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Solutions of two open problems on inequalities involving trigonometric and hyperbolic functions

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ABSTRACT

In 2019, Bagul *et al.* posed two open problems dealing with inequalities involving trigonometric and hyperbolic functions and an adjustable parameter. This article is an attempt to solve these open problems. The results are supported by three-dimensional graphics, taking into account the variation of the parameter involved.

RESUMEN

En 2019, Bagul *et al.* propusieron dos problemas relacionados con desigualdades que involucran funciones trigonométricas e hiperbólicas y un parámetro ajustable. Este artículo es un intento de resolver estos problemas abiertos. Los resultados están apoyados con gráficas tridimensionales, tomando en consideración la variación del parámetro involucrado.

Keywords and Phrases: Trigonometric inequalities, series expansion, open problems.

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1 Introduction

When studying mathematical inequalities, it is often useful to find generalizations of known results. These generalizations can provide deep insights into the structure of inequalities and their applications in various areas of mathematics. They can be established using integer series expansions of well-known elementary functions. For a more rigorous treatment of this topic, see [2,5,7,9,11–15].

A brief discussion of recent progress in the inequalities of some trigonometric and hyperbolic functions is given below. In 2021, Bagul *et al.* [2] studied the following inequalities: For $r > 0$ and $x \in (0, r)$, we have

$$\left(1+\frac{x^2}{\pi^2}\right)e^{ax^2} < \frac{\sinh x}{x} < \left(1+\frac{x^2}{\pi^2}\right)e^{bx^2},$$

where $a = \ln\left[\pi^2(\sinh r)/r(\pi^2+r^2)\right]/r^2$ and $b = 1/6 - 1/\pi^2$ are the best possible constants in the exponential term. To prove these inequalities, the author used the concept of series expansion. For the details, see [2]. Later, in 2023, Li *et al.* [8] presented the proof of the following inequalities involving sine and hyperbolic sine functions using power series expansion: For $|x| < \pi/2$, we have

$$\frac{4}{15}\left(\cos x+\frac{11}{4}\right)^2-\frac{3}{4} \le \frac{\sin(2x)}{2x}+2\frac{\sin x}{x} \le \frac{4}{15}\left(\cos x+\frac{11}{4}\right)^2-\frac{3}{4}+\frac{1}{1260}x^6$$

and, for $x \in \mathbb{R}$ and an integer $n \ge 2$, we have

$$1+2\cosh x+\sum_{k=2}^{n} b_k x^{2k} \le \frac{\sinh(2x)}{2x}+2\frac{\sinh x}{x} \le \frac{4}{15}\left(\cosh x+\frac{11}{4}\right)^2-\frac{3}{4},$$

where $b_k = (2^{2k}-4k)/(2k+1)!$ for $k = 2, 3, \ldots, n$.

In 2018, Malešević, *et al.* [10] gave the following generalized inequalities: For $x \in (0, \pi/2)$ and an integer $n \ge 1$, we have

$$\frac{2+\cos x}{3}+\sum_{k=2}^{2n}(-1)^{k+1}B(k)x^{2k} < \frac{\sin x}{x} < \frac{2+\cos x}{3}+\sum_{k=2}^{2n+1}(-1)^{k+1}B(k)x^{2k},$$

where $B(k) = 2(k-1)/[2(2k+1)!]$.

The following result gives us sharper bounds on the above inequalities established by Bagul *et al.* [4]: For an integer $n \ge 1$, $m = 2n-1$, and $x \in (0, \pi)$, we have

$$F(x) < \frac{\sin x}{x} < G(x),$$

where

$$F(x) = \frac{2m + \cos x}{2m+1} + \frac{2}{2m+1} \sum_{k=1}^{m+1} \frac{k-m}{(2k+1)!} (-1)^{k+1} x^{2k}$$

and

$$G(x) = \frac{(2m+1) + \cos x}{2m+3} + \frac{2}{2m+3} \sum_{k=1}^{m+2} \frac{k-m-1}{(2k+1)!} (-1)^{k+1} x^{2k}.$$

In parallel to these remarkable modern results, there are some open problems on similar functions. For example, in 2019, Bagul *et al.* [3] posed the following open problems on some trigonometric and hyperbolic functions:

(1) For $x \in (0, \pi/2)$ and $p \geq 2$, we have

$$p + (\cos x)^p > \frac{\sin(px)}{px} + p\left(\frac{\sin x}{x}\right).$$

(2) For $x \in (0, \pi/2)$ and $p \in (0, 2]$, we have

$$\frac{\sin(px)}{px} + p\left(\frac{\sin x}{x}\right) > 1 + p\cos x.$$

(3) For $x \in \mathbb{R} - \{0\}$ and $p \in (0, 2]$, we have

$$p + (\cosh x)^p > \frac{\sinh(px)}{px} + p\left(\frac{\sinh x}{x}\right).$$

(4) For $x \in \mathbb{R} - \{0\}$ and $p \geq 2$, we have

$$\frac{\sinh(px)}{px} + p\left(\frac{\sinh x}{x}\right) > 1 + p\cosh x.$$

This article is an attempt to prove two open problems, namely those presented in Items 2 and 4, which are further listed in the main results. Our focus is to show that these inequalities hold for a wide range of p. It is important to note that while the inequalities we prove are useful for a wide range of p and x, we do not claim that these inequalities are optimal in the sense of sharpness. There is scope in the future to find sharper bounds on these inequalities for particular values of p. In particular, the first inequality we prove holds for $p \geq 2$, and the second for $p \in (0, 2)$, which may also be useful to researchers for further development.

The plan is as follows: First, Section 2 gives some preliminary remarks that will be useful for the gradual development of this article. Section 3 deals with our main results, supported by graphics, and Section 4 is the concluding part.

2 Preliminaries

Well-known power series expansions derived from $\sinh x$ and $\cosh x$ are the following formulas:

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \tag{2.1}$$

$$\sinh(px) = \sum_{n=0}^{\infty} \frac{p^{2n+1}x^{2n+1}}{(2n+1)!}, \tag{2.2}$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

and, an immediate consequence of the previous formula,

$$x\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n)!}. \tag{2.3}$$

We may refer to [1] and [6].

3 Main results

In this section, using power series expansion and some trigonometric identities, we present the proof of two inequalities.

Theorem 3.1. *For $x > 0$ and $p \geq 2$, we have*

$$\frac{\sinh(px)}{px} + p\left(\frac{\sinh x}{x}\right) > 1 + p\cosh x.$$

Proof. To prove this result, let us consider the following function:

$$f(x) = \sinh(px) + p^2\sinh x - px - p^2 x\cosh x.$$

A differentiation work gives

$$\begin{aligned} f'(x) &= p\cosh(px) + p^2\cosh x - p - p^2\{\cosh x + x\sinh x\} \\ &= p\cosh(px) + p^2\cosh x - p - p^2\cosh x - p^2 x\sinh x \\ &= p\cosh(px) - p - p^2 x\sinh x \end{aligned}$$

and

$$f''(x) = p^2\sinh(px) - p^2\{\sinh x + x\cosh x\}.$$

From Equations (2.1), (2.2) and (2.3), we can decompose $f''(x)$ as

$$\begin{aligned} f''(x) &= p^2\left[\sum_{n=0}^{\infty}\frac{p^{2n+1}x^{2n+1}}{(2n+1)!}-\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}-\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n)!}\right] \\ &= p^2\left[\sum_{n=0}^{\infty}\frac{p^{2n+1}x^{2n+1}}{(2n+1)!}-\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}-\sum_{n=0}^{\infty}\frac{(2n+1)x^{2n+1}}{(2n+1)!}\right] \\ &= p^2\left[\sum_{n=0}^{\infty}\frac{p^{2n+1}x^{2n+1}}{(2n+1)!}-\sum_{n=0}^{\infty}\frac{(2+2n)x^{2n+1}}{(2n+1)!}\right] = p^2\sum_{n=0}^{\infty}\frac{\left[p^{2n+1}-(2+2n)\right]x^{2n+1}}{(2n+1)!}. \end{aligned}$$

For $p \geq 2$, the Bernoulli inequality gives

$$p^{2n+1}-(2+2n) \geq (1+1)^{2n+1}-(2+2n) \geq 1+(2n+1)-(2+2n) = 0.$$

Therefore, for any $x > 0$, we have $f''(x) > 0$. Hence, we conclude that, for $x > 0$, $f'(x)$ is strictly increasing. As a result, we have $f'(x) > f'(0)$ with $f'(0) = p - p = 0$. This implies that $f(x)$ is strictly increasing, so $f(x) > f(0)$ with $f(0) = 0$. By taking into account the definition of $f(x)$, we find

$$\frac{\sinh(px)}{px} + p\left(\frac{\sinh x}{x}\right) > 1 + p\cosh x.$$

The proof ends. □

Thus, through Theorem 3.1, we provide the solution to one of the open problems in Bagul *et al.* [3]. Figures 1 and 2 illustrate the validity of Theorem 3.1 by considering the following bivariate function with respect to x and p:

$$f_\star(x,p) = \frac{\sinh(px)}{px} + p\left(\frac{\sinh x}{x}\right) - 1 - p\cosh x,$$

with $x > 0$ and $p \geq 2$.

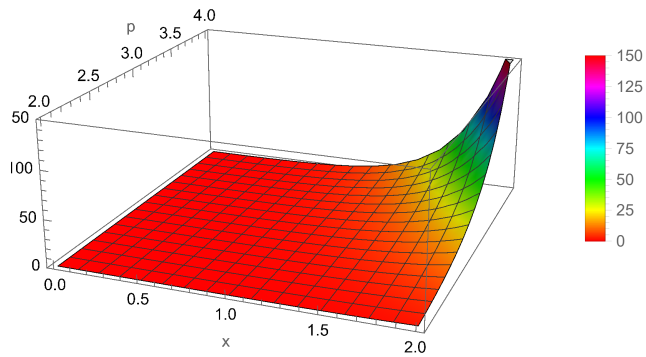

Figure 1: Three-dimensional shape plots of the function $f_{\star}(x,p)$ for $x \in (0,2)$ and $p \in [2,4)$.

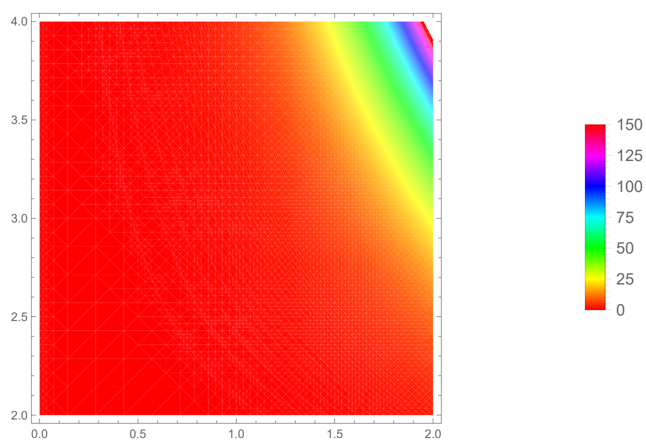

Figure 2: Three-dimensional intensity plots of the function $f_{\star}(x,p)$ for $x \in (0,2)$ and $p \in [2,4)$.

It is clear that the zone corresponding to the negative values is never reached, implying that $f_{\star}(x,p) > 0$ for the considered configuration, which is consistent with Theorem 3.1 as expected.

The next result concerns another open problem in Bagul *et al.* [3].

Theorem 3.2. *For $x \in (0, \pi/2)$ and $p \in (0, 2]$, we have*

$$\frac{\sin(px)}{px} + p\left(\frac{\sin x}{x}\right) > 1 + p\cos x.$$

Proof. To prove this theorem, let us consider the following function:

$$g(x) = \sin(px) + p^2 \sin x - px - p^2 x \cos x.$$

A differentiation work gives

$$g'(x) = p\cos(px) + p^2 \cos x - p - p^2(\cos x - x\sin x) = p\cos(px) - p + p^2 x \sin x$$

and

$$g''(x) = -p^2 \sin(px) + p^2(\sin x + x\cos x) = p^2[\sin x + x\cos x - \sin(px)].$$

Owing to basic trigonometric identities, we obtain

$$\sin(px) = \sin[(p-1)x + x] = \cos[(p-1)x]\sin x + \sin[(p-1)x]\cos x.$$

Hence, we can rewrite $g''(x)$ as

$$\begin{aligned} g''(x) &= p^2\{\sin x + x\cos x - \cos[(p-1)x]\sin x - \sin[(p-1)x]\cos x\} \\ &= p^2[\{1 - \cos[(p-1)x]\}\sin x + \{x - \sin[(p-1)x]\}\cos x]. \end{aligned}$$

We know that, for $x \in (0, \pi/2)$, we have $\sin x > 0$ and $\cos x > 0$. Also, for any $p \in (0, 2]$ we have $\cos[(p-1)x] \leq 1$, implying that $1 - \cos[(p-1)x] \geq 0$.

Now, let us discuss the sign of the term $x - \sin[(p-1)x]$ by distinguishing the cases $p \in (0, 1]$ and $p \in (1, 2]$.

For $p \in (0, 1]$ and $x \in (0, \pi/2)$, it is immediate that

$$-\sin[(p-1)x] = \sin[(1-p)x] \geq 0.$$

Hence, we can conclude that $x - \sin[(p-1)x] > 0$.

Now for $p \in (1, 2]$ and $x \in (0, \pi/2)$, thanks to the classical sine inequality: $\sin y < y$ for $y > 0$, we have

$$\sin[(p-1)x] < (p-1)x \leq x.$$

Thus, we have $x - \sin[(p-1)x] > 0$.

As a result, we can conclude that $g''(x) > 0$. Thus, for $x \in (0, \pi/2)$, $g'(x)$ is strictly increasing. As a result, we have $g'(x) > g'(0)$ with $g'(0) = p - p = 0$. This implies that $g(x)$ is strictly increasing, so $g(x) > g(0)$ with $g(0) = 0$. Thanks to the definition of $g(x)$, we establish that

$$\frac{\sin(px)}{px} + p\left(\frac{\sin x}{x}\right) > 1 + p\cos x.$$

This achieves the proof. □

Hence, through Theorem 3.2, we offer a solution to one of the open problems in Bagul *et al.* [3]. Figures 3 and 4 illustrate the validity of Theorem 3.2 by considering the following bivariate function with respect to x and p:

$$g_\star(x,p) = \frac{\sin(px)}{px} + p\left(\frac{\sin x}{x}\right) - 1 - p\cos x,$$

with $x \in (0, \pi/2)$ and $p \in (0, 2]$.

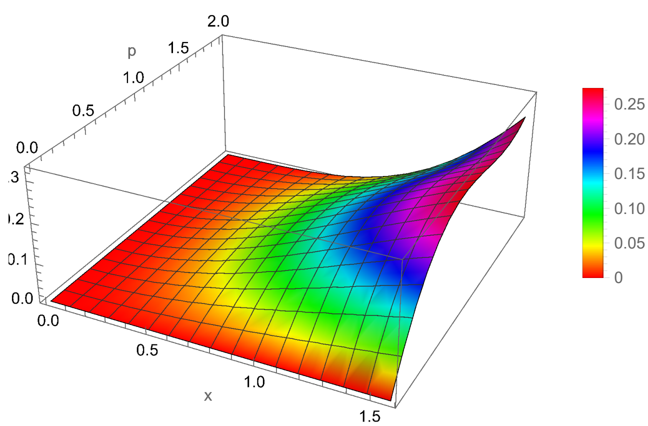

Figure 3: Three-dimensional shape plots of the function $g_\star(x,p)$ for $x \in (0, \pi/2)$ and $p \in (0, 2]$.

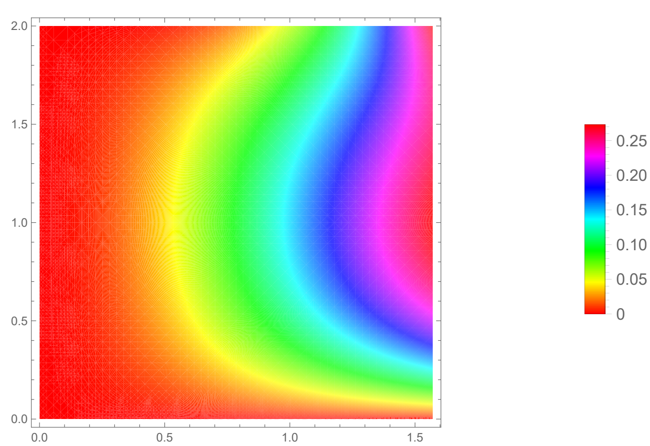

Figure 4: Three-dimensional intensity plots of the function $g_\star(x,p)$ for $x \in (0, \pi/2)$ and $p \in (0, 2]$.

We note that the zone associated with the negative values is never reached, suggesting that $g_\star(x,p) > 0$ for the configuration under consideration, which is in expected agreement with Theorem 3.2.

During our graphical investigation, we found that Theorem 3.2 can be conjectured to be valid for $x \in (0,\pi)$ instead of just $x \in (0,\pi/2)$, as shown in Figure 5 with the absence of a negative value zone. The rigorous proof, however, remains a new challenge to be investigated in the future.

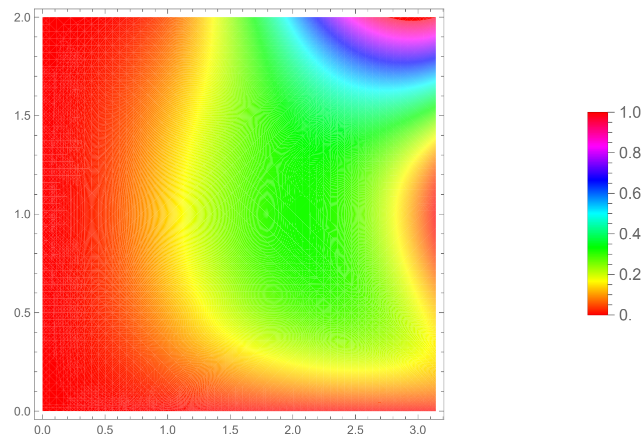

Figure 5: Three-dimensional intensity plots of the function $g_\star(x,p)$ for $x \in (0,\pi)$ and $p \in (0,2]$.

4 Conclusion

In this article, we have given simple and elegant proofs for two open problems posed by Bagul *et al.* in 2019 [3], which concern inequalities related to trigonometric and hyperbolic functions for a large range of p and x. The presented inequalities generalize existing results for large values of p and provide researchers with valuable insights and tools for further developments in this area.

Availability of data and material

Not applicable.

Conflict of interests

The authors declare no conflict of interest.

Ethics approval

The authors declare that the article was not submitted or published anywhere.

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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, ser. National Bureau of Standards Applied Mathematics Series. U.S. Government Printing Office, Washington, DC, 1964, vol. 55.

[2] Y. J. Bagul, R. M. Dhaigude, M. Kostić, and C. Chesneau, "Polynomial-exponential bounds for some trigonometric and hyperbolic functions," *Axioms*, vol. 10, no. 4, 2021, Art. ID 308, doi: 10.3390/axioms10040308.

[3] Y. J. Bagul and C. Chesneau, "Two double sided inequalities involving sinc and hyperbolic sinc functions," *Int. J. Open Problems Compt. Math.*, vol. 12, no. 4, pp. 15–20, 2019.

[4] Y. J. Bagul and C. Chesneau, "Refined forms of Oppenheim and Cusa-Huygens type inequalities," *Acta Comment. Univ. Tartu. Math.*, vol. 24, no. 2, pp. 183–194, 2020.

[5] A. R. Chouikha, "New sharp inequalities related to classical trigonometric inequalities," *J. Inequal. Spec. Funct.*, vol. 11, no. 4, pp. 27–35, 2020.

[6] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 8th ed. Elsevier/Academic Press, Amsterdam, 2015.

[7] R. Klén, M. Visuri, and M. Vuorinen, "On Jordan type inequalities for hyperbolic functions," *J. Inequal. Appl.*, 2010, Art. ID 362548, doi: 10.1155/2010/362548.

[8] W.-H. Li and B.-N. Guo, "Several inequalities for bounding sums of two (hyperbolic) sine cardinal functions," *Filomat*, vol. 38, no. 11, pp. 3937–3943, 2024.

[9] Y. Lv, G. Wang, and Y. Chu, "A note on Jordan type inequalities for hyperbolic functions," *Appl. Math. Lett.*, vol. 25, no. 3, pp. 505–508, 2012, doi: 10.1016/j.aml.2011.09.046.

[10] B. Malešević, T. Lutovac, M. Rašajski, and C. Mortici, "Extensions of the natural approach to refinements and generalizations of some trigonometric inequalities," *Adv. Difference Equ.*, 2018, Art. ID 90, doi: 10.1186/s13662-018-1545-7.

[11] Z.-H. Yang and Y.-M. Chu, "Jordan type inequalities for hyperbolic functions and their applications," *J. Funct. Spaces*, 2015, Art. ID 370979, doi 10.1155/2015/370979.

[12] Z.-H. Yang and Y.-M. Chu, "A sharp double inequality involving trigonometric functions and its applications," *J. Math. Inequal.*, vol. 10, no. 2, pp. 423–432, 2016, doi: 10.7153/jmi-10-33.

[13] Z.-H. Yang, Y.-M. Chu, Y.-Q. Song, and Y.-M. Li, "A sharp double inequality for trigonometric functions and its applications," *Abstr. Appl. Anal.*, 2014, Art. ID 592085, doi: 10.1155/2014/592085.

[14] Z.-H. Yang, Y.-L. Jiang, Y.-Q. Song, and Y.-M. Chu, "Sharp inequalities for trigonometric functions," *Abstr. Appl. Anal.*, pp. Art. ID 601 839, 18, 2014, doi: 10.1155/2014/601839.

[15] L. Zhu, "New Masjed Jamei–type inequalities for inverse trigonometric and inverse hyperbolic functions," *Mathematics*, vol. 10, no. 16, 2022, Art. ID 2972, doi: 10.3390/math10162972.