Cubo A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 26, no. 3, pp. 407–430, December 2024
DOI: 10.56754/0719-0646.2603.407

Fractional Sobolev space: Study of Kirchhoff-Schrödinger systems with singular nonlinearity

Elhoussain Arhrrabi[1,2]
Hamza El-Houari[1,✉]

[1] *Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Faculty of Sciences and Technology, Béni Mellal, Morocco.*
arhrrabi.elhoussain@gmail.com

[2] *ENSI - School of New Sciences and Engineering, Tangier, Morocco.*
h.elhouari94@gmail.com✉

ABSTRACT

This study extensively investigates a specific category of Kirchhoff-Schrödinger systems in fractional Sobolev space with Dirichlet boundary conditions. The main focus is on exploring the existence and multiplicity of non-negative solutions. The non-linearity of the problem generally exhibits singularity. By employing minimization arguments involving the Nehari manifold and a variational approach, we establish the existence and multiplicity of positive solutions for our problem with respect to the parameters η and ζ in suitable fractional Sobolev spaces. Our key findings are novel and contribute significantly to the literature on coupled systems of Kirchhoff-Schrödinger system with Dirichlet boundary conditions.

RESUMEN

Este estudio investiga en detalle una categoría específica de sistemas de Kirchhoff-Schrödinger en espacios de Sobolev fraccionarios con condiciones de borde de Dirichlet. El objetivo principal es explorar la existencia y multiplicidad de soluciones no-negativas. La no-linealidad del problema generalmente exhibe singularidades. Empleando argumentos de minimización que involucran la variedad de Nehari y un enfoque variacional, establecemos la existencia y multiplicidad de soluciones positivas para nuestro problema con respecto a los parámetros η y ζ en espacios de Sobolev fraccionarios apropiados. Nuestros hallazgos principales son novedosos y contribuyen significativamente a la literatura de sistemas de Kirchhoff-Schrödinger acoplados con condiciones de borde de Dirichlet.

Keywords and Phrases: Fractional p-Laplacian operator, Kirchhoff-Schrödinger system, Nehari manifold, fibering map approach.

2020 AMS Mathematics Subject Classification: 35R11, 35A15, 47G20.

Published: 08 October, 2024
Accepted: 04 September, 2024
Received: 18 January, 2024

1 Introduction

The Kirchhoff-Schrödinger problem is a class of partial differential equations that combines aspects of Kirchhoff and Schrödinger equations. It takes the form:

$$M\left(\int_Q |\nabla u|^2 dx\right)(-\Delta u) + V(x)u = f(x,u),$$

where M is a function representing the Kirchhoff-type nonlinearity, $V(x)$ is a potential function, and $f(x,u)$ denotes the nonlinearity in the system. This type of system generalizes the classical Schrödinger equation by incorporating a nonlinear term dependent on the integral of the gradient, reflecting the influence of the entire domain on the local behavior of the solution.

The Kirchhoff-Schrödinger system arises in various physical contexts, such as the study of quantum mechanical systems, nonlinear optics, and the dynamics of elastic strings and membranes. These systems are particularly challenging due to their nonlocal nature and the potential presence of singularities in the nonlinearity $f(x,u)$, which can complicate both theoretical analysis and numerical simulations.

In this study, we consider the following fractional Kirchhoff-Schrödinger equations with singular nonlinearity,

$$\begin{cases} \mathbf{K}\left(\int_Q V(\kappa)|w|^p d\kappa + \int_{Q\times Q} \frac{|w(\kappa)-w(y)|^p}{|\kappa-y|^{d+sp}} d\kappa\, dy\right)\left[(-\Delta)_p^s w + V(\kappa)|w|^{p-2}w\right] \\ \qquad = \eta\alpha(\kappa)|w|^{q-2}u + \frac{1-\varrho}{2-\varrho-\tau}\xi(\kappa)|w|^{-\varrho}|v|^{1-\tau}, \qquad \text{in } Q, \\ \mathbf{K}\left(\int_Q V(\kappa)|v|^p d\kappa + \int_{Q\times Q} \frac{|v(\kappa)-v(y)|^p}{|\kappa-y|^{d+sp}} d\kappa\, dy\right)\left[(-\Delta)_p^s v + V(\kappa)|v|^{p-2}v\right] \\ \qquad = \zeta\beta(\kappa)|v|^{q-2}v + \frac{1-\tau}{2-\varrho-\tau}\xi(\kappa)|w|^{1-\varrho}|v|^{-\tau}, \qquad \text{on } Q, \\ w = v = 0, \qquad \text{on } \mathbb{R}^d \setminus Q, \end{cases} \tag{1.1}$$

where $Q \subset \mathbb{R}^d$ $(d \geq 3)$ is a bounded domain with smooth boundary, $s \in (0,1)$, $0 < \tau < 1$, $0 < \varrho < 1$, $d > ps$, $2-\varrho-\tau < p \leq p\sigma < q < p_s^* = \frac{dp}{d-sp}$, $\alpha, \beta, \xi \in C(\overline{Q})$ are non-negative weight functions, η, ζ are two parameters, $(-\Delta)_p^s$ is the fractional p-Laplacian operator defined as (see [10])

$$(-\Delta)_p^s w(\kappa) = 2\lim_{\epsilon\searrow 0}\int_{Q\setminus B_\epsilon} \frac{|w(\kappa)-w(y)|^{p-2}(w(\kappa)-w(y))}{|\kappa-y|^{d+sp}}\,\mathrm{d}y, \quad \kappa\in\mathbb{R}^d,$$

and $\mathbf{K} : (0,+\infty) \longrightarrow (0,+\infty)$ is the continuous Kirchhoff function defined by

$$\mathbf{K}(t) = k + lt^{\sigma-1} \quad \text{with } k>0,\ l,\ \sigma \geq 1. \tag{1.2}$$

Recently, there has been a lot of interest in examining non local problems of this kind. For an interesting one, we refer to learn more about Kirchhoff problems, specifically those dealing with the Laplace operator and a singular term, in references like [19–22]. Additionally, the study of the fractional Kirchhoff problem, which involves a singular term like $u^{-\gamma}$, can be found in [14]. This research combines a variational approach with a specific truncation argument. For more details on the fractional system, you can check out [23, 36].

These problems involve studying how things spread unevenly in complicated environments. This happens because of random movements, like jumps, where entities can move to nearby places or make longer trips using a specific kind of flight pattern called Lévy flights. These issues are also used to model things like turbulence, chaotic movements, plasma physics, and financial dynamics. Check [1, 7] and references therein for more information.

The system expressed in (1.1) without a Kirchhoff function and potential function has been thoroughly explored in recent years. For the case involving the fractional p-Laplacian, the existence results have been investigated using Morse theory, as discussed in [18]. Perera-Squassina-Yang [25] introducing a novel abstract result based on a pseudo-index associated with the $\mathbb{Z}_2$-cohomological index. These constraints are employed to establish the existence within a certain range of the Palais-Smale condition. It is worth noting that, in this study, bifurcation and multiplicity results are obtained with specific limitations on the parameter η. Additionally, the investigation into the multiplicity of solutions is conducted through the Nehari manifold and fibering maps in works like [6, 15, 29, 31].

In a distinct context, the investigation of the problem was undertaken in [6].

$$\begin{cases} (-\Delta)_p^s u = \eta|w|^{q-2}u + \dfrac{2\varrho}{\varrho+\tau}|w|^{\varrho-2}u|v|^{\tau}, \quad \text{in } Q, \\ \\ (-\Delta)_p^s v = \zeta|v|^{q-2}v + \dfrac{2\tau}{\varrho+\tau}|w|^{\varrho}|v|^{\tau-2}v, \quad \text{in } Q, \\ \\ u = v = 0, \quad \text{on } \mathbb{R}^d \setminus Q, \end{cases}$$

where Q is a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂Q, $d > sp$, $s \in (0,1)$, $p < \varrho + \tau < p_s^*$, η, ζ are two parameters. The scholars investigated the Nehari manifold associated with the problem, employing fibering maps, and established the existence of solutions under certain conditions for the parameter pair (η, ζ).

The problem expressed in (1.1) without a Kirchhoff coefficient has been thoroughly explored in recent years. For the case involving the fractional p-Laplacian, the existence results have been investigated using Morse theory, as discussed in [24]

$$\begin{cases} (-\Delta)_p^s u + |w|^{q-2} u = \dfrac{H_w(\kappa, u, v)}{|\kappa|^\gamma}, & \text{in } \mathbb{R}^d, \\ \\ (-\Delta)_p^s v + |w|^{q-2} u = \dfrac{H_w(\kappa, u, v)}{|\kappa|^\gamma} & \text{in } \mathbb{R}^d, \end{cases}$$

where $d \geq 1$, $0 < s < 1$, $d = ps$, $\gamma \in (0, d)$ and H has exponential growth. By using a version of the mountain pass theorem without (PS) condition, they established the existence of nontrivial solution to the above system. In [35] the authors studied the existence of solutions to the following quasi linear Schrödinger system

$$\begin{cases} (-\Delta)_p^s u + \alpha(\kappa)|w|^{q-2} u = H_w(\kappa, u, v) & \text{in } \mathbb{R}^d, \\ \\ (-\Delta)_p^s v + \beta(\kappa)|w|^{q-2} u = H_w(\kappa, u, v) & \text{in } \mathbb{R}^d, \end{cases}$$

where $1 < q \leq p$, $sp < d$, they used the critical approach, to obtain the existence of nontrivial and non negative solutions for the above system.

Following this, the issue has been explored by various authors in the context of Laplacian, p-Laplacian, and fractional N-Laplacian operators, employing either the technique employed in this paper or employing critical point methods. Noteworthy references encompass [2, 4, 5, 9, 12, 16, 28, 30, 34].

Motivated by the results above, by using minimization arguments and implicit function theorem together with variational approach, we prove the existence and multiplicity of nontrivial, non-negative solutions for the singular fractional Kirchhoff-Schrödinger system described in (1.1) within suitable fractional Sobolev spaces.

This paper is organized as follow: In the second section, we discuss familiar properties and results related to fractional Sobolev spaces. In the third section, we show the existence theorem and its proof, which uses the Nehari manifold and fibering map approach. In the fourth section, we demonstrate the existence of multiple nontrivial positive solutions for our problem (1.1).

2 Preliminaries

In this paper, $Q \subset \mathbb{R}^d$ represents a bounded domain with a smooth boundary, and $\langle \cdot, \cdot \rangle$ denotes the standard duality between X and its dual space X^*.

Let $u : Q \times Q \longrightarrow \mathbb{R}$ be a measurable function,

$$[w]_{s,p} = \left(\int_{Q\times Q} \frac{|w(\kappa) - w(y)|^p}{|\kappa - y|^{d+ps}} d\kappa \, dy \right)^{1/p},$$

is the Gagliardo seminorm. We denote by $\mathbb{W}^{s,p}(Q)$ the fractional Sobolev space given by

$$\mathbb{W}^{s,p}(Q) := \{u \in \mathbb{L}^p(Q) : \ [w]_{s,p} < \infty\},$$

with the norm

$$\|w\|_{s,p} := \left(\|w\|^p_{\mathbb{L}^p(Q)} + [w]^p_{s,p} \right)^{1/p},$$

where

$$\|w\|_{\mathbb{L}^p(Q)} = \left(\int_Q |w|^p d\kappa \right)^{1/p}.$$

For our analysis, we assume the following assumption:

(V) $V \in \mathbb{L}^\infty_{loc}(Q)\backslash\{0\}$, ess $\inf_{\kappa\in Q} V(\kappa) > 0$ and meas$(\{x \in q : V(x) \leq L\}) < \infty$, for all $L > 0$,

where meas$(\cdot)$ denotes the Lebesgue measure in Q.

When V satisfies **(V)**, the basic space

$$\mathbb{W}_s(Q) := \left\{ w \in \mathbb{W}^{s,p}(Q) : \ V|w|^p \in \mathbb{L}^1(Q); \quad u = 0 \ \text{ in } \ \mathbb{R}^d\backslash Q \right\}$$

denotes the completion of $C_0^\infty(Q)$ with respect to the norm

$$\|w\|_{\mathbb{W}_s} := \left(\|w\|^p_{\mathbb{L}^p(V,Q)} + [w]^p_{s,p} \right)^{1/p},$$

where

$$\|w\|_{\mathbb{L}^p(V,Q)} = \left(\int_Q V(\kappa)|w|^p d\kappa \right)^{1/p}.$$

In $\mathbb{W}_s$ we have the following embedding

Lemma 2.1 ([33])**.** *Let $0 < s < 1 < p < +\infty$ with $ps < d$ and suppose that the assumption **(V)** holds. Then,*

$$\mathbb{W}_s(Q) \hookrightarrow \mathbb{L}^q(Q) \quad \textit{for all } q \in [p, p_s^*). \tag{2.1}$$

When $r+r' \in (p,p^*)$, then, for any $u \in \mathbb{W}_s$, we obtain

$$\|w\|_{\mathbb{L}^{r+r'}(Q)} \leq S\|w\|_{\mathbb{W}_s}. \tag{2.2}$$

Let us define the functional $\Psi_{s,p} : \mathbb{W}_s \longrightarrow \mathbb{R}$ by

$$\Psi_{s,p}(w) = \int_{Q\times Q} \frac{|w(\kappa)-w(y)|^p}{|\kappa-y|^{d+ps}} d\kappa\, dy + \int_Q V(\kappa)|w|^p d\kappa.$$

At this point, we introduce our working space $\mathbb{W} = \mathbb{W}_s \times \mathbb{W}_s$, which is a reflexive Banach space endowed with the norm

$$\|(w,v)\|_{\mathbb{W}} = \Big(\Psi_{s,p}(w) + \Psi_{s,p}(v)\Big)^{1/p}. \tag{2.3}$$

We say that $(w,v) \in \mathbb{W}$ is a weak solution to system (1.1) if $u,v>0$ in Q, one has

$$\begin{aligned}
&\mathbf{K}\big(\|w\|_{\mathbb{W}_s}\big)\left(\int_Q V(\kappa)|w|^{p-2}u\phi d\kappa + \int_Q \frac{|w(\kappa)-w(y)|^{p-2}(w(\kappa)-w(y))(\phi(\kappa)-\phi(y))}{|\kappa-y|^{d+sp}}\, d\kappa\, dy\right)\\
&+\mathbf{K}\big(\|v\|_{\mathbb{W}_s}\big)\left(\int_Q V(\kappa)|v|^{p-2}v\phi d\kappa + \int_Q \frac{|v(\kappa)-v(y)|^{p-2}(v(\kappa)-v(y))(\psi(\kappa)-\psi(y))}{|\kappa-y|^{d+sp}}\, d\kappa\, dy\right)\\
&= \int_Q \big(\eta\alpha(\kappa)|w|^{q-2}u\phi + \zeta\beta(\kappa)|v|^{q-2}v\phi\big)d\kappa + \frac{1-\varrho}{2-\varrho-\tau}\int_Q \xi(\kappa)u^{-\varrho}v^{1-\tau}\psi\, d\kappa\\
&\quad + \frac{1-\tau}{2-\varrho-\tau}\int_Q \xi(\kappa)u^{1-\varrho}v^{-\tau}\psi\, d\kappa,
\end{aligned}$$

for all $(\phi,\psi) \in \mathbb{W}$.

Now, with the essential tools in place, we are ready to state our main results, which take the following form:

Theorem 2.2. *There exists*

$$\Lambda_0 = \left(\frac{q+\varrho+\tau-2}{\|\zeta\|_\infty k(q-p)}\right)^{\frac{p}{p+\varrho+\tau-2}} \left(\frac{2-\varrho-\tau-q}{k(2-\varrho-\tau-p)}|Q|^{\frac{p_s^*-q}{p_s^*}}\right)^{-\frac{p}{p-q}} S^{\frac{2-\varrho-\tau}{p+\varrho+\tau-2}},$$

such that if

$$0 < \big(\eta\|\alpha\|_\infty\big)^{\frac{p}{p-q}} + \big(\zeta\|\beta\|_\infty\big)^{\frac{p}{p-q}} < \Lambda_0,$$

then system (1.1) *has at least two nontrivial positive solutions.*

3 Nehari manifold & fibering map analysis

In this part, we gather basic information about a Nehari manifold and discuss fibering maps.

Obviously, the energy functional $\mathfrak{J}_{\eta,\zeta} : \mathbb{W}_s \longrightarrow \mathbb{R}$ associated with problem (1.1) is given by

$$\mathfrak{J}_{\eta,\zeta}(w,v) = \frac{1}{p}\Big(\hat{\mathbf{K}}(\|w\|^p_{\mathbb{W}_s}) + \hat{\mathbf{K}}(\|v\|^p_{\mathbb{W}_s})\Big) - \frac{1}{q}\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)d\kappa \\ - \frac{1}{2-\varrho-\tau}\int_Q \xi(\kappa)(w^+)^{1-\varrho}(v^+)^{1-\tau}\,d\kappa,$$

where $\hat{\mathbf{K}}(t) = \int_0^t \mathbf{K}(\varrho)d\varrho$. This together with (1.2) gets to

$$\begin{aligned}\mathfrak{J}_{\eta,\zeta}(w,v) &= \frac{k}{p}\Big(\Psi_{s,p}(w) + \Psi_{s,p}(v)\Big)^p + \frac{l}{p\sigma}\Big(\Psi_{s,p}(w) + \Psi_{s,p}(v)\Big)^{p\sigma} \\ &\quad - \frac{1}{q}\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)d\kappa - \frac{1}{2-\varrho-\tau}\int_Q \xi(\kappa)(w^+)^{1-\varrho}(v^+)^{1-\tau}\,d\kappa \\ &= \frac{k}{p}\|(w,v)\|^p_{\mathbb{W}} + \frac{l}{p\sigma}\|(w,v)\|^{p\sigma}_{\mathbb{W}} - \frac{1}{q}\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)d\kappa \\ &\quad - \frac{1}{2-\varrho-\tau}\int_Q \xi(\kappa)(w^+)^{1-\varrho}(v^+)^{1-\tau}\,d\kappa, \end{aligned} \tag{3.1}$$

where $r^+ = \max\{r,0\}$ and $r^- = \max\{-r,0\}$ for $r \in \mathbb{R}$.

Keep in mind that $\mathfrak{J}_{\eta,\zeta}$ does not behave smoothly in $\mathbb{W}$. So, standard variational methods will not work here. If (w,v) is a weak solution for the problem (1.1), it means that both w and v are positive in Q and satisfy the equation

$$\mathbf{K}\big(\|w\|_{\mathbb{W}_s}\big)\Psi_{s,p}(w) + \mathbf{K}\big(\|v\|_{\mathbb{W}_s}\big)\Psi_{s,p}(\mathbf{V}) - \eta\int_Q \alpha(\kappa)|w|^q\,d\kappa \\ - \zeta\int_Q \beta(\kappa)|v|^q\,d\kappa - \int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa = 0,$$

which implies by using (1.2) that

$$k\|(w,v)\|^p_{\mathbb{W}} + l\|(w,v)\|^{p\sigma}_{\mathbb{W}} - \eta\int_Q \alpha(\kappa)|w|^q\,d\kappa - \zeta\int_Q \beta(\kappa)|v|^q\,d\kappa - \int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa = 0. \tag{3.2}$$

It is simple to confirm that the energy functional $\mathfrak{J}_{\eta,\zeta}(w,v)$ is not bounded below in the space $\mathbb{W}$. However, we will demonstrate that on the Nehari manifold, defined below, $\mathfrak{J}_{\eta,\zeta}(w,v)$ is bounded below. We will establish a solution by minimizing this functional over specific subsets. The Nehari manifold is defined as follows:

$$
\begin{aligned}
\mathbf{N}_{\eta,\zeta} = \Bigg\{ &(w,v) \in \mathbb{W} \setminus \{(0,0)\};\ \frac{k}{p}\|(w,v)\|_{\mathbb{W}}^p + \frac{l}{k}\|(w,v)\|_{\mathbb{W}}^{p\sigma} - \eta \int_Q \alpha(\kappa)|w|^q \, d\kappa \\
&-\zeta \int_Q \beta(\kappa)|v|^q \, d\kappa - \int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau} \, d\kappa = 0 \Bigg\}.
\end{aligned}
$$

Now, understanding that the Nehari manifold is intricately connected to a fibering maps which is introduced by Drábek and Pohozaev in [11]. The form of the fibrering maps is as follows, $\Upsilon_{w,v} : t \mapsto \mathfrak{J}_{\eta,\zeta}(tw,tv)$ for $t > 0$ defined by

$$
\begin{aligned}
\Upsilon_{w,v}(t) = &\frac{1}{p}\left(\hat{\mathbf{K}}\big(t^P ||w||_{\mathbb{W}_s}^p\big) + \hat{\mathbf{K}}\big(t^p ||v||_{\mathbb{W}_s}^p\big)\right) - \frac{t^q}{q}\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\, d\kappa \\
&- \frac{t^{2-\varrho-\tau}}{2-\varrho-\tau}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\, d\kappa.
\end{aligned}
$$

The first and second derivative of Υ respectively, is given by

$$
\begin{aligned}
\Upsilon'_{w,v}(t) = &kt^{p-1}\|(w,v)\|_{\mathbb{W}}^p + lt^{p\sigma-1}\|(w,v)\|_{\mathbb{W}}^{p\sigma} - t^{q-1}\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\, d\kappa \\
&- t^{1-\varrho-\tau}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\, d\kappa
\end{aligned} \tag{3.3}
$$

and

$$
\begin{aligned}
\Upsilon''_{w,v}(t) = &(p-1)kt^{p-2}\|(w,v)\|_{\mathbb{W}}^p + l(p\sigma-1)t^{p\sigma-2}\|(w,v)\|_{\mathbb{W}}^{p\sigma} \\
&- (q-1)t^{q-2}\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\, d\kappa \\
&- (1-\varrho-\tau)t^{-\varrho-\tau}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\, d\kappa.
\end{aligned} \tag{3.4}
$$

Now, we prove some useful inequality. Using Hölder's and Sobolev inequalities, one has

$$
\begin{aligned}
\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big) d\kappa &\le |Q|^{\frac{p_s^*-q}{p_s^*}} \Big(\eta\|\alpha\|_\infty \|w\|_{p_s^*}^q + \zeta\|\beta\|_\infty \|v\|_{p_s^*}^q\Big) \\
&\le |Q|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}} \Big(\eta\|\alpha\|_\infty \|w\|^q + \zeta\|\beta\|_\infty \|v\|^q\Big) \\
&\le |Q|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}} \Big((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + \zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}} (\|w\|^q + \|v\|^q) \\
&\le C|Q|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}} \Big((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}} \|(w,v)\|_{\mathbb{W}}^q
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\, d\kappa &\le \|\zeta\|_\infty \left(\frac{1-\varrho}{2-\varrho-\tau}\int_Q |w|^{2-\varrho-\tau}\, d\kappa + \frac{1-\tau}{2-\varrho-\tau}\int_Q |v|^{2-\varrho-\tau}\, d\kappa\right) \\
&\le \|\zeta\|_\infty S^{-\frac{2-\varrho-\tau}{p}} \|(w,v)\|_{\mathbb{W}}^{2-\varrho-\tau}.
\end{aligned} \tag{3.6}
$$

Lemma 3.1. *Let $(w,v) \in \mathbb{W} \setminus \{(0,0)\}$. Then $(tw,tv) \in \mathbf{N}_{\eta,\zeta}$ if and only if $\Upsilon'_{w,v}(t) = 0$.*

Proof. The conclusion is derived from the observation that

$$\begin{aligned}\Upsilon'_{w,v}(t) &= \langle \mathfrak{J}'_{\eta,\zeta}(w,v),(w,v)\rangle \\ &= kt^{p-1}\|(w,v)\|^p_{\mathbb{W}} + lt^{p\sigma-1}\|(w,v)\|^{p\sigma}_{\mathbb{W}} - t^{q-1}\left(\int_Q \eta\alpha(\kappa)|w|^q\,d\kappa - \int_Q \zeta\beta(\kappa)|v|^q\,d\kappa\right) \\ &\quad - t^{1-\varrho-\tau}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa = 0\end{aligned}$$

if and only if $(tw,tv) \in \mathbf{N}_{\eta,\zeta}$. □

Due to Lemma 3.1, we have $(w,v) \in \mathbf{N}_{\eta,\zeta}$ are associated with stationary points of $\Upsilon_{w,v}(tw,tv)$ and in particular, $(w,v) \in \mathbf{N}_{\eta,\zeta}$ if and only if $\Upsilon'_{w,v}(1) = 0$. Hence, we split $\mathbf{N}_{\eta,\zeta}$ into three parts:

$$\begin{aligned}\mathbf{N}^+_{\eta,\zeta} &= \Big\{(w,v) \in \mathbf{N}_{\eta,\zeta} : \Upsilon''_{w,v}(1) > 0\Big\} = \Big\{(tw,tv) \in \mathbb{W}\setminus\{0,0\} : \Upsilon'_{w,v}(t) = 0, \Upsilon''_{w,v}(t) > 0\Big\},\\ \mathbf{N}^-_{\eta,\zeta} &= \Big\{(w,v) \in \mathbf{N}_{\eta,\zeta} : \Upsilon''_{w,v}(1) < 0\Big\} = \Big\{(tw,tv) \in \mathbb{W}\setminus\{0,0\} : \Upsilon'_{w,v}(t) = 0, \Upsilon''_{w,v}(t) < 0\Big\},\\ \mathbf{N}^0_{\eta,\zeta} &= \Big\{(w,v) \in \mathbf{N}_{\eta,\zeta} : \Upsilon''_{w,v}(1) = 0\Big\} = \Big\{(tw,tv) \in \mathbb{W}\setminus\{0,0\} : \Upsilon'_{w,v}(t) = 0, \Upsilon''_{w,v}(t) = 0\Big\}.\end{aligned}$$

For the proof of the following lemma we refer to [32].

Lemma 3.2. *If (w,v) is a minimizer of $\mathfrak{J}_{\eta,\zeta}$ on $\mathbf{N}_{\eta,\zeta}$ such that $(w,v) \notin \mathbf{N}^0_{\eta,\zeta}$. Then, (w,v) is a critical point for $\mathfrak{J}_{\eta,\zeta}$.*

Our initial result is as follows:

Lemma 3.3. *$\mathfrak{J}_{\eta,\zeta}$ is bounded below on $\mathbf{N}_{\eta,\zeta}$ and coercive.*

Proof. As $(w,v) \in \mathbf{N}_{\eta,\zeta}$, then using (3.2) and the embedding of $\mathbb{W}_s$ in $\mathbb{L}^{2-\varrho-\tau}(Q)$, we obtain

$$\mathfrak{J}_{\eta,\zeta}(w,v) = k\left(\frac{1}{p}-\frac{1}{q}\right)\|(w,v)\|^p_{\mathbb{W}} + l\Big(\frac{1}{p\sigma}-\frac{1}{q}\Big)\|(w,v)\|^{p\sigma}_{\mathbb{W}} - \left(\frac{1}{2-\varrho-\tau}-\frac{1}{q}\right)\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa.$$

Then by (3.6), we obtain

$$\begin{aligned}\mathfrak{J}_{\eta,\zeta}(w,v) &\geq k\left(\frac{1}{p}-\frac{1}{q}\right)\|(w,v)\|^p_{\mathbb{W}} + l\left(\frac{1}{p\sigma}-\frac{1}{q}\right)\|(w,v)\|^{p\sigma}_{\mathbb{W}}\\ &\quad - \left(\frac{1}{2-\varrho-\tau}-\frac{1}{q}\right)\|\zeta\|_\infty S^{-\frac{2-\varrho-\tau}{2}}\|(w,v)\|^{2-\varrho-\tau}_{\mathbb{W}}.\end{aligned}$$

Since $2-\varrho-\tau < p \leq p\sigma$, it follows that $\mathfrak{J}_{\eta,\zeta}$ is coercive and bounded below on $\mathbf{N}_{\eta,\zeta}$. □

Lemma 3.4. *For every $(w,v) \in \mathbf{N}^-_{\eta,\zeta}$ (respectively $\mathbf{N}^+_{\eta,\zeta}$) with $u,v \geq 0$, and all $(\phi,\psi) \in \mathbf{N}_{\eta,\zeta}$ with $(\phi,\psi) \geq 0$, there exist $\varepsilon > 0$ and a continuous function $h = h(r) > 0$ such that for all $r \in \mathbb{R}$ with $|r| < \varepsilon$ we have $h(0) = 1$ and $h(r)(w + r\phi, v + r\psi) \in \mathbf{N}^-_{\eta,\zeta}$ (respectively $\mathbf{N}^+_{\eta,\zeta}$).*

Proof. First, let us introduce the function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} f(t,r) &= kt^{p+\varrho+\tau-2}\|(w+r\phi, v+r\psi)\|^p + lt^{p\sigma+\varrho+\tau-2}\|(w+r\phi, v+r\psi)\|^{p\sigma} \\ &\quad - (q+\varrho+\tau-2)t^{q+\varrho+\tau-3}\int_Q \Big(\eta\alpha(\kappa)(w+r\phi)^q + \zeta\beta(\kappa)(v+r\psi)^q\Big)d\kappa \\ &\quad - \int_Q \xi(\kappa)(w+r\phi)^{1-\varrho}(v+r\psi)^{1-\tau}d\kappa. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{df}{dt}(t,r) &= k\big(p+\varrho+\tau-\big)t^{p+\varrho+\tau-3}\|(w+r\phi, v+r\psi)\|^p \\ &\quad + l\big(p\sigma+\varrho+\tau-2\big)t^{p\sigma+\varrho+\tau-3}\|(w+r\phi, v+r\psi)\|^{p\sigma} \\ &\quad - t^{q+\varrho+\tau-2}\int_Q \Big(\eta\alpha(\kappa)(w+r\phi)^q + \zeta\beta(\kappa)(v+r\psi)^q\Big)d\kappa. \end{aligned}$$

Hence, $\frac{df}{dt}$ is continuous. Recall that $(w,v) \in \mathbf{N}^-_{\eta,\zeta} \subset \mathbf{N}_{\eta,\zeta}$, we have $f(1,0) = 0$, and

$$\begin{aligned} \frac{df}{dt}(1,0) &= k\big(p+\varrho+\tau-2\big)\|(w,v)\|^p_{\mathbb{W}} + l\big(p\sigma+\varrho+\tau-2\big)\|(w,v)\|^{p\sigma}_{\mathbb{W}} \\ &\quad - \big(q+\varrho+\tau-2\big)\int_Q \Big(\eta\alpha(\kappa)w^q + \zeta\beta(\kappa)v^q\Big)d\kappa < 0. \end{aligned}$$

Thus, by applying the implicit function theorem to the function f at the point $(1,0)$, we deduce the existence of $\delta > 0$ and a positive continuous function $h = h(r) > 0$, defined for $r \in \mathbb{R}$ with $|r| < \delta$, satisfying:

$$h(0) = 1 \quad \text{and} \quad h(r)(w+r\phi, v+r\psi) \in \mathbf{N}_{\eta,\zeta}, \quad \text{for all } r \in \mathbb{R}, \quad |r| < \delta.$$

Hence, for a small possible $\varepsilon > 0$ ($\varepsilon < \delta$), we obtain

$$h(r)(w+r\phi, v+r\psi) \in \mathbf{N}^-_{\eta,\zeta}, \quad \forall r \in \mathbb{R},\ |r| < \varepsilon.$$

Similarly, we prove the other case. □

Lemma 3.5. *There exists*

$$\Lambda_0 = \left(\frac{q+\varrho+\tau-2}{\|\zeta\|_\infty k(q-p)}\right)^{\frac{p}{p+\varrho+\tau-2}} \left(\frac{2-\varrho-\tau-q}{k(2-\varrho-\tau-p)}|Q|^{\frac{2_s^*-q}{2_s^*}}\right)^{-\frac{p}{p-q}} S^{\frac{2-\varrho-\tau}{p+\varrho+\tau-2}},$$

such that for $0 < \left(\eta\|\alpha\|_\infty\right)^{\frac{p}{p-q}} + \left(\zeta\|\beta\|_\infty\right)^{\frac{p}{p-q}} < \Lambda_0$ *we have:*

(i) If $\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\,d\kappa > 0$*, then, there exist a unique* $T_l > 0$ *and* $t_0 < T_l < t_1$ *such that*

$$\Upsilon_{w,v}(t_0) = \Upsilon_{w,v}(t_1),$$
$$\Upsilon'_{w,v}(t_0) < 0 < \Upsilon'_{w,v}(t_1);$$

that is, $(t_0w, t_0v) \in \mathbf{N}^+_{\eta,\zeta}$*,* $(t_1w, t_1v) \in \mathbf{N}^-_{\eta,\zeta}$ *and*

$$\mathfrak{J}_{\eta,\zeta}(t_0w, t_0v) = \min_{0\le t\le t_1} \mathfrak{J}_{\eta,\zeta}(tw, tv),$$
$$\mathfrak{J}_{\eta,\zeta}(t_1w, t_1v) = \max_{t\ge T_l} \mathfrak{J}_{\eta,\zeta}(tw, tv).$$

(ii) If $\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\,d\kappa < 0$*, then there exists a unique* $T_l > 0$ *such that* $(T_lw, T_lv) \in \mathbf{N}^-_{\eta,\zeta}$ *and* $\mathfrak{J}_{\eta,\zeta}(T_lw, T_lv) = \max_{t\ge 0} \mathfrak{J}_{\eta,\zeta}(tw, tv)$.

Proof. (*i*) Suppose that $\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\,d\kappa > 0$. Define the function $\psi_{w,v} : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$\psi_{w,v}(t) = kt^{p-q}\|(w,v)\|^p_{\mathbb{W}} + lt^{p\sigma-q}\|(w,v)\|^{p\sigma}_{\mathbb{W}} - t^{2-\varrho-\tau-q}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa.$$

Note that $(tw, tv) \in \mathbf{N}_{\eta,\zeta}$ if and only if

$$\psi_{w,v}(t) = \int_Q \left(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\right)d\kappa.$$

Now, the first derivative of the function ψ is

$$\begin{aligned}\psi'_{w,v}(t) &= k(p-q)t^{p-q-1}\|(w,v)\|^p_{\mathbb{W}} + (p\sigma-q)lt^{p\sigma-q-1}\|(w,v)\|^{p\sigma}_{\mathbb{W}}\\ &\quad - (2-\varrho-\tau-q)t^{1-\varrho-\tau-q}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa\\ &= t^{-q-1}\Big(k(p-q)t^p\|(w,v)\|^p_{\mathbb{W}} + (p\sigma-q)lt^{p\sigma}\|(w,v)\|^{p\sigma}_{\mathbb{W}}\\ &\quad - (2-\varrho-\tau-q)t^{-\varrho-\tau+2}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa\Big).\end{aligned} \tag{3.7}$$

It is clear that $\psi_{w,v}(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$. Moreover, using (3.7), it is simple to see that $\lim_{t\longrightarrow 0^+} \psi'_{w,v}(t) > 0$ and $\lim_{t\longrightarrow\infty} \psi'_{w,v}(t) < 0$. Thus, there exists $T_l > 0$ such that $\psi_{w,v}(t)$ is decreasing on (T_l,∞), increasing on $(0,T_l)$, and $\psi'_{w,v}(T_l) = 0$. Thus,

$$\psi_{w,v}(T_l) = kT_l^{p-q}\|(w,v)\|_{\mathbb{W}}^p + lT_l^{p\sigma-q}\|(w,v)\|_{\mathbb{W}}^{p\sigma} - T_l^{2-\varrho-\tau-q}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa,$$

where T_l is the solution of

$$\begin{aligned} k(p-q)t^p\|(w,v)\|_{\mathbb{W}}^p &+ (p\sigma-q)lt^{p\sigma}\|(w,v)\|_{\mathbb{W}}^{p\sigma} \\ &- (2-\varrho-\tau-q)t^{-\varrho-\tau+2}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa = 0. \end{aligned} \tag{3.8}$$

Then, using (3.8), we obtain

$$T_0 := \left(\frac{(2-\varrho-\tau-q)\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa}{k(p-q)\|(w,v)\|_{\mathbb{W}}^p}\right)^{\frac{1}{p+\tau+\varrho-2}} \le T_l. \tag{3.9}$$

From inequality (3.9), we can find a constant $C = C(p,q,\varrho,\tau) > 0$ such that

$$\begin{aligned} \psi_{w,v}(T_l) &\ge \psi_{w,v}(T_0) \\ &\ge kT_0^{p-q}\|(w,v)\|_{\mathbb{W}}^p - T_0^{2-\varrho-\tau-q}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa \\ &\ge k\Big(\frac{\varrho+\tau}{q+\varrho+\tau-2}\Big)\Big(\frac{q+\varrho+\tau-2}{k(q-2)}\Big)^{\frac{2-q}{\tau+\varrho}} \frac{\|(w,v)\|_{\mathbb{W}}^{2\frac{q+\varrho+\tau-2}{\tau+\varrho}}}{\big(\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa\big)^{\frac{q-2}{\tau+\varrho}}} \\ &\quad - |Q|^{\frac{2_s^*-q}{2_s^*}} S^{-\frac{q}{2}}\Big((\eta\|\alpha\|_\infty)^{\frac{2}{2-q}} + (\zeta\|\beta\|_\infty)^{\frac{2}{2-q}}\Big)^{\frac{2-q}{2}}\|(w,v)\|_{\mathbb{W}}^q > 0, \end{aligned}$$

if and only if

$$\begin{aligned} &(\eta\|\alpha\|_\infty)^{\frac{2}{2-q}} + (\zeta\|\beta\|_\infty)^{\frac{2}{2-q}} \\ &\quad < \left(\frac{k(q-2)}{\|\zeta\|_\infty(q+\varrho+\tau-2)}\right)^{-\frac{2}{\varrho+\tau}} \left(\frac{q+\varrho+\tau-2}{k(\varrho+\tau)}|Q|^{\frac{2_s^*-q}{2_s^*}}\right)^{-\frac{2}{2-q}} S^{\frac{\varrho+\tau-2}{\varrho+\tau}+\frac{q}{2-q}} = \Lambda_0. \end{aligned}$$

Then, there exist exactly two points $t_0 < T_l$ and $t_1 > T_l$ with

$$\psi'_{w,v}(t_0) = \int_Q \big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\big)\,d\kappa = \psi'_{w,v}(t_1).$$

Also, $\psi'_{w,v}(t_0) > 0$ and $\psi'_{w,v}(t_1) < 0$. That is, $(t_0u,t_0v) \in \mathbf{N}^+_{\eta,\zeta}$ and $(t_1u,t_1v) \in \mathbf{N}^-_{\eta,\zeta}$. Since

$$\Upsilon'_{w,v}(t) = t^q\left(\psi_{w,v}(t) - \int_Q \big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\big)\,d\kappa\right).$$

Thus, $\Upsilon'_{w,v}(t) < 0$ for all $t \in [0, t_0)$ and $\Upsilon'_{w,v}(t) > 0$ for all $t \in (t_0, t_1)$. Hence $\mathfrak{J}_{\eta,\zeta}(t_0 w, t_0 v) = \min_{0\le t\le t_1} \mathfrak{J}_{\eta,\zeta}(tw, tv)$. In the same way, $\Upsilon'_{w,v}(t) > 0$ for all $t \in (t_0, t_1)$, $\Upsilon'_{w,v}(t) = 0$ and $\Upsilon'_{w,v}(t) < 0$ for all $t \in (t_1, \infty)$ that is $\mathfrak{J}_{\eta,\zeta}(t_1 w, t_1 v) = \max_{t\ge T_l} \mathfrak{J}_{\eta,\zeta}(tw, tv)$.

(*ii*) Suppose that $\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\, d\kappa < 0$. So $\psi_{w,v}(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$. Therefore, for all (η, ζ) there exists $T_l > 0$ such that $(T_l w, T_l v) \in \mathbf{N}^-_{\eta,\zeta}$ and $\mathfrak{J}_{\eta,\zeta}(T_l w, T_l v) = \max_{t\ge 0} \mathfrak{J}_{\eta,\zeta}(tw, tv)$. □

The consequence of Lemma 3.5 is summarized in the following Lemma.

Lemma 3.6. *There exists*

$$\Lambda_0 = \left(\frac{q+\varrho+\tau-2}{\|\zeta\|_\infty k(q-p)}\right)^{\frac{p}{p+\varrho+\tau-2}} \left(\frac{2-\varrho-\tau-q}{k(2-\varrho-\tau-p)}|Q|^{\frac{p_s^*-q}{p_s^*}}\right)^{-\frac{p}{p-q}} S^{\frac{2-\varrho-\tau}{p+\varrho+\tau-2}},$$

such that for $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, *we have* $\mathbf{N}^\pm_{\eta,\zeta} \neq \emptyset$ *and* $\mathbf{N}^0_{\eta,\zeta} = \emptyset$.

Proof. From Lemma 3.4, we infer that $\mathbf{N}^\pm_{\eta,\zeta}$ are non-empty for all (η, ζ) with $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$. Next, we employ a proof by contradiction to show that $\mathbf{N}^0_{\eta,\zeta} = \emptyset$ for all (η, ζ), with $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$. Let $(w, v) \in \mathbf{N}^0_{\eta,\zeta}$. Then, we have two cases:

Case 1: $(w, v) \in \mathbf{N}^+_{\eta,\zeta}$ and $\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\, d\kappa = 0$. Using (3.3) and (3.4) with $t = 1$, it follows that

$$\begin{aligned}(p-1)k\|(w,v)\|^p_{\mathbb{W}} &+ l(p\sigma-1)\|(w,v)\|^{p\sigma}_{\mathbb{W}} - (1-\varrho-\tau)\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\, d\kappa \\ &= (p+\varrho+\tau-2)k\|(w,v)\|^p_{\mathbb{W}} + l(p\sigma+\varrho+\tau-2)\|(w,v)\|^{p\sigma}_{\mathbb{W}} > 0,\end{aligned}$$

which is a contradiction.

Case 2: Let $(w, v) \in \mathbf{N}^-_{\eta,\zeta}$ and $\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\, d\kappa = 0$. Using (3.3) and (3.4) with $t = 1$, it follows that

$$(p-q)k\|(w,v)\|^p_{\mathbb{W}} + l(p\sigma-q)\|(w,v)\|^{p\sigma}_{\mathbb{W}} = -(q+\varrho+\tau)\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\, d\kappa, \tag{3.10}$$

$$\begin{aligned}(2-\varrho-\tau-p)k\|(w,v)\|^p_{\mathbb{W}} &+ l(2-\varrho-\tau-p\sigma)\|(w,v)\|^{p\sigma}_{\mathbb{W}} \\ &= (2-\varrho-\tau-q)\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\, d\kappa.\end{aligned} \tag{3.11}$$

Now, define $\mathfrak{E}_{\eta,\zeta} : \mathbf{N}_{\eta,\zeta} \longrightarrow \mathbb{R}$ as follows

$$\mathfrak{E}_{\eta,\zeta}(w,v) = \frac{2-\varrho-\tau-p}{2-\varrho-\tau-q} k\|(w,v)\|_{\mathbb{W}}^{p} + \frac{2-\varrho-\tau-p\sigma}{2-\varrho-\tau-q} l\|(w,v)\|_{\mathbb{W}}^{p\sigma} - \int_Q \left(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\right) d\kappa.$$

Therefore, from (3.11), $\mathfrak{E}_{\eta,\zeta}(w,v) = 0$ for all $(w,v) \in \mathbf{N}^0_{\eta,\zeta}$. Furthermore,

$$\begin{aligned}
\mathfrak{E}_{\eta,\zeta}(w,v) &\geq \frac{2-\varrho-\tau-p}{2-\varrho-\tau-q} k\|(w,v)\|_{\mathbb{W}}^{p} - \int_Q \left(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\right) d\kappa \\
&\geq \frac{2-\varrho-\tau-p}{2-\varrho-\tau-q} k\|(w,v)\|_{\mathbb{W}}^{p} \\
&\quad - C|Q|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}} \left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} \|(w,v)\|_{\mathbb{W}}^{q} \\
&\geq \|(w,v)\|_{\mathbb{W}}^{q} \left(\frac{2-\varrho-\tau-p}{2-\varrho-\tau-q} k\|(w,v)\|_{\mathbb{W}}^{p-q}\right. \\
&\quad \left. - C|Q|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}} \left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\right).
\end{aligned}$$

Then, utilizing (3.6) and (3.10), we get

$$\|(u,v)\| \geq \frac{1}{\|\zeta\|_\infty} S^{-\frac{2-\varrho-\tau}{p(p+\varrho+\tau-2)}} \left(\frac{k(p-q)}{2-\varrho-\tau-q}\right)^{-\frac{1}{p+\varrho+\tau-2}}. \tag{3.12}$$

From (3.12) we get

$$\begin{aligned}
\mathfrak{E}_{\eta,\zeta}(w,v) \geq \|(w,v)\|_{\mathbb{W}}^{q} &\left(\frac{2-\varrho-\tau-p}{2-\varrho-\tau-q} k \left(k(p-q)\|\zeta\|_\infty S^{\frac{2-\varrho-\tau}{p(p+\varrho+\tau-2)}}\right) \left(\frac{k(p-q)}{2-\varrho-\tau-q}\right)^{\frac{q-p}{p+\varrho+\tau-2}}\right. \\
&\left. - C|Q|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}} \left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\right).
\end{aligned}$$

This implies that for $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, we have $\mathfrak{E}_{\eta,\zeta}(w,v) > 0$, for all $(w,v) \in \mathbf{N}^0_{\eta,\zeta}$. The proof is complete. □

Due to Lemmas 3.3 and 3.4, for $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, we can write $\mathbf{N}_{\eta,\zeta} = \mathbf{N}^+_{\eta,\zeta} \cup \mathbf{N}^-_{\eta,\zeta}$ and define

$$c^+_{\eta,\zeta} = \inf_{(w,v)\in\mathbf{N}^+_{\eta,\zeta}} \mathfrak{J}_{\eta,\zeta}(w,v), \quad c^-_{\eta,\zeta} = \inf_{(w,v)\in\mathbf{N}^-_{\eta,\zeta}} \mathfrak{J}_{\eta,\zeta}(w,v).$$

3.1 Existence of a minimizer on $\mathbf{N}^+_{\eta,\zeta}$.

In this subsection, we establish that the minimum of $\mathfrak{J}_{\eta,\zeta}$ is found within $\mathbf{N}^+_{\eta,\zeta}$. Furthermore, we demonstrate that this minimizer also serves as a solution to problem (1.1).

Lemma 3.7. *If* $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$*, then for all* $(w,v) \in \mathbf{N}^+_{\eta,\zeta}$*, we have* $c^+_{\eta,\zeta} < 0$.

Proof. Let $(w_0^+, v_0^+) \in \mathbf{N}^+_{\eta,\zeta}$, then $\Upsilon''_{(w_0^+,v_0^+)}(1) > 0$ which from (3.2) gives

$$\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa < \frac{k(p-q)}{2-\varrho-\tau-q}\|(w,v)\|^p_{\mathbb{W}} + \frac{l(p\sigma-q)}{2-\varrho-\tau-q}\|(w,v)\|^{p\sigma}_{\mathbb{W}}. \tag{3.13}$$

Thus, according to (3.2) with (3.13), we obtain

$$\begin{aligned}\mathfrak{J}_{\eta,\zeta}(w,v) &\le k\Big(\frac{1}{p}-\frac{1}{q}\Big)\|(w,v)\|^p_{\mathbb{W}} + l\left(\frac{1}{p\sigma}-\frac{1}{q}\right)\|(w,v)\|^{p\sigma}_{\mathbb{W}} \\ &\quad - \left(\frac{1}{2-\varrho-\tau}-\frac{1}{q}\right)\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa \\ &\le \left[k\left(\frac{1}{p}-\frac{1}{q}\right) - \left(\frac{1}{2-\varrho-\tau}-\frac{1}{q}\right)\frac{k(p-q)}{2-\varrho-\tau-q}\right]\|(w,v)\|^p_{\mathbb{W}} \\ &\quad + \left[l\left(\frac{1}{p\sigma}-\frac{1}{q}\right) - \left(\frac{1}{2-\varrho-\tau}-\frac{1}{q}\right)\frac{l(p\sigma-q)}{2-\varrho-\tau-q}\right]\|(w,v)\|^{p\sigma}_{\mathbb{W}}.\end{aligned} \tag{3.14}$$

Hence, using (3.14), we get

$$\mathfrak{J}_{\eta,\zeta}(w,v) < -\left(\frac{k(q-p)(p+\varrho+\tau-2)}{pq(2-\varrho-\tau)}\|(w,v)\|^p + \frac{l(q-p)(p+\varrho+\tau-2)}{pq(2-\varrho-\tau)}\|(w,v)\|^{p\sigma}\right) < 0.$$

Therefore, the definition of $c^+_{\eta,\zeta}$ owing to $c^+_{\eta,\zeta} < 0$. □

Theorem 3.8. *If* $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$*, then there exists* (w_0^+, v_0^+) *in* $\mathbf{N}^+_{\eta,\zeta}$ *satisfying* $\mathfrak{J}_{\eta,\zeta}(w_0^+, v_0^+) = \inf_{(w,v)\in\mathbf{N}^+_{\eta,\zeta}} \mathfrak{J}_{\eta,\zeta}(w,v)$.

Proof. From the fact that $\mathfrak{J}_{\eta,\zeta}$ is bounded below on $\mathbf{N}_{\eta,\zeta}$, then it bounded on $\mathbf{N}^+_{\eta,\zeta}$. Thus, there exists $\{(w_n^+, v_n^+)\} \subset \mathbf{N}^+_{\eta,\zeta}$ a sequence such that

$$\mathfrak{J}_{\eta,\zeta}(w_n^+, v_n^+) \longrightarrow \inf_{(w,v)\in\mathbf{N}^+_{\eta,\zeta}} \mathfrak{J}_{\eta,\zeta}(w,v) \ \text{ as } n \longrightarrow \infty.$$

Since $\mathfrak{J}_{\eta,\zeta}$ is coercive, $\{w_n, v_n\}$ is bounded in $\mathbb{W}$. Then, there exists a sub-sequence, still denoted by (w_n^+, v_n^+) and $(w_0^+, v_0^+) \in \mathbb{W}$ such that, as $n \longrightarrow \infty$,

$$w_n^+ \rightharpoonup w_0^+,\ v_n^+ \rightharpoonup v_0^+ \quad \text{weakly in } \mathbb{W}_s(Q),$$
$$w_n^+ \longrightarrow w_0^+,\ v_n^+ \longrightarrow v_0^+ \quad \text{strongly in } \mathbb{L}^r(Q) \text{ for } 1 \leq r < p_s^*,$$
$$w_n^+ \longrightarrow w_0^+,\ v_n^+ \longrightarrow v_0^+ \quad \text{a.e. in } Q.$$

Claim:

$$\lim_{n\longrightarrow\infty} \int_Q \alpha(\kappa)|w_n^+|^{1-\varrho} d\kappa = \int_Q \alpha(\kappa)|w_0^+|^{1-\varrho} d\kappa. \tag{3.15}$$

Indeed, due to Vitali's theorem (see [26, pp. 133]), we only need to prove that

$$\left\{\int_Q \alpha(\kappa)|w_n^+|^{1-\varrho} d\kappa, n \in N\right\} \quad \text{is equi-absolutely-continuous.}$$

Since $\{w_n\}$ is bounded, by the Sobolev embedding theorem, there exists a constant $C > 0$ such that $|w_n|_{p_s^*} \leq C < \infty$. Moreover, by the Hölder inequality we have

$$\int_Q \alpha(\kappa) w^{1-\varrho} d\kappa \leq \|\alpha\|_\infty \int_Q |w|^{1-\varrho} d\kappa \leq \|\alpha\|_\infty |Q|^{\frac{p_s^*}{p_s^*+\varrho-1}} |w|_{p_s^*}^{1-\varrho}. \tag{3.16}$$

From (3.16), for every $\varepsilon > 0$, setting

$$\delta = \left(\frac{\varepsilon}{\|\alpha\|_\infty C^{1-\varrho}}\right)^{\frac{p_s^*}{p_s^*+\gamma-1}},$$

when $A \subset Q$ with $\operatorname{meas}(A) < \delta$, we have

$$\int_A \alpha(\kappa)|w_n^+|^{1-\varrho} d\kappa \leq \|\alpha\|_\infty \|w\|_{p_s^*}^{1-\varrho} \big(\operatorname{meas}(A)\big)^{\frac{p_s^*+\varrho-1}{p_s^*}} \leq \|\alpha\|_\infty C^{1-\varrho} \delta^{\frac{p_s^*+\varrho-1}{p_s^*}} < \varepsilon.$$

Thus, our claim is true. Similarly, we claim that

$$\lim_{n\longrightarrow\infty} \int_Q \beta(\kappa)|v_n^+|^{1-\tau} d\kappa = \int_Q \beta(\kappa)|v_0^+|^{1-\tau} d\kappa. \tag{3.17}$$

On the other hand, by [3] there exists $l \in \mathbb{L}^r(\mathbb{R}^d)$ such that

$$|w_n^+(\kappa)| \leq l(\kappa), \quad |v_n^+(\kappa)| \leq l(\kappa), \quad \text{as } k \longrightarrow \infty$$

for $1 \leq r < p_s^*$. Therefore by the dominated convergence theorem,

$$\int_Q \Big(\eta|w_n^+|^q + \zeta|v_n^+|^q\Big) d\kappa \longrightarrow \int_Q \Big(\eta|w_0^+|^q + \zeta|v_0^+|^q\Big) d\kappa.$$

Furthermore, from Lemma 3.5, there exists t_0 such that $(t_0 w_0^+, t_0 v_0^+) \in \mathbf{N}_{\eta,\zeta}^+$. Now, we shall prove that $w_n^+ \longrightarrow w_0^+$ strongly in $\mathbb{W}_s$, $v_n^+ \longrightarrow v_0^+$ strongly in $\mathbb{W}_s$. Suppose otherwise, then

$$\|(w_0^+, v_0^+)\|_{\mathbb{W}} \leq \liminf_{n\longrightarrow\infty} \|(w_n^+, v_n^+)\|_{\mathbb{W}}.$$

On the other hand, since $(w_n^+, v_n^+) \in \mathbf{N}^+_{\eta,\zeta}$, one has

$$\begin{aligned}\lim_{n\longrightarrow\infty} \Upsilon'_{w_n^+,v_n^+}(t_0) &= \lim_{n\longrightarrow\infty} \Big(kt_0^{p-1}\|(w_n^+, v_n^+)\|^p + lt_0^{p\sigma-1}\|(w_n^+, v_n^+)\|^{p\sigma} \\ &\quad - t_0^{q-1}\int_Q \big(\eta\alpha(\kappa)|w_n^+|^q + \zeta\beta(\kappa)|v_n^+)|^q\big)\,d\kappa - t_0^{1-\varrho-\tau}\int_Q \xi(\kappa)|w_n^+|^{1-\varrho}|v_n^+|^{1-\tau}\,d\kappa\Big) \\ &> kt_0^{p-1}\|(w_0^+, v_0^+)\|^p + lt_0^{p\sigma-1}\|(w_0^+, v_0^+)\|^{p\sigma} \\ &\quad - t_0^{q-1}\int_Q \big(\eta\alpha(\kappa)|w_0^+|^q + \zeta\beta(\kappa)|v_0^+)|^q\big)\,d\kappa - t_0^{1-\varrho-\tau}\int_Q \xi(\kappa)|w_0^+|^{1-\varrho}|v_0^+|^{1-\tau}\,d\kappa \\ &= \Upsilon'_{w_0^+,v_0^+}(t_0) = 0.\end{aligned}$$

Therefore, $\Upsilon'_{w_n^+,v_n^+}(t_0) > 0$ for n large enough. Furthermore, $(w_n^+, v_n^+) \in \mathbf{N}^+_{\eta,\zeta}$, and we can see for all n that $\Upsilon'_{w_n^+,v_n^+}(t) < 0$ for $t \in (0,t)$ and $\Upsilon'_{w_n^+,v_n^+}(1) = 0$. Thus we must have $t_0 > 1$. Moreover $\Upsilon_{w_n^+,v_n^+}(1)$ is decreasing for $t \in (0,t_0)$ and that is

$$\mathfrak{J}_{\eta,\zeta}(t_0w_0^+, t_0v_0^+) < \mathfrak{J}_{\eta,\zeta}(w_0^+, v_0^+) = \lim_{n\longrightarrow\infty} \mathfrak{J}_{\eta,\zeta}(w_n^+, v_n^+) = \inf_{(w,v)\in\mathbf{N}^+_{\eta,\zeta}} \mathfrak{J}_{\eta,\zeta}(w,v)$$

which gives a contradiction. Thus, $w_n^+ \longrightarrow w_0^+$ strongly in $\mathbb{W}_s$, $v_n^+ \longrightarrow v_0^+$ strongly in $\mathbb{W}_s$ and $\mathfrak{J}_{\eta,\zeta}(w_0^+, v_0^+) = \inf_{(w,v)\in\mathbf{N}^+_{\eta,\zeta}} \mathfrak{J}_{\eta,\zeta}(w,v)$. The proof of Theorem 3.8 is complete. □

3.2 Existence of a minimizer on $\mathbf{N}^-_{\eta,\zeta}$.

In this subsection, we aim to establish the existence of a solution to problem (1.1) by demonstrating the existence of a minimizer for $\mathfrak{J}_{\eta,\zeta}$ within the set $\mathbf{N}^-_{\eta,\zeta}$.

Lemma 3.9. *If* $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, *then for all* $(w,v) \in \mathbf{N}^+_{\eta,\zeta}$, *one has* $c^-_{\eta,\zeta} > d_0$ *for some* $d_0 = d_0\big(\varrho, \tau, p, q, \alpha, \beta, \eta, \zeta, |Q|\big) > 0$.

Proof. Let $(w_0^-, v_0^-) \in \mathbf{N}^-_{\eta,\zeta}$, then we have $\Upsilon''_{w_0^-,v_0^-}(1) < 0$ which from (3.2) gives

$$\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa > \frac{k(p-q)}{2-\varrho-\tau-q}\|(w,v)\|^p_{\mathbb{W}} + \frac{l(p\sigma-q)}{2-\varrho-\tau-q}\|(w,v)\|^{p\sigma}_{\mathbb{W}}. \tag{3.18}$$

Hence, using (3.6), we get

$$\|(w,v)\|_{\mathbb{W}} > \frac{1}{\|\zeta\|_\infty} S^{-\frac{2-\varrho-\tau}{p(p+\varrho+\tau-2)}} \left(\frac{k(p-q)}{2-\varrho-\tau-q}\right)^{-\frac{1}{p+\varrho+\tau-2}}. \tag{3.19}$$

Therefore, by (3.5) and (3.19), we obtain

$$\begin{aligned}
\mathfrak{J}_{\eta,\zeta}(w,v) &\geq k\left(\frac{1}{p}-\frac{1}{2-\varrho-\tau}\right)\|(w,v)\|_{\mathbb{W}}^{p}-\left(\frac{1}{q}-\frac{1}{2-\varrho-\tau}\right)|Q|^{\frac{p_s^*-q}{p_s^*}} \\
&\quad\times S^{-\frac{q}{p}}\left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}}+(\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\|(w,v)\|_{\mathbb{W}}^{q} \\
&= \|(w,v)\|_{\mathbb{W}}^{q}\Bigg[k\left(\frac{1}{p}-\frac{1}{2-\varrho-\tau}\right)\|(w,v)\|_{\mathbb{W}}^{p-q}-\left(\frac{1}{q}-\frac{1}{2-\varrho-\tau}\right)|Q|^{\frac{p_s^*-q}{p_s^*}} \\
&\quad\times S^{-\frac{q}{p}}\left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}}+(\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\Bigg] \\
&> \|(w,v)\|_{\mathbb{W}}^{q}\Bigg[k\left(\frac{1}{p}-\frac{1}{2-\varrho-\tau}\right)S^{\frac{(p-q)}{p}}\left(\frac{p-q}{2-\varrho-\tau-q}\right)^{\frac{q-p}{p+\varrho+\tau-2}} \\
&\quad-\left(\frac{1}{q}-\frac{1}{2-\varrho-\tau}\right)|Q|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}}+(\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\Bigg].
\end{aligned}$$

Hence, if $0<(\eta\|\alpha\|_\infty)^{\frac{p}{p-q}}+(\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}<\Lambda_0$, then $\mathfrak{J}_{\eta,\zeta}(w,v)>d_0$ for all $(w,v)\in\mathbf{N}_{\eta,\zeta}^{-}$ for some $d_0=d_0\big(\varrho,\tau,p,q,\alpha,\beta,\eta,\zeta,|Q|\big)>0$. Therefore $c_{\eta,\zeta}^{-}>d_0$ follows from the definition $c_{\eta,\zeta}^{-}$. ☐

Theorem 3.10. *If $0<(\eta\|\alpha\|_\infty)^{\frac{p}{p-q}}+(\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}<\Lambda_0$, then there exists (w_0^-,v_0^-) in $\mathbf{N}_{\eta,\zeta}^{-}$ satisfying $\mathfrak{J}_{\eta,\zeta}(w_0^-,v_0^-)=\inf\limits_{(w,v)\in\mathbf{N}_{\eta,\zeta}^{-}}\mathfrak{J}_{\eta,\zeta}(w,v)$.*

Proof. As $\mathfrak{J}_{\eta,\zeta}$ is bounded below on $\mathbf{N}_{\eta,\zeta}$ and then on $\mathbf{N}_{\eta,\zeta}^{-}$. Thus, there exists $\{(w_n^-,v_n^-)\}\subset\mathbf{N}_{\eta,\zeta}^{-}$, a sequence such that

$$\mathfrak{J}_{\eta,\zeta}(w_n^-,v_n^-)\longrightarrow\inf_{(w,v)\in\mathbf{N}_{\eta,\zeta}^{-}}\mathfrak{J}_{\eta,\zeta}(w,v)\quad\text{as } n\longrightarrow\infty.$$

Since $\mathfrak{J}_{\eta,\zeta}$ is coercive, $\{(w_n,v_n)\}$ is bounded in $\mathbb{W}$. Then there exists a sub-sequence, still denoted by (w_n^-,v_n^-) and $(w_0^-,v_0^-)\in\mathbb{W}$ such that, as $n\longrightarrow\infty$,

$$\begin{gathered}
w_n^+\rightharpoonup w_0^-,\ v_n^-\rightharpoonup v_0^-\quad\text{weakly in } \mathbb{W}_s(Q),\\
w_n^-\longrightarrow w_0^-,\ v_n^-\longrightarrow v_0^-\quad\text{strongly in } \mathbb{L}^r(Q) \text{ for } 1\leq r<p_s^*,\\
w_n^-\longrightarrow w_0^-,\ v_n^-\longrightarrow v_0^-\quad\text{a.e. in } Q.
\end{gathered}$$

Furthermore, similar to the proof in Lemma 3.8, we have

$$\begin{gathered}
\lim_{n\longrightarrow\infty}\int_Q|w_n^-|^{1-\varrho}d\kappa=\int_Q|w_0^-|^{1-\varrho}d\kappa,\\
\lim_{n\longrightarrow\infty}\int_Q|v_n^-|^{1-\tau}d\kappa=\int_Q|v_0^-|^{1-\tau}d\kappa,\\
\int_Q\Big(\eta\alpha(\kappa)|w_n^+|^q+\zeta\beta(\kappa)|v_n^+|^q\Big)d\kappa\longrightarrow\int_Q\Big(\eta\alpha(\kappa)|w_0^+|^q+\zeta\beta(\kappa)|v_0^+|^q\Big)d\kappa.
\end{gathered}$$

Moreover, by Lemma 3.5, there exists t_1 such that $(t_1 w_0^-, t_1 v_0^-) \in \mathbf{N}_{\eta,\zeta}^-$. Now, we prove that $w_n^- \longrightarrow w_0^-$ strongly in $\mathbb{W}_s$, $v_n^- \longrightarrow v_0^-$ strongly in $\mathbb{W}_s$. Suppose otherwise, then

$$\|(w_0^-, v_0^-)\|_{\mathbb{W}} \leq \liminf_{n \longrightarrow \infty} \|(w_n^-, v_n^-)\|_{\mathbb{W}}.$$

Thus, since $(w_n^-, v_n^-) \in \mathbf{N}_{\eta,\zeta}^-$ and $\mathfrak{J}_{\eta,\zeta}(tw_0^-, tv_0^-) \leq \mathfrak{J}_{\eta,\zeta}(w_0^-, v_0^-)$, for all $t \geq 0$ we have

$$\mathfrak{J}_{\eta,\zeta}(t_1 w_0^-, t_1 v_0^-) < \lim_{n \longrightarrow \infty} \mathfrak{J}_{\eta,\zeta}(t_1 u_n^-, t_1 v_n^-) \leq \lim_{n \longrightarrow \infty} \mathfrak{J}_{\eta,\zeta}(w_n^-, v_n^-) = c_{\eta,\zeta}^-,$$

which gives a contradiction. Hence, $w_n^- \longrightarrow w_0^-$ strongly in $\mathbb{W}_s(Q)$, $v_n^- \longrightarrow v_0^-$ strongly in $\mathbb{W}_s(Q)$ and $\mathfrak{J}_{\eta,\zeta}(w_0^-, v_0^-) = \inf_{(w,v) \in \mathbf{N}_{\eta,\zeta}^-} \mathfrak{J}_{\eta,\zeta}(w, v)$. Which complete the proof. □

4 Multiple solutions

In this section, we shall prove Theorem (2.2), which gives the multiplicity of solutions for problem (1.1).

Proof of Theorem 2.2. To begin, let us establish the existence of non-negative solutions. Initially, according to Theorems 3.8 and 3.10, there exist$(w_0^+, v_0^+) \in \mathbf{N}_{\eta,\zeta}^+$, $(w_0^-, v_0^-) \in \mathbf{N}_{\eta,\zeta}^-$ satisfying

$$\mathfrak{J}_{\eta,\zeta}(w_0^+, v_0^+) = \inf_{(w,v) \in \mathbf{N}_{\eta,\zeta}^+} \mathfrak{J}_{\eta,\zeta}(w, v),$$

$$\mathfrak{J}_{\eta,\zeta}(w_0^-, v_0^-) = \inf_{(w,v) \in \mathbf{N}_{\eta,\zeta}^-} \mathfrak{J}_{\eta,\zeta}(w, v).$$

Also, from the fact that $\mathfrak{J}_{\eta,\zeta}(w_0^+, v_0^+) = \mathfrak{J}_{\eta,\zeta}(|w_0^+|, |v_0^+|)$ and $(|w_0^+|, |v_0^+|) \in \mathbf{N}_{\eta,\zeta}^+$. Similarly we have $\mathfrak{J}_{\eta}(w_0^-, v_0^-) = \mathfrak{J}_{\eta,\zeta}(|w_0^-|, |v_0^-|)$ and $(|w_0^-|, |v_0^-|) \in \mathbf{N}_{\eta,\zeta}^-$, thus we can assume $(w_0^\pm, v_0^\pm) \geq 0$. Due to Lemma 3.2, $(w_0^\pm, v_0^\pm)$ are the nontrivial non-negative solutions of problem (1.1). Finally, we need to establish that the solutions obtained in Theorems 3.8 and 3.10 are distinct. Given that $\mathbf{N}_{\eta,\zeta}^- \cap \mathbf{N}_{\eta,\zeta}^+ = \emptyset$, it follows that $(w_0^\pm, v_0^\pm)$ are indeed distinct. This completes the proof. □

Authors contributions

All authors of this manuscript contributed equally to this work.

Funding

Not applicable.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

References

[1] D. Applebaum, *Lévy processes and stochastic calculus*, 2nd ed., ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2009, vol. 116, doi: 10.1017/CBO9780511809781.

[2] G. M. Bisci, N. V. Thin, and L. Vilasi, "On a class of nonlocal Schrödinger equations with exponential growth," *Adv. Differential Equations*, vol. 27, no. 9-10, pp. 571–610, 2022.

[3] H. Brezis, *Analyse fonctionnelle*, ser. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983, théorie et applications. [Theory and applications].

[4] F. Cammaroto and L. Vilasi, "Weak solutions for a fractional equation in $\mathbb{R}^n$ with Kirchhoff terms," *Complex Var. Elliptic Equ.*, vol. 61, no. 10, pp. 1362–1374, 2016, doi: 10.1080/17476933.2016.1177029.

[5] L. S. Chadli, H. El-Houari, and H. Moussa, "Multiplicity of solutions for nonlocal parametric elliptic systems in fractional Orlicz-Sobolev spaces," *J. Elliptic Parabol. Equ.*, vol. 9, no. 2, pp. 1131–1164, 2023, doi: 10.1007/s41808-023-00238-4.

[6] W. Chen and S. Deng, "The Nehari manifold for a fractional p-Laplacian system involving concave-convex nonlinearities," *Nonlinear Anal. Real World Appl.*, vol. 27, pp. 80–92, 2016, doi: 10.1016/j.nonrwa.2015.07.009.

[7] A. Cotsiolis and N. K. Tavoularis, "Best constants for Sobolev inequalities for higher order fractional derivatives," *J. Math. Anal. Appl.*, vol. 295, no. 1, pp. 225–236, 2004, doi: 10.1016/j.jmaa.2004.03.034.

[8] M. G. Crandall, P. H. Rabinowitz, and L. Tartar, "On a Dirichlet problem with a singular nonlinearity," *Comm. Partial Differential Equations*, vol. 2, no. 2, pp. 193–222, 1977, doi: 10.1080/03605307708820029.

[9] R. Dhanya, J. Giacomoni, S. Prashanth, and K. Saoudi, "Global bifurcation and local multiplicity results for elliptic equations with singular nonlinearity of super exponential growth in $\mathbb{R}^2$," *Adv. Differential Equations*, vol. 17, no. 3-4, pp. 369–400, 2012.

[10] E. Di Nezza, G. Palatucci, and E. Valdinoci, "Hitchhiker's guide to the fractional Sobolev spaces," *Bull. Sci. Math.*, vol. 136, no. 5, pp. 521–573, 2012, doi: 10.1016/j.bulsci.2011.12.004.

[11] P. Drábek and S. I. Pohozaev, "Positive solutions for the p-Laplacian: application of the fibering method," *Proc. Roy. Soc. Edinburgh Sect. A*, vol. 127, no. 4, pp. 703–726, 1997, doi: 10.1017/S0308210500023787.

[12] H. El-Houari, L. S. Chadli, and H. Moussa, “Existence of solution to M-Kirchhoff system type,” in *2021 7th International Conference on Optimization and Applications (ICOA)*, 2021, doi: 10.1109/ICOA51614.2021.9442669.

[13] H. El-Houari, L. S. Chadli, and M. Hicham, “Nehari manifold and fibering map approach for fractional $p(.)$-Laplacian Schrödinger system,” *SeMA Journal*, 2023, doi: 10.1007/s40324-023-00343-3.

[14] A. Fiscella, “A fractional Kirchhoff problem involving a singular term and a critical nonlinearity,” *Adv. Nonlinear Anal.*, vol. 8, no. 1, pp. 645–660, 2019, doi: 10.1515/anona-2017-0075.

[15] A. Ghanmi and K. Saoudi, “A multiplicity results for a singular problem involving the fractional p-Laplacian operator,” *Complex Var. Elliptic Equ.*, vol. 61, no. 9, pp. 1199–1216, 2016, doi: 10.1080/17476933.2016.1154548.

[16] A. Ghanmi and K. Saoudi, “The Nehari manifold for a singular elliptic equation involving the fractional Laplace operator,” *Fract. Differ. Calc.*, vol. 6, no. 2, pp. 201–217, 2016, doi: 10.7153/fdc-06-13.

[17] M. Ghergu and V. D. Rădulescu, *Singular elliptic problems: bifurcation and asymptotic analysis*, ser. Oxford Lecture Series in Mathematics and its Applications. The Clarendon Press, Oxford University Press, Oxford, 2008, vol. 37.

[18] A. Iannizzotto, S. Liu, K. Perera, and M. Squassina, “Existence results for fractional p-Laplacian problems via Morse theory,” *Adv. Calc. Var.*, vol. 9, no. 2, pp. 101–125, 2016, doi: 10.1515/acv-2014-0024.

[19] C.-Y. Lei, J.-F. Liao, and C.-L. Tang, “Multiple positive solutions for Kirchhoff type of problems with singularity and critical exponents,” *J. Math. Anal. Appl.*, vol. 421, no. 1, pp. 521–538, 2015, doi: 10.1016/j.jmaa.2014.07.031.

[20] J.-F. Liao, X.-F. Ke, C.-Y. Lei, and C.-L. Tang, “A uniqueness result for Kirchhoff type problems with singularity,” *Appl. Math. Lett.*, vol. 59, pp. 24–30, 2016, doi: 10.1016/j.aml.2016.03.001.

[21] J.-F. Liao, P. Zhang, J. Liu, and C.-L. Tang, “Existence and multiplicity of positive solutions for a class of Kirchhoff type problems with singularity,” *J. Math. Anal. Appl.*, vol. 430, no. 2, pp. 1124–1148, 2015, doi: 10.1016/j.jmaa.2015.05.038.

[22] R.-Q. Liu, C.-L. Tang, J.-F. Liao, and X.-P. Wu, “Positive solutions of Kirchhoff type problem with singular and critical nonlinearities in dimension four,” *Commun. Pure Appl. Anal.*, vol. 15, no. 5, pp. 1841–1856, 2016, doi: 10.3934/cpaa.2016006.

[23] G. Molica Bisci, V. D. Radulescu, and R. Servadei, *Variational methods for nonlocal fractional problems*, ser. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2016, vol. 162, doi: 10.1017/CBO9781316282397.

[24] T. V. Nguyen, "Existence of solution to singular Schrödinger systems involving the fractional p-Laplacian with Trudinger-Moser nonlinearity in $\mathbb{R}^N$," *Math. Methods Appl. Sci.*, vol. 44, no. 8, pp. 6540–6570, 2021, doi: 10.1002/mma.7208.

[25] K. Perera, M. Squassina, and Y. Yang, "Bifurcation and multiplicity results for critical fractional p-Laplacian problems," *Math. Nachr.*, vol. 289, no. 2-3, pp. 332–342, 2016, doi: 10.1002/mana.201400259.

[26] W. Rudin, *Real and complex analysis.* McGraw-Hill Book Co., New York-Toronto-London, 1966.

[27] K. Saoudi, "A fractional Kirchhoff system with singular nonlinearities," *Anal. Math. Phys.*, vol. 9, no. 3, pp. 1463–1480, 2019, doi: 10.1007/s13324-018-0251-7.

[28] K. Saoudi and M. Kratou, "Existence of multiple solutions for a singular and quasilinear equation," *Complex Var. Elliptic Equ.*, vol. 60, no. 7, pp. 893–925, 2015, doi: 10.1080/17476933.2014.981169.

[29] K. Saoudi, "A critical fractional elliptic equation with singular nonlinearities," *Fract. Calc. Appl. Anal.*, vol. 20, no. 6, pp. 1507–1530, 2017, doi: 10.1515/fca-2017-0079.

[30] K. Saoudi, "Existence and non-existence of solutions for a singular problem with variable potentials," *Electron. J. Differential Equations*, 2017, Art. ID 291.

[31] K. Saoudi, M. Kratou, and E. Al Zahrani, "Uniqueness and existence of solutions for a singular system with nonlocal operator via perturbation method," *J. Appl. Anal. Comput.*, vol. 10, no. 4, pp. 1311–1325, 2020.

[32] G. Tarantello, "On nonhomogeneous elliptic equations involving critical Sobolev exponent," *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, vol. 9, no. 3, pp. 281–304, 1992, doi: 10.1016/S0294-1449(16)30238-4.

[33] C. E. Torres Ledesma, "Existence and symmetry result for fractional p-Laplacian in $\mathbb{R}^n$," *Commun. Pure Appl. Anal.*, vol. 16, no. 1, pp. 99–113, 2017, doi: 10.3934/cpaa.2017004.

[34] M. Xiang, B. Zhang, and V. D. Rădulescu, "Superlinear Schrödinger-Kirchhoff type problems involving the fractional p-Laplacian and critical exponent," *Adv. Nonlinear Anal.*, vol. 9, no. 1, pp. 690–709, 2020, doi: 10.1515/anona-2020-0021.

[35] M. Xiang, B. Zhang, and Z. Wei, "Existence of solutions to a class of quasilinear Schrödinger systems involving the fractional p-Laplacian," *Electron. J. Qual. Theory Differ. Equ.*, 2016, Art. ID 107, doi: 10.14232/ejqtde.2016.1.107.

[36] J. Yang, H. Chen, and Z. Feng, "Multiple positive solutions to the fractional Kirchhoff problem with critical indefinite nonlinearities," *Electron. J. Differential Equations*, 2020, Art. ID 101.