating into two cases:

- If n is odd, then $P_n(t+a)$ is an Eisenstein polynomial for 2 (with the respective choice of a).
- If n is even, then $P_n(t+x_0)$ is an Eisenstein polynomial for 2. □

From now on, we assume

Assumption 2.14. *K_n is a quadratic extension of K_{n-1} and the ring $\mathcal{O}$ is a totally real.*

Lemma 2.15 ([7, Lemma 2.19])**.** *Let r be any real number and $a, b \in \mathcal{O}_{n-1}$ with $n \geq 1$. For $n = 1$, if $0 \ll a + bx_1 \ll 2r$, then $|b| < \frac{r}{x_1}$. For $n \geq 2$, if $0 \ll a + bx_n \ll 2r$, then $|b^\sigma| < \frac{r}{\sqrt{\nu - \lambda x_{n-1}}}$ for every embedding σ of $\mathcal{O}_n$.*

3 Increasing case

Assumption 3.1. *For this section, let us assume $\nu \geq 2\lambda^2 + 2$, $x_0 \geq \frac{\lambda}{4}$ and the sequence (x_n) is strictly increasing.*

Definition 3.2. *For each $n \geq 1$, let k_n be the only rational integer such that*

$$\lceil \alpha \rceil - (k_n + 1) < x_n < \lceil \alpha \rceil - k_n.$$

Remember that x_1 is not a rational integer by Assumption 2.14 and note that the sequence (k_n) is (non strictly) decreasing, hence the k_n take only finitely many values, and since the sequence (x_n) tends to α, eventually k_n is 0.

The main result we use to compute the JR number in the increasing case is the following lemma:

Lemma 3.3. *Assume $x \in \mathcal{O}$. We have $0 \ll x \ll 2\lceil \alpha \rceil$ if and only if $x \in X$.*

The set X is defined as follows:

$$\begin{aligned} X_0 &= \{1, 2, \ldots, 2\lceil \alpha \rceil - 1\}, \\ X_n &= X_0 \cup \{\lceil \alpha \rceil \pm j \pm x_s \colon 0 \leq j \leq k_s \text{ and } 1 \leq s \leq n\}, \\ X &= \bigcup_{n \geq 0} X_n. \end{aligned}$$

Lemma 3.4. *If $\lambda \geq 2$, then $x_1 + x_2 + \lceil x_1 \rceil > 2\lceil \alpha \rceil$.*

Proof. It is enough to prove that we have $x_2 + 2x_1 > 2(\alpha + 1)$. We have

$$2\sqrt{\nu + \lambda x_0} + \sqrt{\nu + \lambda\sqrt{\nu + \lambda x_0}} \geq \sqrt{4\nu + \lambda^2} + \sqrt{2\lambda^2 + 2 + \lambda\sqrt{2\lambda^2 + 2 + \frac{\lambda^2}{4}}}$$
$$\geq \sqrt{4\nu + \lambda^2} + \sqrt{\lambda^2 + 4\lambda + 4} = 2(\alpha + 1),$$

where the first inequality is by Assumption 3.1. □

Lemma 3.5. *Let $n \geq 1$. If $0 \ll a \pm bx_n \ll 2\lceil \alpha \rceil$, with $a, b \in \mathcal{O}_{n-1}$, then $|b| < 2$ (in the inequality, the plus-minus means that both inequalities hold).*

Proof. Since $\nu \geq 2\lambda^2 + 2$ and $\nu \in \mathbb{N}$, we can write $\nu = 2\lambda^2 + k$, for some $k \geq 2$. Since $0 < a \pm bx_n < 2\lceil \alpha \rceil$, combining both inequalities we obtain $|b| < \frac{\lceil \alpha \rceil}{x_n}$. So, we have

$$|b| < \frac{\lceil \alpha \rceil}{x_n} \leq \frac{\alpha + 1}{\sqrt{2\lambda^2 + k + \lambda x_{n-1}}} = \frac{\lambda + \sqrt{\lambda^2 + 4(2\lambda^2 + k)} + 2}{2\sqrt{2\lambda^2 + k + \lambda x_{n-1}}}$$
$$\leq \frac{\lambda + 2 + \sqrt{\lambda^2} + \sqrt{8\lambda^2 + 4k}}{\sqrt{8\lambda^2 + 4k}} \leq 1 + \frac{2\lambda + 2}{\sqrt{8\lambda^2 + 4k}} \leq 1 + \frac{2\lambda + 2}{\sqrt{8\lambda^2 + 8}} \leq 2,$$

where the last inequality is true because $2\lambda + 2 \leq \sqrt{8\lambda^2 + 8}$ for every $\lambda \geq 1$. □

Lemma 3.6. *We have $\nu - \lambda\alpha > 1$.*

Proof. Since $\nu \geq 2\lambda^2 + 2$ and $\nu \in \mathbb{N}$, we can write $\nu = 2\lambda^2 + k$, for some $k \geq 2$. Hence, we have

$$(2\lambda^2 + k) - \lambda\left(\frac{\lambda + \sqrt{\lambda^2 + 4(2\lambda^2 + k)}}{2}\right) > 1 \iff 3\lambda^2 + 2k - 2 > \lambda\sqrt{9\lambda^2 + 4k}$$
$$\iff 4k^2 + 12k\lambda^2 - 8k + 9\lambda^4 - 12\lambda^2 + 4 > 9\lambda^4 + 4k\lambda^2 \iff 4k^2 + (8\lambda^2 - 8)k + 4 - 12\lambda^2 > 0,$$

and since $k \geq 0$, the latter is true for

$$k > \frac{8 - 8\lambda^2 + \sqrt{64\lambda^4 + 64\lambda^2}}{8} = 1 - \lambda^2 + \sqrt{\lambda^4 + \lambda^2}.$$

We consider the continuous function $x \mapsto 1 - x^2 + \sqrt{x^4 + x^2}$. The line $y = \frac{3}{2}$ is an horizontal asymptote for this function, hence we have

$$1 - \lambda^2 + \sqrt{\lambda^4 + \lambda^2} < \frac{3}{2},$$

for every $\lambda \geq 1$. □

Lemma 3.7. *Let $x = a + bx_1 \in \mathcal{O}_1$, with $a, b \in \mathbb{Z}$. If $0 < a \pm bx_1 < 2\lceil\alpha\rceil$, then $x \in X_1$.*

Proof. By Lemma 3.5, we have $b = \pm 1$ or $b = 0$.

- If $a \leq \lceil\alpha\rceil - (k_1 + 1)$, then $b = 0$. Indeed, if $|b| = 1$, by choosing σ such that $x^\sigma = a - |b|x_1$, we obtain:
$$a - |b|x_1 \leq \lceil\alpha\rceil - (k_1 + 1) - x_1 \leq 0,$$
by the definition of k_1, contradicting our hypothesis.

- If $a \geq \lceil\alpha\rceil + (k_1 + 1)$, then $b = 0$. If $|b| = 1$, by choosing σ such that $x^\sigma = a + |b|x_1$, we obtain:
$$a + |b|x_1 \geq \lceil\alpha\rceil + (k_1 + 1) + x_1 \geq 2\lceil\alpha\rceil,$$
again contradicting our hypothesis.

Therefore, we have either $|a - \lceil\alpha\rceil| \geq k_1 + 1$ and $b = 0$, or $|a - \lceil\alpha\rceil| < k_1 + 1$ and $|b| \leq 1$. In both cases, we have $x \in X_1$. □

Lemma 3.8. *Assume $n > m \geq 1$ and $\lambda \geq 2$.*

(1) *We have $\lceil\alpha\rceil \pm j + x_m + x_n \geq 2\lceil\alpha\rceil$ for every $0 \leq j \leq k_m$.*

(2) *We have $\lceil\alpha\rceil \pm j - x_m - x_n \leq 0$ for every $0 \leq j \leq k_m$.*

Proof.

(1) Note that $\lceil x_1\rceil = \lceil\alpha\rceil - k_1$. By Lemma 3.4, and using the fact that (x_n) is increasing, we have
$$x_m + x_n + \lceil\alpha\rceil - k_1 \geq 2\lceil\alpha\rceil,$$
for each $n > m \geq 1$. Since $k_1 \geq k_m$ for each $m \geq 1$, we have
$$x_m + x_n + \lceil\alpha\rceil \pm j \geq 2\lceil\alpha\rceil,$$
for every $0 \leq j \leq k_m$.

(2) For every $0 \leq j \leq k_m$, we have $\lceil\alpha\rceil \pm j - x_m - x_n \leq 0$ if and only if $x_m + x_n + \lceil\alpha\rceil \pm j \geq 2\lceil\alpha\rceil$. So we can conclude by item (1). □

Lemma 3.9. *Assume $\lambda \geq 2$. We have $\lceil x_n \rceil + x_n \geq \lceil \alpha \rceil + 2$ for each $n \geq 1$. In particular, we have $x_n \geq k_n + 2$ for each $n \geq 1$.*

Proof. Since (x_n) is increasing, it is enough to prove that we have $x_1 + \lceil x_1 \rceil > \alpha + 3$. If $\lambda = 2$, then we have (recalling that we have $x_0 \geq 1$ and $\nu \geq 10$ by Assumption 3.1)

$$x_1 + \lceil x_1 \rceil \geq \sqrt{\nu+2} + \lceil \sqrt{\nu+2} \rceil > \sqrt{\nu+1} + \lceil \sqrt{12} \rceil = 3 + \alpha.$$

For $\lambda \geq 3$, we have

$$2x_1 + 2\lceil x_1 \rceil \geq \sqrt{4\nu + \lambda^2} + \sqrt{9\lambda^2 + 8} > \sqrt{4\nu + \lambda^2} + \lambda + 6 = 2(\alpha + 3),$$

where the first inequality is by Assumption 3.1, and the second inequality is because $\lambda \geq 3$. In particular, using $\lceil x_n \rceil = \lceil \alpha \rceil - k_n$ for each $n \geq 1$, we have $\lceil x_n \rceil + x_n \geq \lceil \alpha \rceil + 2$ if and only if $x_n \geq k_n + 2$. □

Lemma 3.10. *Let $x \in \mathcal{O}$. If $0 \ll x \ll 2\lceil \alpha \rceil$, then $x \in X$.*

Proof. For $\lambda = 1$, this is [7, Lemma 4.9]. For $\lambda \geq 2$, which we now assume, we start as in [7, Lemma 4.9]. We prove by induction on n that if $x \in \mathcal{O}_n$ is such that $0 \ll x \ll 2\lceil \alpha \rceil$, then $x \in X_n$. This is clear for $n = 0$. For $n = 1$, we have $x \in X_1$ by Lemma 3.7. Assume $n \geq 2$. Let us fix $x = a + bx_n \in \mathcal{O}_n$ with $a, b \in \mathcal{O}_{n-1}$. By Lemma 2.15, we have $0 \ll a \ll 2\lceil \alpha \rceil$, so $a \in X_{n-1}$ by induction hypothesis. Also, by Lemma 2.15, we have

$$\overline{|b|} < \frac{\lceil \alpha \rceil}{\sqrt{\nu - \lambda x_{n-1}}} < \frac{\lceil \alpha \rceil}{\sqrt{\nu - \lambda \alpha}} \leq \lceil \alpha \rceil,$$

since $\sqrt{\nu - \lambda\alpha} \geq 1$ by Lemma 3.6. Hence, we have $0 \ll \lceil \alpha \rceil + b \ll 2\lceil \alpha \rceil$, and by induction hypothesis we have $\lceil \alpha \rceil + b \in X_{n-1}$. From the definition of X_{n-1}, we have either $b \in \mathbb{Z}$, or $|b| = |j \pm x_s|$ for some $1 \leq s \leq n-1$ and $0 \leq j \leq k_s$. In the first case, we have either $b = 0$ or $b = \pm 1$ by Lemma 3.5. In the second case, we have, also by Lemma 3.5, either $|j + x_s| < 2$ or $|x_s - j| < 2$. If $|j + x_s| < 2$, then $x_s < 2 - j \leq 2$ and we have a contradiction by Lemma 3.9. If $|x_s - j| < 2$, then $x_s < j + 2 \leq k_s + 2$, which is a contradiction, again by Lemma 3.9. Therefore, we have $b \in \{-1, 0, 1\}$. For $b = 0$, there is nothing to prove, as we already know that $x = a$ lies in X_{n-1}. Assume $|b| = 1$. We can write $x = a \pm x_n$, and since $a \in X_{n-1}$, we have either $a \in \{1, \ldots, 2\lceil \alpha \rceil - 1\}$, or $a = \lceil \alpha \rceil \pm j \pm x_s$ for some $1 \leq s \leq n-1$ and $0 \leq j \leq k_s$.

- If $a \in \{1, \ldots, \lceil \alpha \rceil - (k_n + 1)\}$, then we can choose an embedding σ such that:

$$x^\sigma = a - x_n \leq \lceil \alpha \rceil - (k_n + 1) - x_n < 0,$$

by definition of k_n, which contradicts our hypothesis.

- If $a \in \{\lceil\alpha\rceil + (k_n+1), \dots, 2\lceil\alpha\rceil - 1\}$, then again we can choose σ such that

$$x^\sigma = a + x_n \geq \lceil\alpha\rceil + (k_n+1) + x_n > 2\lceil\alpha\rceil,$$

which again contradicts our hypothesis on x.

- If $a = \lceil\alpha\rceil \pm j + x_s$, with $0 \leq j \leq k_s$, then

$$a + x_n = \lceil\alpha\rceil \pm j + x_s + x_n \geq 2\lceil\alpha\rceil,$$

by Lemma 3.8, a contradiction.

- If $a = \lceil\alpha\rceil \pm j - x_s$, with $0 \leq j \leq k_s$, then

$$a - x_n = \lceil\alpha\rceil \pm j - x_s - x_n \leq 0,$$

also by Lemma 3.8, again a contradiction.

So, we have $a \in \{\lceil\alpha\rceil - k_n, \dots, \lceil\alpha\rceil + k_n\}$. Therefore, if $|b| = 1$, then x is of the form $\lceil\alpha\rceil \pm j \pm x_n$ where $0 \leq j \leq k_n$. In all the cases we obtain $x \in X$. □

Proof Lemma 3.3. Thanks to Lemma 3.10, we need only to prove the lemma from right to left. Assume $x \in X$. For $x \in X_0$, there is nothing to prove. Assume $x \in X_n$ for some $n \geq 1$, so that $x = \lceil\alpha\rceil \pm j \pm x_s$ for some s and j such that $1 \leq s \leq n$ and $0 \leq j \leq k_s$. By definition of k_s, we have $x_s + k_s < \lceil\alpha\rceil$. Hence, we have

$$\lceil\alpha\rceil \pm j + x_s \leq \lceil\alpha\rceil + k_s + x_s < 2\lceil\alpha\rceil,$$

and

$$\lceil\alpha\rceil \pm j - x_s \geq \lceil\alpha\rceil - k_s - x_s > 0.$$

Thus, we have $0 < x^\sigma < 2\lceil\alpha\rceil$ for every embedding σ of $\mathcal{O}_s$ since $\overline{|x_s|} = x_s$ by Lemma 2.7. □

Proposition 3.11. *The ring $\mathcal{O}$ has the JR property and $JR(\mathcal{O}) = \lceil\alpha\rceil + \alpha$.*

Proof. For each n we have $x_n + \lceil\alpha\rceil < \alpha + \lceil\alpha\rceil$. By Theorem 2.7, we have $\overline{|x_n|} = x_n$, and hence, there are infinitely many $x \in \mathcal{O}$ such that $0 \ll x \ll \lceil\alpha\rceil + \alpha$. Since the sequence (x_n) is increasing and converges to α, for each $\varepsilon > 0$, there are only finitely many n such that $x_n + \lceil\alpha\rceil < \alpha + \lceil\alpha\rceil - \varepsilon$. Moreover, almost all n we have $k_n = 0$. Hence, there are only finitely many elements of the form $x_n + \lceil\alpha\rceil + j$ where $0 \leq j \leq k_n$ and $k_n \geq 1$. In particular, only finitely many of them satisfy

$0 \ll x_n + \lceil \alpha \rceil + j \ll \lceil \alpha \rceil + \alpha$. Therefore, by Lemma 3.3, for each $\varepsilon > 0$, there are only finitely many $x \in \mathcal{O}$ such that $0 \ll x \ll \lceil \alpha \rceil + \alpha - \varepsilon$. □

4 Decreasing case

Assumption 4.1. *For this section, let us assume that the sequence (x_n) is strictly decreasing.*

We define the following sets:

$$X_0 = \{1, 2, \ldots, 2\lfloor \alpha \rfloor + 1\}$$
$$X_n = X_0 \cup \{\lfloor \alpha \rfloor + 1 \pm x_k : \ 1 \leq k \leq n\}$$
$$X = \bigcup_{n \geq 0} X_n.$$

The following lemma and theorem are exactly as [7, Lemma 3.2, Proposition 3.4 and Proposition 3.5], changing their hypothesis $\nu - x_1 \geq 1$ by $\nu - \lambda x_1 \geq 1$. For this reason, we will omit the proof.

Lemma 4.2 ([7, Lemma 3.2])**.** *Assume $\nu - \lambda x_1 \geq 1$ and $x_1 < \lfloor \alpha \rfloor + 1$. For each $n \geq 0$, if $x \in \mathcal{O}_n$ and $0 \ll x \ll 2\lfloor \alpha \rfloor + 2$, then $x \in X_n$.*

Theorem 4.3 ([7, Propositions 3.4 and 3.5])**.** *Assume $\nu - \lambda x_1 \geq 1$ and $x_1 < \lfloor \alpha \rfloor + 1$. The JR number of $\mathcal{O}$ is $\lfloor \alpha \rfloor + \alpha + 1$ and $\mathcal{O}$ satisfies the isolation property.*

The following proposition proves that there are infinitely many pairs (ν, x_0) for which Theorem 4.3 holds.

Proposition 4.4. *For any λ congruent to 1 or 3 modulo 4, there are infinitely many distinct values of α corresponding to pairs (ν, x_0) of rational integers such that*

(1) $\nu < x_0^2 - \lambda x_0$,

(2) $\sqrt{\nu + \lambda x_0}$ *is not a rational integer,*

(3) *For every $n \geq 1$, we have $[K_n \colon K_{n-1}] = 2$,*

(4) $\lambda^3 x_0 < \nu^2 - \lambda^2 \nu$,

(5) $\nu - \lambda x_1 \geq 1$,

(6) $\sqrt{\nu + \lambda x_0} < \lfloor \alpha \rfloor + 1$.

Proof. For any $\lambda \geq 1$ which is congruent to 1 or 3 modulo 4, we choose $\nu = 4\lambda^4$ and $x_0 = 2\lambda^2 + \lambda$. The first two conditions are immediate. The condition 3 holds by Proposition 2.13. The condition

4 holds because $\nu^2 > \lambda^2\nu + \lambda^3 x_0$ iff $16\lambda^4 > 4\lambda^2 + 2\lambda + 1$ which is true for all $\lambda \geq 1$. For the condition 5 we have

$$\nu - \lambda x_1 > \nu - \lambda x_0 = 4\lambda^4 - 2\lambda^3 - \lambda^2 \geq 1.$$

for each $\lambda \geq 1$. Finally, we have

$$\alpha = \frac{\lambda + \sqrt{\lambda^2 + 4\nu}}{2} = \frac{\lambda + \sqrt{16\lambda^4 + \lambda^2}}{2} = \frac{\lambda + 4\lambda^2 + \varepsilon}{2} = \frac{\lambda - 1}{2} + 2\lambda^2 + \frac{1}{2} + \frac{\varepsilon}{2}$$

for some $0 < \varepsilon < 1$. Since λ is congruent to 1 or 3 modulo 4, we have $\lfloor\alpha\rfloor = 2\lambda^2 + \frac{\lambda-1}{2}$. Therefore, we have

$$(\lfloor\alpha\rfloor + 1)^2 = 4\lambda^4 + 2\lambda^3 + 2\lambda^2 + \left(\frac{\lambda+1}{2}\right)^2 > 4\lambda^4 + 2\lambda^3 + \lambda^2 = \nu + \lambda x_0,$$

so the last condition is satisfied. □

For $\lambda = 1$, M. Castillo [1, Theorem 1] was able to remove the hypothesis $\nu - x_1 \geq 1$ and $x_1 < \lfloor\alpha\rfloor + 1$, and obtain the following theorem:

Theorem 4.5. *Assuming $\lambda = 1$ and $\nu > 3$, $\mathcal{O}$ has JR number $\lfloor\alpha\rfloor + \alpha + 1$ and it satisfies the isolation property.*

Now we will present some new results for $\lambda = 3$. The same proof can be easily adapted to the case $\lambda = 2, 4, 5\ldots$ We could not find the general pattern that would let us write a general proof since for each value of λ there are cases that must be studied independently.

We will prove the following theorem at the end of this section.

Theorem 4.6. *Assume $\lambda = 3$. If $x_1 < \lfloor\alpha\rfloor + 1$ and $\nu \neq 19$, then $\mathcal{O}$ has JR number $\lfloor\alpha\rfloor + \alpha + 1$ and it satisfies the isolation property.*

Assumption 4.7. *For the following lemmas we assume that $\lambda = 3$.*

Lemma 4.8. *If $x_1 < \lfloor\alpha\rfloor + 1$ and $\nu \neq 19$, then $\nu - 3x_2 \geq 1$.*

Proof. Since $x_1 < \lfloor\alpha\rfloor + 1$, we have

$$\nu - 3x_2 > \nu - 3(\lfloor\alpha\rfloor + 1) \geq \nu - 3\alpha - 3.$$

Therefore, it suffices to prove $\nu - 3\alpha - 3 \geq 1$. This is satisfied if and only if $2\nu - 17 \geq 3\sqrt{9 + 4\nu}$, which is true for every $\nu \geq 24$. By Lemma 2.9, we have $\nu \geq 3$, so we must analyze the cases when $\nu \in \{3, \ldots, 23\}$. A simple calculation shows that for $\nu \in \{3, \ldots, 18\}$, there is no x_0 that satisfies the inequalities given in Theorem 2.7. Hence, $\nu \in \{19, \ldots, 23\}$, and again solving the inequalities given in Theorem 2.7, we obtain the following cases:

ν	x_0	x_1	x_2	$\nu - 3x_1$	$\nu - 3x_2$
19	7	$\sqrt{40}$	$\sqrt{19+3\sqrt{40}}$	0.03	0.51
20	7	$\sqrt{41}$	$\sqrt{20+3\sqrt{41}}$	0.79	1.21
	8	$\sqrt{44}$	$\sqrt{20+3\sqrt{44}}$	0.10	1.05
21	7	$\sqrt{42}$	$\sqrt{21+3\sqrt{42}}$	1.56	1.92
	8	$\sqrt{45}$	$\sqrt{21+3\sqrt{45}}$	0.88	1.76
	9	$\sqrt{48}$	$\sqrt{21+3\sqrt{48}}$	0.22	1.61
22	7	$\sqrt{43}$	$\sqrt{22+3\sqrt{43}}$	2.33	2.63
	8	$\sqrt{46}$	$\sqrt{22+3\sqrt{46}}$	1.65	2.48
	9	7	$\sqrt{22+3\sqrt{49}}$	1	2.33
	10	$\sqrt{52}$	$\sqrt{22+3\sqrt{52}}$	0.37	2.18
23	7	$\sqrt{44}$	$\sqrt{23+3\sqrt{44}}$	3.10	3.35
	8	$\sqrt{47}$	$\sqrt{23+3\sqrt{47}}$	2.43	3.20
	9	$\sqrt{50}$	$\sqrt{23+3\sqrt{50}}$	1.79	3.05
	10	$\sqrt{53}$	$\sqrt{23+3\sqrt{53}}$	1.16	2.91
	11	$\sqrt{56}$	$\sqrt{23+3\sqrt{56}}$	0.55	2.78

Table 1: Approximate values of $\nu - 3x_1$ and $\nu - 3x_2$ for $\nu \in \{19, \ldots, 23\}$. □

Lemma 4.9. *Let $x \in \mathcal{O}_1$ be such that $0 \ll x \ll 2\lfloor\alpha\rfloor+2$. If $(\nu, x_0) \in \{(20,7),(20,8),(21,8),(21,9)\}$, then $x \in X_1$.*

Proof. Let $x = a + bx_1 \in \mathcal{O}_1$, with $a, b \in \mathbb{Z}$, be such that $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$. Note that in all cases we have $\lfloor\alpha\rfloor + 1 = 7$ and $x_1 \geq \sqrt{41}$. Since $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$, by Lemma 2.15, we have $a \in \{1, \ldots, 13\}$ and

$$|b| < \frac{\lfloor\alpha\rfloor + 1}{x_1} \leq \frac{7}{\sqrt{41}},$$

so we have $b \in \{-1, 0, 1\}$. Finally, using a computer program (we used SageMath 9.2, see below) we can analyze all the cases to see that x is indeed in X_1. □

```
def cases_X1(x_0, nu, l):
    x_1=sqrt(nu+l*x_0)
    floor_alpha=math.floor((l+sqrt(l**2+4*nu))/2)
    for a in srange(1,2*floor_alpha+2,1):
        for b in [-1,0,1]:
            if 0<a+b*x_1<2*floor_alpha+2 and
            0<a-b*x_1<2*floor_alpha+2:
                print(a+b*x_1)
cases_X1(7,20,3)
cases_X1(8,20,3)
cases_X1(8,21,3)
cases_X1(9,21,3)
```

Lemma 4.10. *Let $x \in \mathcal{O}_2$ be such that $0 \ll x \ll 2\lfloor\alpha\rfloor+2$. If $(\nu, x_0) \in \{(20,7),(20,8),(21,8),(21,9)\}$, then $x \in X_2$.*

Proof. Let $x = a + bx_2 \in \mathcal{O}_2$, with $a, b \in \mathcal{O}_1$. Note that in all cases we have $x_1 \geq \sqrt{41}$, $x_2 \geq \sqrt{20+3\sqrt{41}}$ and $\lfloor\alpha\rfloor + 1 = 7$. Since $0 \ll a + bx_2 \ll 2\lfloor\alpha\rfloor + 2$, by Lemma 2.15 we have $0 \ll a \ll 2\lfloor\alpha\rfloor + 2$. Hence, $a \in \{1, \ldots, 13\} \cup \{7 \pm x_1\}$ by Lemma 4.9. We will prove that we have $\overline{|b|} < 1.2$. Assume, for the sake of contradiction, that this is not the case. We will see that for whatever choice of a, there is an embedding σ such that x^σ is either negative or larger than 14, contradicting our hypothesis.

- Assume first $a \in \{1,2,3,4,5,6\}$: We choose σ such that $x^\sigma = a - |b|x_2$, so that we have
$$x^\sigma = a - |b|x_2 \leq 6 - x_2 < 0.$$
- Assume $a \in \{8,9,10,11,12,13\}$: We choose σ such that $x^\sigma = a + |b|x_2$, so that we have
$$(a + bx_2)^\sigma = a + |b|x_2 \geq 8 + x_2 > 14.$$
- Assume $a = 7 + x_1$: We choose σ such that $x^\sigma = a + |b|x_2$, so that we have
$$a + \overline{|b|}x_2 \geq 7 + x_1 + x_2 > 14.$$
- Assume $a = 7 - x_1$: We choose σ such that $x^\sigma = a - |b|x_2$, so that we have
$$a - \overline{|b|}x_2 \leq 7 - x_1 - x_2 < 0.$$
- Assume $a = 7$. We choose σ such that $x^\sigma = a - \overline{|b|}x_2$, so that we have
$$a + |b|x_2 \geq 7 + 1.2x_2 \geq 7 + 1.2\sqrt{20+3\sqrt{41}} > 14.$$

We conclude $\overline{|b|} < 1.2$.

Write $b = b_1 + b_2x_1$, with $b_1, b_2 \in \mathbb{Z}$, so that
$$\overline{|b_1 + b_2x_1|} < 1.2.$$

Hence in particular, we have $|b_1| < 1.2$ and $|b_2| < \frac{1.2}{\sqrt{41}}$. The only choices for b_1 and b_2 are $(b_1, b_2) \in \{(-1,0),(0,0),(1,0)\}$. Therefore, if $a \in \{1,\ldots,6\} \cup \{8,\ldots,13\} \cup \{7 \pm x_1\}$, then $b = 0$ by the first four cases above. Otherwise, if $a = 7$, then we can have either $x = 7 - x_2$, or $x = 7 + x_2$, or $x = 7$. In all cases we obtain $x \in X_2$. □

Lemma 4.11. *Assume* $x_1 < \lfloor\alpha\rfloor+1$ *and* $\nu \neq 19$*. For each* $n \geq 0$*, if* $x \in \mathcal{O}_n$ *and* $0 \ll x \ll 2\lfloor\alpha\rfloor+2$*, then* $x \in X_n$*.*

Proof. If $\nu - 3x_1 \geq 1$, then we are done by Lemma 4.2. Assume $\nu - 3x_1 < 1$. By Lemma 4.8, the only cases where $\nu - 3x_1 < 1$ are when

$$(\nu, x_0) \in \{(20,7),(20,8),(21,8),(21,9),(22,10),(23,11)\}$$

(see Table 1). However, when $(\nu, x_0) \in \{(22,10),(23,11)\}$, a simple calculation shows that $x_1 > \lfloor\alpha\rfloor + 1$, so we may assume $(\nu, x_0) \in \{(20,7),(20,8),(21,8),(21,9)\}$. We will prove by induction on n that if $x \in \mathcal{O}_n$ is such that $0 \ll x \ll 2\lceil\alpha\rceil + 2$, then $x \in X_n$. It is clear for $n = 0$. For $n = 1$ and $n = 2$ we are done by Lemmas 4.9 and 4.10 respectively. Assume $n \geq 3$. By Lemmas 2.15 and 4.8 we have

$$|b^\sigma| < \frac{\lfloor\alpha\rfloor + 1}{\sqrt{\nu - 3x_{n-1}}} \leq \frac{\lfloor\alpha\rfloor + 1}{\sqrt{\nu - 3x_2}} \leq \lfloor\alpha\rfloor + 1$$

for every $n \geq 3$. The rest of the proof goes exactly as the proof of [7, Lemma 3.2]. □

Lemma 4.12. *Assume* $x_1 < \lfloor\alpha\rfloor + 1$ *and* $\nu \neq 19$*. Let* $x \in \mathcal{O}$*. We have* $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$ *if and only if* $x \in X$*.*

Proof. By Lemma 4.11, we need only to prove the lemma from right to left. Let $x \in X$. If $x \in X_0$, then there is nothing to prove. Assume $x \in X_n$ for some $n \geq 1$, so that $x = \lfloor\alpha\rfloor + 1 \pm x_k$ for some $1 \leq k \leq n$. Since the sequence (x_n) is decreasing, we have

$$\lfloor\alpha\rfloor + 1 + x_k < 2\lfloor\alpha\rfloor + 2,$$

and

$$\lfloor\alpha\rfloor + 1 - x_k > 0$$

for every $1 \leq k \leq n$. Therefore, we have $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$ since $\overline{|x_k|} = x_k$ by Theorem 2.7. □

Proof Theorem 4.6. We will prove that $\lfloor\alpha\rfloor + \alpha + 1$ is the JR number of $\mathcal{O}$ and that it satisfies the isolation property. Since (x_n) is a decreasing sequence and converges to α, for every $\varepsilon > 0$ there exist infinitely many n such that

$$x_n + \lfloor\alpha\rfloor + 1 < \lfloor\alpha\rfloor + \alpha + 1 + \varepsilon.$$

So, by Lemma 4.12 and Theorem 2.7, for every $\varepsilon > 0$, there exist infinitely many $x \in \mathcal{O}$ such that $0 \ll x \ll \lfloor\alpha\rfloor + \alpha + 1 + \varepsilon$. Also, for each $n \geq 1$, we have $\lfloor\alpha\rfloor + 1 + x_n > \lfloor\alpha\rfloor + 1 + \alpha$. Hence, if $x \in \mathcal{O}$ is such that $0 \ll x \ll \lfloor\alpha\rfloor + \alpha + 1$, by Lemma 4.12, then we have $x \in \{1, \ldots, 2\lfloor\alpha\rfloor + 1\}$.

Therefore, $\lfloor\alpha\rfloor + \alpha + 1$ is the JR number of $\mathcal{O}$, and it is not a minimum. We now show that it satisfies the isolation property. Let $M = \lfloor\alpha\rfloor + 1 - \alpha$ and $x \in \mathcal{O}$ be such that

$$0 \ll x \ll \mathrm{JR}(\mathcal{O}) + M = 2\lfloor\alpha\rfloor + 2.$$

By Lemma 4.12, we have

$$x \in \{1, 2, \ldots, 2\lfloor\alpha\rfloor + 1\} \cup \{\lfloor\alpha\rfloor + 1 \pm x_n : n \geq 1\}$$

and since (x_n) is decreasing with limit α, we have

$$\lfloor\alpha\rfloor + 1 + x_n \geq \lfloor\alpha\rfloor + 1 + \alpha + \varepsilon$$

for only finitely many n. □

References

[1] M. Castillo, "On the Julia Robinson number of rings of totally real algebraic integers in some towers of nested square roots," Ph.D. dissertation, Universidad de Concepción, 2018, Available: http://dmat.cfm.cl/dmat/wp-content/uploads/2018/05/castillomarianela_prog4208.pdf.

[2] P. Gillibert and G. Ranieri, "Julia Robinson numbers," *Int. J. Number Theory*, vol. 15, no. 8, pp. 1565–1599, 2019, doi: 10.1142/S1793042119500908.

[3] M. Jarden and C. R. Videla, "Undecidability of families of rings of totally real integers," *Int. J. Number Theory*, vol. 4, no. 5, pp. 835–850, 2008, doi: 10.1142/S1793042108001705.

[4] F. Pazuki, N. Technau, and M. Widmer, "Northcott numbers for the house and the Weil height," *Bull. Lond. Math. Soc.*, vol. 54, no. 5, pp. 1873–1897, 2022, doi: 10.1090/S0002-9939-2015-12592-0.

[5] J. Robinson, "The undecidability of algebraic rings and fields," *Proc. Amer. Math. Soc.*, vol. 10, pp. 950–957, 1959, doi: 10.1090/S0002-9939-2015-12592-0.

[6] J. Robinson, "On the decision problem for algebraic rings," in *Studies in mathematical analysis and related topics*, ser. Stanford Studies in Mathematics and Statistics. Stanford Univ. Press, Stanford, CA, 1962, vol. IV, pp. 297–304.

[7] X. Vidaux and C. R. Videla, "Definability of the natural numbers in totally real towers of nested square roots," *Proc. Amer. Math. Soc.*, vol. 143, no. 10, pp. 4463–4477, 2015, doi: 10.1090/S0002-9939-2015-12592-0.