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Congruences of finite semidistributive lattices

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ABSTRACT

We show that there are finite distributive lattices that are not the congruence lattice of any finite semidistributive lattice. For $0 \leq k \leq 2$, the distributive lattice $(\mathbf{B}_k)_{++} = \mathbf{2} + \mathbf{B}_k$, where $\mathbf{B}_k$ denotes the boolean lattice with k atoms, is not the congruence lattice of any finite semidistributive lattice. Neither can these lattices be a filter in the congruence lattice of a finite semidistributive lattice. However, each $(\mathbf{B}_k)_{++}$ with $k \geq 3$ is the congruence lattice of a finite semidistributive lattice, say $\mathbf{L}_k$. These lattices $\mathbf{L}_k$ cannot be bounded (in the sense of McKenzie), as no $(\mathbf{B}_k)_{++}$ $(k \geq 0)$ is the congruence lattice of a finite bounded lattice. A companion paper shows that every $(\mathbf{B}_k)_{++}$ $(k \geq 0)$ can be represented as the congruence lattice of an infinite semidistributive lattice. We also find sufficient conditions for a finite distributive lattice to be representable as the congruence lattice of a finite bounded (and hence semidistributive) lattice.

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RESUMEN

Mostramos que existen reticulados distributivos finitos que no son el reticulado de congruencia de cualquier reticulado semidistributivo finito. Para $0 \leq k \leq 2$, el reticulado distributivo $(\mathbf{B}_k)_{++} = \mathbf{2} + \mathbf{B}_k$, donde $\mathbf{B}_k$ denota el reticulado booleano con k átomos, no es el reticulado de congruencia de cualquier reticulado semidistributivo finito. Estos reticulados tampoco pueden ser un filtro en el reticulado de congruencia de un reticulado semidistributivo finito. De todas formas, cada $(\mathbf{B}_k)_{++}$ con $k \geq 3$ es el reticulado de congruencia de un reticulado semidistributivo finito, digamos $\mathbf{L}_k$. Estos reticulados $\mathbf{L}_k$ no pueden ser acotados (en el sentido de McKenzie), puesto que ningún $(\mathbf{B}_k)_{++}$ $(k \geq 0)$ es el reticulado de congruencia de un reticulado finito acotado. Un artículo acompañante muestra que todo $(\mathbf{B}_k)_{++}$ $(k \geq 0)$ puede ser representado como el reticulado de congruencia de un reticulado infinito semidistributivo. También encontramos condiciones suficientes para que un reticulado finito distributivo sea representable como el reticulado de congruencia de un reticulado finito acotado (y por lo tanto semidistributivo).

Keywords and Phrases: Distributive lattice, semidistributive lattice, congruence lattice.

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1 Introduction

R. P. Dilworth proved in the 1940's that every finite distributive lattice is the congruence lattice of a finite lattice. Not every finite distributive lattice is isomorphic to the congruence lattice of a finite join semidistributive (or meet semidistributive) lattice, but it turns out that there is only one restriction; see Theorem 1.1 below, from [1]. This note shows that there is at least one additional restriction on a finite distributive lattice $\mathbf{D}$ in order for $\mathbf{D}$ to be the congruence lattice of a finite (meet and join) semidistributive lattice; see Theorem 3.1.

In the paper K. Adaricheva *et al.* [1] it was shown that a finite distributive lattice $\mathbf{D} \cong \mathcal{O}(\mathbf{P})$ is the congruence lattice of a finite *join* semidistributive lattice if and only if every non-maximal element of $\mathbf{P}$ is below at least two maximal elements. In fact, the equivalence of five conditions is proved in that paper.

Theorem 1.1. *The following are equivalent for a finite distributive lattice* $\mathbf{D}$. *Let* $\mathbf{D} \cong \mathcal{O}(\mathbf{P})$ *for an ordered set* $\mathbf{P}$ *(isomorphic to* $\mathrm{J}(\mathbf{D})$*).*

(1) $\mathbf{D} \cong \mathrm{Con}\ \mathbf{L}$ *for a finite join semidistributive lattice* $\mathbf{L}$.

(2) $\mathbf{D} \cong \mathrm{Con}\ \mathbf{S}$ *for a finite lower bounded lattice* $\mathbf{S}$.

(3) $\mathbf{D} \cong \mathrm{Con}\ \mathbf{G}$ *for a finite convex geometry* $\mathbf{G}$.

(4) $\mathbf{D} \cong \mathrm{Con}\ \mathbf{A}$ *for a finite, lower bounded, atomistic convex geometry* $\mathbf{A}$.

(5) *Every non-maximal element of* $\mathbf{P}$ *is below at least two maximal elements.*

(6) *The three-element chain is not a filter in* $\mathbf{D}$.

We will show that there is at least one additional restriction for the congruence lattice $\mathrm{Con}\ \mathbf{K}$ when $\mathbf{K}$ is a finite lattice that is both join and meet semidistributive. The restrictions are perhaps best expressed in terms of the lattices $(\mathbf{B}_k)_{++}$ obtained by adjoining a new zero *twice* to a boolean lattice with k atoms. Theorem 3.1 is that neither $(\mathbf{B}_0)_{++}$ nor $(\mathbf{B}_2)_{++}$ can be a filter in the congruence lattice of a finite semidistributive lattice. (Since $(\mathbf{B}_0)_{++}$ is a three-element chain and $(\mathbf{B}_1)_{++}$ is a four-element chain, excluding the latter is redundant.) We can show that every $(\mathbf{B}_k)_{++}$ with $k \geq 3$ is the congruence lattice of a finite semidistributive lattice (Theorem 4.8). However, a lattice $\mathbf{K}$ with $\mathrm{Con}\ \mathbf{K} \cong (\mathbf{B}_k)_{++}$ for $\mathbf{K}$ finite, semidistributive and $k \geq 3$ cannot be bounded in the sense of McKenzie (Theorem 4.2). To complicate matters, it turns out that every lattice $(\mathbf{B}_k)_{++}$ with $k \geq 0$ is isomorphic to the congruence lattice of an *infinite* semidistributive lattice, as shown by the author and G. Grätzer [11]. It remains open whether every finite distributive lattice is the congruence lattice of an infinite semidistributive lattice.

2 Preliminaries on congruence lattices and semidistributivity

A lattice is *join semidistributive* if it satisfies the condition

$$x \vee y = x \vee z \text{ implies } x \vee y = x \vee (y \wedge z)\,.$$

The dual is called *meet semidistributive*, and a lattice is *semidistributive* if it is both join and meet semidistributive. This notion was introduced in Jónsson [12] as a basic property of free lattices; summaries of semidistributive lattices can be found in [3, 7].

A lattice homomorphism $h : \mathbf{K} \to \mathbf{L}$ is *lower bounded* if for every $a \in L$, $h^{-1}(\uparrow a)$ is either empty or has a least element. Dually, h is an *upper bounded* homomorphism if $h^{-1}(\downarrow a)$ has a greatest element whenever it is nonempty. A homomorphism that is both lower and upper bounded is called *bounded*.

A finitely generated lattice is said to be *bounded* if it is a bounded homomorphic image of a free lattice. The basic historical sources are R. McKenzie [15] and A. Day [5]; again more recent summaries can be found in [3, 7]. Bounded lattices inherit semidistributivity from free lattices.

For $k \geq 0$, $\mathbf{B}_k$ denotes the boolean lattice with k atoms; in particular, $\mathbf{B}_0$ is a one-element lattice. Given a lattice $\mathbf{K}$, let $\mathbf{K}_+$ denote the lattice obtained by adjoining a new zero element. The lattices $(\mathbf{B}_k)_{++}$ will play an important role in this paper.

For finite subsets X, Y of a lattice $\mathbf{L}$, we say that X refines Y, written $X \ll Y$, if for each $x \in X$ there exists $y \in Y$ such that $x \leq y$. An inclusion $p \leq \bigvee Q$, where $p \in L$ and $Q \subseteq L$ is a finite nonempty subset, is a *minimal nontrivial join cover* if $p \nleq q$ for all $q \in Q$ and Q cannot be properly refined, *i.e.*, if $p \leq \bigvee R$ and $R \ll Q$, then $Q \subseteq R$. When $p \leq \bigvee Q$ is a minimal nontrivial join cover, then Q is an antichain of join irreducible elements. We say that a minimal nontrivial join cover $p \leq \bigvee Q$ is *doubly minimal* if there is no minimal nontrivial join cover S with $p \leq \bigvee S < \bigvee Q$.

A join irreducible element p in a finite lattice has a unique lower cover, denoted p_*. A finite lattice $\mathbf{L}$ is meet semidistributive if and only if for each join irreducible element $p \in \mathrm{J}(\mathbf{L})$, there is a unique element $\kappa(p)$ that is maximal with respect to the property of being above p_* and not above p; see *e.g.* Theorem 2.56 of [7]. Thus $x \leq \kappa(p)$ if and only if $p_* \vee x \ngeq p$. Indeed, $\kappa(p)$ will be meet irreducible with the unique upper cover $\kappa(p)^* = p \vee \kappa(p)$. Note that if $p \leq \bigvee Q$ is a minimal nontrivial join cover and $q \in Q$, then $\bigvee(Q \setminus \{q\}) \leq \kappa(q)$; else q could be replaced by q_* for a proper refinement.

Let us review congruence lattices of finite lattices and the special properties of bounded ones.

Define five relations on the set of join irreducible elements $\mathrm{J}(\mathbf{L})$, the first three requiring meet semidistributivity.

- $p\,A\,q$ if $q < p < q \vee \kappa(q)$,
- $p\,B\,q$ if $p \neq q$, $p \leq p_* \vee q$, $p \nleq p_* \vee q_*$, or equivalently, $p \neq q$, $q \nleq \kappa(p)$, $q_* \leq \kappa(p)$,
- $C = A \cup B$,
- $p\,D\,q$ if $q \in Q$ for some minimal nontrivial join cover Q of p,
- $p\,E\,q$ if $q \in R$ for some doubly minimal nontrivial join cover R of p.

Now in a finite semidistributive lattice $E \subseteq C \subseteq D$, and the containments can be proper (Theorem 2.59 of [7]). Form the reflexive, transitive closures of the last three: $\overline{C}$, $\overline{D}$, $\overline{E}$. These are quasi-orders.

An *order ideal* of a quasi-ordered set $(\mathbf{Q}, \leq)$ is a subset $I \subseteq Q$ such that $s \leq t \in I$ implies $s \in I$. The order ideals of $\mathbf{Q}$ form a distributive lattice $\mathcal{O}(\mathbf{Q}, \leq)$. A standard result is that for any finite lattice, $\mathrm{Con}\,\mathbf{L} \cong (\,\mathrm{J}(\mathbf{L}), \overline{D})$, see Chapter 10 of [17] or Section II.3 of [7]. But for bounded finite lattices, we also have $\mathrm{Con}\,\mathbf{L} \cong \mathcal{O}(\mathrm{J}(\mathbf{L}), \overline{E})$, see Section 6.6 of [3] or Section 9 of [4]. (This does not mean that D and E are the same, but their reflexive, transitive closures $\overline{D}$ and $\overline{E}$ are.)

We assume a familiarity with the following basic facts of lattice theory.

- Every finite distributive lattice is isomorphic to the lattice of order ideals of its join irreducible elements, $\mathbf{D} \cong \mathcal{O}(\mathbf{P})$ where $\mathbf{P} = (\mathrm{J}(\mathbf{D}), \leq)$. By convention, $\mathcal{O}(\mathbf{P})$ includes the empty ideal.
- Equivalently, $\mathbf{D}$ is isomorphic to the lattice of order filters of meet irreducible elements, $\mathbf{D} \cong \mathcal{F}(\mathbf{Q})$, where $\mathbf{Q} = (\mathrm{M}(\mathbf{D}), \leq)$ and filters are ordered by *reverse* set inclusion.
- For disjoint unions of ordered sets, $\mathcal{O}(\mathbf{P} \,\dot\cup\, \mathbf{Q}) \cong \mathcal{O}(\mathbf{P}) \times \mathcal{O}(\mathbf{Q})$, while lattices satisfy $\mathrm{Con}(\mathbf{K} \times \mathbf{L}) \cong (\mathrm{Con}\,\mathbf{K}) \times (\mathrm{Con}\,\mathbf{L})$. Hence we may restrict our attention to connected finite ordered sets.
- For any finite lattice $\mathbf{L}$, the congruence lattice $\mathrm{Con}\,\mathbf{L}$ is isomorphic to the ideal lattice of the quasi-ordered set $\mathbf{Q} = (\mathrm{J}(\mathbf{L}), \overline{D})$, *i.e.*, $\mathrm{Con}\,\mathbf{L} \cong \mathcal{O}(\mathrm{J}(\mathbf{L}), \overline{D})$.
- In particular, maximal members of $\mathbf{Q}$ correspond to simple homomorphic images of $\mathbf{L}$.
- The two-element lattice $\mathbf{2}$ is the only finite simple semidistributive lattice. (Infinite simple semidistributive lattices exist; see [8].)
- Thus for a finite semidistributive lattice, coatoms of $\mathrm{Con}\,\mathbf{L}$ correspond to maximal members of $\mathbf{Q} = (\mathrm{J}(\mathbf{L}), \overline{D})$, which in turn correspond to join prime elements of $\mathbf{L}$. That is, the maximal proper congruences on a finite semidistributive lattice are exactly those with two classes, $\uparrow p$ and $\downarrow \kappa(p)$, where p is a join prime element.

- Every element in a finite join semidistributive lattice has a canonical join representation [13]. This canonical representation is the unique non-refinable join representation of the element, and refines every other join representation. Thus if $a = \bigvee B$ canonically and also $a = \bigvee C$, then $B \ll C$.

- In a finite join semidistributive lattice, the canonical joinands of $1_{\mathbf{L}}$ are join prime.

- The atoms of a finite meet semidistributive lattice are join prime.

While the whole theory of Day doubling of intervals is relevant to bounded lattices, for this paper we need only double points, which is easily described; see [5, 6, 10]. If $\mathbf{L}$ is a lattice and $a \in L$, let $\mathbf{L}[a]$ be the lattice on the set $L \setminus \{a\} \cup \{(a, 0), (a, 1)\}$ with the order $\leq'$ such that, for x, $y \in L \setminus \{a\}$ and $i \in \{0, 1\}$,

- $x \leq' y$ iff $x \leq y$,

- $(a, 0) \leq' (a, 1)$,

- $x \leq' (a, i)$ iff $x \leq a$,

- $(a, i) \leq' y$ iff $a \leq y$.

Note that $(a, 1)$ is join irreducible in $\mathbf{L}[a]$. Doubling intervals, and in particular points, preserves both meet and join semidistributivity, and both lower and upper boundedness [5].

3 Congruence lattices of finite semidistributive lattices

Consider the two ordered sets in Figure 1.

Theorem 3.1. *The distributive lattices $\mathcal{O}(\mathbf{2})$ and $\mathcal{O}(\mathbf{Y})$ are not the congruence lattice of a finite semidistributive lattice.*

Recall that the homomorphic images of a finite semidistributive lattice $\mathbf{L}$ are semidistributive. (More generally, bounded homomorphisms preserve semidistributivity; see the proof of Theorem 2.20 in [7].) It follows that neither $\mathbf{2}$ nor $\mathbf{Y}$ can be a filter in $(\mathrm{J}(\mathbf{L}), \overline{D})$ when $\mathbf{L}$ is a finite semidistributive lattice. Note that $\mathcal{O}(\mathbf{2}) = \mathbf{3}$ is the three-element chain $(\mathbf{B}_0)_{++}$, while $\mathcal{O}(\mathbf{Y}) = (\mathbf{B}_2)_{++}$. The four-element chain $\mathbf{4} = (\mathbf{B}_1)_{++}$ has $\mathbf{3}$ as a filter, so neither is it the congruence lattice of a finite semidistributive lattice. See also Lemma 3.3 below.

Proof. Elements of $(\mathrm{J}(\mathbf{L}), \overline{D})$ may be equivalence classes induced by the quasi-order $\overline{D}$, but maximal elements of $(\mathrm{J}(\mathbf{L}), \overline{D})$ correspond to singleton classes with one join prime element. This is because

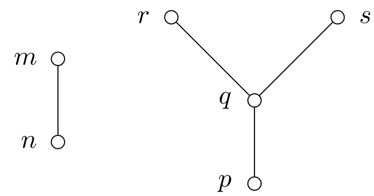

Figure 1: Ordered sets **2** and **Y**

every finite nontrivial semidistributive lattice contains join prime elements, and a join prime element p has no nontrivial join cover, making $p\,D\,q$ impossible when p is maximal in $(\mathrm{J}(\mathbf{L}), \overline{D})$. Now $\mathbf{2}$ is the only finite semidistributive lattice with only one join prime element, and its congruence lattice is $\mathbf{2} = \mathcal{O}(\mathbf{1})$, not $\mathbf{3} = \mathcal{O}(\mathbf{2})$. We conclude that Con $\mathbf{L} \cong \mathbf{3}$ cannot occur. (This argument applies with join semidistributivity only; see Theorem 1.1.)

So suppose $\mathbf{L}$ is a finite semidistributive lattice with Con $\mathbf{L} \cong \mathcal{O}(\mathbf{Y})$. Then $\mathbf{L}$ has two join prime elements, which includes its atoms and the canonical joinands of 1. The trivial case with one atom and $1_{\mathbf{L}}$ join prime would give $\mathbf{L} \cong \mathbf{3}$, while Con $\mathbf{3} \cong \mathbf{2} \times \mathbf{2}$, so that does not occur. Thus $\mathbf{L}$ has exactly two atoms, say r and s, and $1_{\mathbf{L}} = r \vee s$. Since r is an atom, $\kappa(r)$ is the largest element not above r, and similarly for $\kappa(s)$. So the coatoms of $\mathbf{L}$ are $\kappa(r)$ and $\kappa(s)$, and they satisfy $\kappa(r) \wedge \kappa(s) = 0_{\mathbf{L}}$. Thus $L = \{0, 1\} \,\dot{\cup}\, [r, \kappa(s)] \,\dot{\cup}\, [s, \kappa(r)]$, as in Figure 2.

Put $\mathbf{U} = [r, \kappa(s)]$ and $\mathbf{V} = [s, \kappa(r)]$. Note $u \vee v = 1$ and $u \wedge v = 0$ for any $u \in U$ and $v \in V$. Hence congruences behave independently on the sublattices $\mathbf{U}$ and $\mathbf{V}$. It follows that Con $\mathbf{L}$ is isomorphic to Con $\mathbf{U}$ $\times$ Con $\mathbf{V}$ with three additional elements on top, as illustrated in Figure 2. If Con $\mathbf{L} \cong \mathcal{O}(\mathbf{Y})$, then Con $\mathbf{U}$ $\times$ Con $\mathbf{V} \cong \mathbf{3} = \mathcal{O}(\mathbf{2})$, which is impossible by the first part. Therefore there is no finite semidistributive lattice with Con $\mathbf{L} \cong \mathcal{O}(\mathbf{Y})$. □

In the preceding argument, $\mathbf{U}$ and $\mathbf{V}$ are intervals of $\mathbf{L}$, and hence finite semidistributive lattices. On the other hand, one or both of these could have only one element. Hence, from the proof we conclude:

Corollary 3.2. *The following are equivalent for a finite distributive lattice* $\mathbf{D}$ *with two coatoms.*

(1) $\mathbf{D} \cong$ Con $\mathbf{L}$ *for some finite semidistributive lattice* $\mathbf{L}$.

(2) $\mathbf{D}$ *is a glued sum* $\mathbf{D} = \mathbf{E} \oplus (\mathbf{2} \times \mathbf{2})$ *where* $\mathbf{E} \cong$ Con $\mathbf{K}$ *for some finite semidistributive lattice* $\mathbf{K}$.

While the corollary applies only to distributive lattices with two coatoms, it allows us to construct a multitude of examples of distributive lattices, both representable and non-representable as congruence lattices of finite semidistributive lattices. The construction in the positive direction mimics Figure 2 with say $\mathbf{U} = \mathbf{K}$ and $|\mathbf{V}| = 1$.

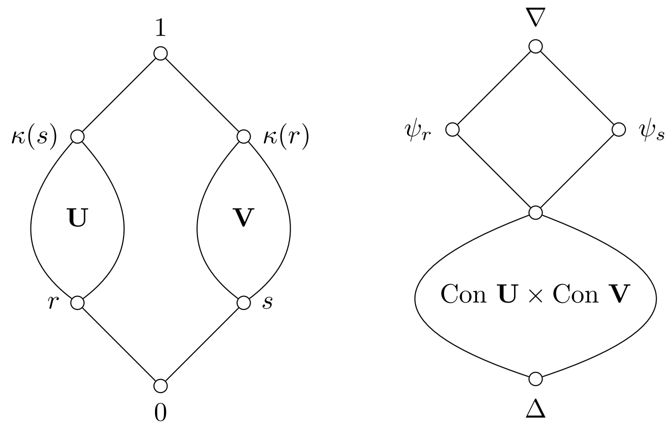

Figure 2: $\mathbf{L}$ and Con $\mathbf{L}$ for a finite semidistributive lattice with exactly two join prime elements, r and s. The congruence ψ_r collapses the intervals $[r, 1]$ and $[0, \kappa(r)]$, the congruence ψ_s collapses the interval $[s, 1]$ and $[0, \kappa(s)]$, so that $\mathbf{L}/(\psi_r \wedge \psi_s) \cong \mathbf{2} \times \mathbf{2}$.

It is currently unknown whether additional restrictions apply to congruence lattices of finite semidistributive lattices. By analogy with situation for finite join semidistributive lattices (Theorem 1.1) we conjecture that the restrictions of Theorem 3.1 are the only ones.

Conjecture: *A finite distributive lattice* $\mathbf{D}$ *is the congruence lattice of a finite semidistributive lattice if and only if neither the* 3*-element chain nor* $\mathcal{O}(\mathbf{Y}) = (\mathbf{B}_2)_{++}$ *is a filter of* $\mathbf{D}$.

With respect to such characterizations, we remind the reader of an elementary fact.

Lemma 3.3. *Let* $\mathbf{S}$ *and* $\mathbf{P}$ *be finite ordered sets. Then* $\mathcal{O}(\mathbf{S})$ *is isomorphic to a filter of* $\mathcal{O}(\mathbf{P})$ *if and only if* $\mathbf{S}$ *is a filter of* $\mathbf{P}$.

Proof. If $\mathbf{S}$ is a filter of $\mathbf{P}$, let $L = P \setminus S$. Clearly $\uparrow L$ is a filter of $\mathcal{O}(\mathbf{P})$ isomorphic to $\mathcal{O}(\mathbf{S})$.

Conversely, assume that $\mathcal{O}(\mathbf{S})$ is isomorphic to a filter of $\mathcal{O}(\mathcal{P})$, say $\mathcal{O}(\mathbf{S}) \cong \uparrow K$. Set $T = P \setminus K$ and $\mathbf{T} = (T, \leq)$ with the order inherited from $\mathbf{P}$. As the complement of an ideal, T is a filter in $\mathbf{P}$. Now $\mathbf{S} \cong \mathrm{J}(\mathcal{O}(\mathbf{S}))$. We want to establish an isomorphism $\nu : \mathbf{T} \cong \mathrm{J}(\uparrow K)$ between $\mathbf{T}$ and the ideals that are join irreducible in the filter $\uparrow K$ (which need not be join irreducible in $\mathcal{O}(\mathbf{P})$).

For $t \in T$, define $\nu(t) = K \cup \downarrow t$. Note that $\nu(t)$ is join irreducible in $\uparrow K$. In fact, for an ideal $L \geq K$, $t \in L$ iff $L \geq \nu(t)$.

On the other hand, if L is join irreducible in $\uparrow K$, then there is a unique ideal $L_\dagger$ with $L \succ L_\dagger \geq K$. There is only one element in $L \setminus L_\dagger$, and it must be in T. Denote this element by $\tau(L)$, so that $\tau(L) \in T$ and $L = L_\dagger \,\dot\cup\, \{\tau(L)\}$.

Now $\tau\nu(t) = t$ because $\nu(t) \succ \nu(t) \setminus \{t\} \geq K$. Let us show that $\nu\tau(L) = K \cup \downarrow \tau(L) = L$ when L is join irreducible in $\uparrow K$. Clearly $K \cup \downarrow \tau(L) \subseteq L$. Suppose the reverse inclusion fails. That means there exists an element $\ell_0 \in L \cap T$ with $\ell_0 \not\leq \tau(L)$. Let $\ell_1 \geq \ell_0$ be maximal in L, so also

$\ell_1 \in T$ and $\ell_1 \ngeq \tau(L)$. Then $L \succ L \setminus \{\ell_1\} \geq K$, yielding another lower cover of L in $\uparrow K$ besides $L_\dagger$, contrary to the assumption that L is join irreducible in the interval. Therefore $\nu(\tau(L)) = L$.

It remains to show that for I, L join irreducible in $\uparrow K$, $\tau(I) \leq \tau(L)$ iff $I \leq L$. But $I = \nu\tau(I) = K \cup \downarrow\tau(I)$ and $L = \nu\tau(L) = K \cup \downarrow\tau(L)$ with $\tau(I)$ and $\tau(L)$ not in K, from which the claim follows immediately. □

4 Congruence lattices of finite bounded lattices

Now we turn to finite lattices that are bounded homomorphic images of a free lattice. These inherit semidistributivity from the free lattice. Finite bounded lattices have many special properties, which we summarize here from [3], Sections 3-2.6 and 3-2.7, or [7], Sections II-4 and II-5, both of which have references to the original sources.

Theorem 4.1. *The following are equivalent for a finite semidistributive lattice* $\mathbf{L}$.

(1) $\mathbf{L}$ *is bounded.*

(2) $\mathbf{L}$ *is lower bounded.*

(3) $\mathbf{L}$ *is upper bounded.*

(4) $\mathrm{J}(\mathbf{L})$ *contains no* D*-cycle*
$$p_0 \, D \, p_1 \, D \, p_2 \, D \, \ldots \, D \, p_{m-1} \, D \, p_0.$$

(5) $\mathrm{J}(\mathbf{L})$ *contains no* E*-cycle*
$$p_0 \, E \, p_1 \, E \, p_2 \, E \, \ldots \, E \, p_{n-1} \, E \, p_0.$$

(6) $|\,\mathrm{J}(\mathrm{Con}\ \mathbf{L})| = |\,\mathrm{J}(\mathbf{L})|$.

Condition (6), from Pudlák and Tůma [19], is particularly important for us: if $\mathbf{L}$ is a finite bounded lattice with $\mathrm{Con}\ \mathbf{L} \cong \mathcal{O}(\mathbf{P})$, then there is a bijection between $\mathrm{J}(\mathbf{L})$ and $\mathbf{P}$. This is not true for unbounded semidistributive lattices in general, because of the presence of D-cycles as in (4).

Moreover, bounded finite lattices have $\mathrm{Con}\ \mathbf{L} \cong \mathcal{O}(\mathrm{J}(\mathbf{L}), \overline{\overline{E}})$, where E is the relation on $\mathrm{J}(\mathbf{L})$ determined by doubly minimal join covers. This need not be true for unbounded lattices.

Recall that $\mathbf{B}_k$ denotes the boolean lattice with k atoms, and $\mathbf{L}_+$ denotes the lattice obtained by adding a new least element 0 to $\mathbf{L}$.

Theorem 4.2. *For* $k \geq 0$*, the distributive lattice* $(\mathbf{B}_k)_{++}$ *is not the congruence lattice of a finite bounded lattice.*

The proof uses a technical lemma, which is Theorem 2.60 in [7].

Lemma 4.3. *Let* $\mathbf{K}$ *be a finite semidistributive lattice, and let* $q \in \mathrm{J}(\mathbf{K})$*. Assume* $q \leq \bigvee R$ *is a doubly minimal nontrivial join cover. Then there is a unique* $r_0 \in R$ *such that* $r_0 \nleq q$*, and* $q\,B\,r_0$ *holds. The remaining* $r \in R \setminus \{r_0\}$ *satisfy* $r < q$ *and* $q\,A\,r$*.*

Proof. Assume that $q \leq \bigvee R$ is doubly minimal, and consider any $r \in R$. Let $S = R \setminus \{r\}$, noting that $\bigvee S \leq \kappa(r)$ else $r \vee \bigvee S = r_* \vee \bigvee S$, contradicting minimality. If $r < q$, then $r < q < r \vee \bigvee S \leq r \vee \kappa(r) = \kappa(r)^*$ so that $q\,A\,r$ holds.

Meanwhile, $q \leq \bigvee R$ implies $\bigvee R \nleq \kappa(q)$. Hence $q \leq q_* \vee r_0$ for at least one $r_0 \in R$. This refines to a minimal nontrivial join cover $q \leq \bigvee T$ with $T \ll \{q_*, r_0\}$. Clearly $\bigvee T \leq q \vee r_0 \leq \bigvee R$; by the double minimality, $\bigvee T = q \vee r_0 = \bigvee R$.

We have $q \leq q_* \vee r_0 = \bigvee R$. Suppose $q \leq q_* \vee r_{0*}$. Then by the double minimality of $\bigvee R$ we get $\bigvee R = q_* \vee r_{0*}$. Put $S = R \setminus \{r_0\}$, noting $\bigvee S \leq q_*$ by the preceding paragraph. Recall that in a join semidistributive lattice, $u = \bigvee a_i = \bigvee b_j$ implies $u = \bigvee_{i,j}(a_i \wedge b_j)$. (This is Theorem 1.21 in [7], from Jónsson and Kiefer [13].) Thus we calculate

$$\bigvee R = \bigvee S \vee r_0 = q_* \vee r_{0*} = \bigvee S \vee (r_0 \wedge q_*) \vee r_{0*} = \bigvee S \vee r_{0*}$$

which contradicts $q \leq \bigvee R$ being a minimal (nonrefinable) join cover. So $q \nleq q_* \vee r_{0*}$, whence $q\,B\,r_0$ holds.

By ($\mathrm{SD}_\vee$), $R = T$ consists of the canonical joinands of $q \vee r_0$, all except one of which, namely r_0, lie below q_*. □

Corollary 4.4. *If* $q\,E\,s$ *and* $s \nleq q$ *in a finite semidistributive lattice, then* $q\,B\,s$*.*

Now we can prove Theorem 4.2.

Proof. We may assume $k \geq 3$, as the cases $0 \leq k \leq 2$ are covered by Theorem 3.1.

Suppose that $\mathbf{L}$ is a bounded lattice and that Con $\mathbf{L} \cong (\mathbf{B}_k)_{++}$. Then $(\mathrm{J}(\mathbf{L}), \overline{E})$ is isomorphic to the ordered set drawn in Figure 3. Note that because $\mathbf{L}$ is bounded, the relation $\overline{E}$ is antisymmetric (as there are no E-cycles), making $\overline{E}$-classes singletons. So each point in Figure 3 represents an element of $\mathrm{J}(\mathbf{L})$.

Moreover, the elements labeled $r_1, \ldots, r_k$ in the top row are join prime in $\mathbf{L}$. Let $R_1 = \{r_1, \ldots, r_\ell\}$ be the join prime elements with $r_i < q$, and let $R_2 = \{r_{\ell+1}, \ldots, r_k\}$ be those with $r_j \nleq q$. As the diagram indicates, we have $p\,E\,q$ and $q\,E\,r_i$ for all i. Since $p\,E\,q$, in $\mathbf{L}$ there is at least one doubly minimal nontrivial join cover $p \leq q \vee \bigvee S$ with $S \subseteq R_1 \cup R_2$.

Clearly $S \cap R_1 = \varnothing$, *i.e.*, we cannot have $s < q$ with both in the same minimal join cover. So $S \subseteq R_2$. But if $s_0 \in R_2$, then $q\,B\,s_0$ by Corollary 4.4, so $q \leq q_* \vee s_0$. Thus $p \leq q_* \vee \bigvee S$, contradicting the minimality of $\{q\} \cup S$. □

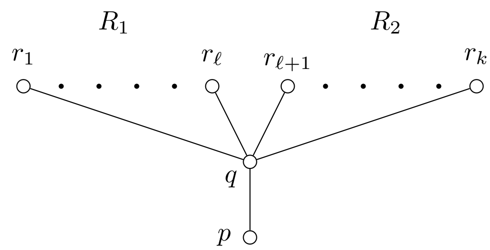

Figure 3: $(\mathrm{J}(\mathbf{L}), \overline{E})$ for $(\mathbf{B}_k)_{++}$

But now we encounter an unexpected surprise. The lattice **U** in Figure 4 is obtained by doubling the point p_0 in a lattice $\mathbf{U}_0$ from [14]. Now **U** is not bounded, because it has the D-cycle $p_0\,A\,p_1\,A\,p_2\,B\,p_3\,B\,p_0$. However, its congruence lattice Con **U** is $(\mathbf{B}_3)_{++}$.

Theorem 4.5. *The lattice* $(\mathbf{B}_3)_{++}$ *is the congruence lattice of a finite semidistributive lattice.*

A couple of lemmas are required to prove this.

When a point a is doubled in a finite lattice **L**, then the principal congruence $\alpha = \mathrm{Cg}((a,0),(a,1))$ has only one nontrivial congruence class, so that α is an atom of Con $\mathbf{L}[a]$ with $\mathbf{L}[a]/\alpha \cong \mathbf{L}$. But we need a little more information as to which congruences lie above α. The calculation is based on the following straightforward lemmas.

Lemma 4.6. *Let* **L** *be a finite lattice. Double a join irreducible element* $a \in \mathrm{J}(\mathbf{L})$*, replacing* a *by* $(a,0)$ *and* $(a,1)$*. Note that both* $(a,0)$ *and* $(a,1)$ *are join irreducible in* $\mathbf{L}[a]$*.*

(1) *If* $a\,D\,b$ *in* **L***, then* $(a,0)\,D\,b$ *and* $(a,1)\,D\,b$ *in* $\mathbf{L}[a]$*.*

(2) *If* $c\,D\,a$ *in* **L***, then* $c\,D\,(a,0)$ *in* $\mathbf{L}[a]$*, but* $c \not{D}\,(a,1)$*.*

(3) *If* **L** *is meet semidistributive and* $\kappa(a) \neq a_*$*, then* $(a,1)\,D\,(a,0)$*.*

If p and q are join irreducible elements with $p\,D\,q$, then we have the congruence inclusion $\mathrm{Cg}(p,p_*) \leq \mathrm{Cg}(q,q_*)$. Thus whenever there is a D-cycle $p_0\,D\,p_1\,D\,\ldots\,D\,p_{n-1}\,D\,p_0$, then $\mathrm{Cg}(p_i,p_{i*}) = \mathrm{Cg}(p_j,p_{j*})$ for all i, j. We refer to this congruence as the *congruence generated by the cycle.*

Lemma 4.7. *Let* **L** *be a finite, subdirectly irreducible, semidistributive lattice with* Con $\mathbf{L} \cong \mathbf{D}$*. Suppose the monolith* μ *of* **L** *is generated by a proper* D*-cycle, and let* a *be a join irreducible element in that cycle. Then* Con $\mathbf{L}[a] \cong \mathbf{D}_+$*.*

The crucial observation is that if $p_0\,D\,p_1\,D\,\ldots\,D\,p_{n-1}\,D\,p_0$ is a D-cycle in **L**, then by Lemma 4.6(1) and (2),

$$p_0\,D\,p_1\,D\,\ldots\,D\,(p_j,0)\,D\,\ldots\,D\,p_{n-1}\,D\,p_0$$

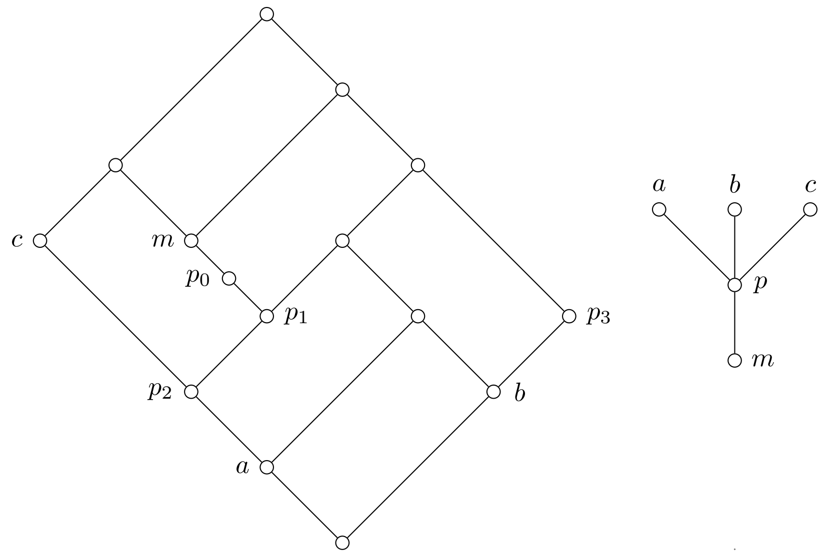

Figure 4: A finite semidistributive lattice with Con $\mathbf{U} \cong (\mathbf{B}_3)_{++}$. On the left $\mathbf{U}$, on the right $(\mathrm{J}(\mathbf{U}), \overline{D})$.

is a D-cycle in $\mathbf{L}[p_j]$.

In our situation, for Theorem 4.5 we have the original lattice from [14] with congruence lattice isomorphic to $(\mathbf{B}_3)_+$. Doubling p_0 to get the element labeled m in $\mathbf{U}$, as in the figure, yields Con $\mathbf{U} \cong (\mathbf{B}_3)_{++}$ and thus Theorem 4.5.

With the lattice $\mathbf{U}$ as a pattern, we can find more examples. The lattice $\mathbf{U}_0$ from [14] has a D-cycle of the form $AABB$ and 3 join prime elements. We would like to find finite, subdirectly irreducible, semidistributive lattices $\mathbf{L}_0$ whose join irreducibles consist of a D-cycle and k join prime elements, so that Con $\mathbf{L}_0 \cong (\mathbf{B}_k)_+$. Then double an element p in the D-cycle to obtain $(\mathbf{B}_k)_{++}$ as the congruence lattice of $\mathbf{L}_0[p]$.

In [18] there is a finite, semidistributive, unbounded lattice $\mathbf{V}_6$ based on a D-cycle of the form $(AB)^3$ that has Con $\mathbf{V}_6 \cong (\mathbf{B}_6)_+$. Doubling a join irreducible in the cycle yields another finite semidistributive lattice $\mathbf{W}_6$ with Con $\mathbf{W}_6 \cong (\mathbf{B}_6)_{++}$. A straightforward generalization of the construction in [18], using a cycle of the form $(AB)^m$ for $m \geq 3$, gives a finite semidistributive lattice $\mathbf{W}_{2m}$ with Con $\mathbf{W}_{2m} \cong (\mathbf{B}_{2m})_{++}$. The general construction to represent all $(\mathbf{B}_n)_{++}$ with $n \geq 4$ is somewhat more complicated.

Theorem 4.8. *For all $k \geq 3$, the lattice $(\mathbf{B}_k)_{++}$ is the congruence lattice of a finite semidistributive lattice.*

As noted earlier, all the lattices $(\mathbf{B}_k)_{++}$ $(k \geq 0)$ can be represented as the congruence lattice of an *infinite* semidistributive lattice [11].

Proof. The case $k = 3$ is Theorem 4.5, so let us consider $n \geq 4$. We will construct a finite semidistributive lattice $\mathbf{X}_n$ whose join irreducible elements have the following properties:

- there is a D-cycle of the form $B^2 A^{n-2}$,

$$p_0 \, B \, p_1 \, B \, p_2 \, A \, p_3 \, A \, \ldots \, A \, p_{n-1} \, A \, p_0 \ ;$$

- there are n join prime elements p_{0*}, p_{1*}, $x_3, \ldots, x_n$;
- for each join prime element q there is a p_j such that $p_j \, D \, q$;
- there are no more join irreducible elements in $\mathbf{X}_n$.

Thus Con $\mathbf{X}_n \cong (\mathbf{B}_n)_+$. Applying Lemma 4.7 to double an element in the cycle yields a lattice $\mathbf{Y}_n$ with Con $\mathbf{Y}_n \cong (\mathbf{B}_n)_{++}$.

A standard duality for finite lattices is to regard $\mathbf{L}$ as a closure system on the ordered set of its join irreducibles $\mathbf{J} = (\mathrm{J}(\mathbf{L}), \leq)$. Given $\mathbf{L}$, the map $a \mapsto \downarrow a \cap \mathbf{J}$ represents the lattice as an intersection-closed collection of subsets of $\mathbf{J}$. The corresponding closure operator γ on $\mathbf{J}$ is given by

$$x \in \gamma(\{y\}) \text{ if } x \leq y,$$
$$x \in \gamma(Y) \text{ if } x \leq \bigvee Y$$

for x, $y \in J$ and $Y \subseteq J$. Then $\mathbf{L}$ is isomorphic to the lattice of γ-closed subsets of $\mathbf{J}$ (which are automatically order ideals by the first rule, including the empty ideal $\varnothing$).

To construct a lattice using the duality, we must specify the ordered set $\mathbf{J}$ and a basis for the desired join operation. Following custom, we write the closure rules as $x \leq y$ and $x \leq \bigvee Y$, respectively. Part of the verification will include checking that $\leq$ is a partial order, and that $\gamma(x) \setminus \{x\} = \downarrow x \setminus \{x\}$ is closed for $x \in J$, so that x is join irreducible.

To construct $\mathbf{X}_n$ with the properties described above, for $n \geq 4$ we take

$$\mathbf{J}_n = \{p_{0*}, p_0, p_{1_*}, p_1, p_2, \ldots, p_{n-1}, x_3, \ldots, x_n\}.$$

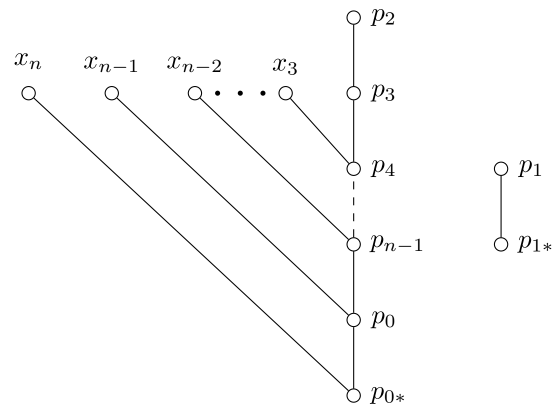

Figure 5: The order on the join irreducibles of $\mathbf{X}_n$

The order on $\mathbf{J}_n$ is given by

$$p_2 > p_3 > \cdots > p_{n-1} > p_0 > p_{0*}$$
$$x_j > p_{j+1} \text{ for } 3 \leq j < n-1$$
$$x_{n-1} > p_0$$
$$x_n > p_{0*}$$
$$p_1 > p_{1*}$$

as illustrated in Figure 5. The defining join covers are

$$(\ddagger) \quad \begin{aligned} &p_0 \leq p_{0*} \vee p_1 \\ &p_1 \leq p_{1*} \vee p_2 \\ &p_2 \leq p_3 \vee x_3 \\ &\cdots \\ &p_{n-2} \leq p_{n-1} \vee x_{n-1} \\ &p_{n-1} \leq p_0 \vee x_n \\ &p_1 \leq p_{1*} \vee x_3 \end{aligned}$$

The last is a bit of a mystery, but is required for meet semidistributivity, and does the job.

Set $\mathbf{X}_n$ to be the lattice of closed ideals of $\mathbf{J}_n$. Routine checks, with multiple cases, show that the elements of $\mathbf{J}_n$ are join irreducible, with the lower covers u_* as indicated in Figure 5, and that the join covers given in the basis ($\ddagger$) are minimal (nonrefinable). Thus for each inclusion $u \leq y \vee z$ in ($\ddagger$) we have $u \, D \, y$ and $u \, D \, z$. These facts give the desired properties from the first paragraph of the proof. It remains to prove that $\mathbf{X}_n$ is semidistributive.

To see that $\mathbf{X}_n$ is meet semidistributive, we must show that every join irreducible element q has a unique element $\kappa(q) \in \mathbf{X}_n$ that is maximal w.r.t. being above q_* and not above q. The elements $p_{0*}, p_{1_*}, x_3, \ldots, x_n$ are join prime, so for them $\kappa(q) = \bigvee\{u \in J : u \ngeq q\}$. For the rest, we calculate as follows.

$$\begin{aligned}
&\kappa(p_0) = p_{1*} \vee x_n \\
&\kappa(p_1) = p_{1*} \vee p_3 \vee \bigvee_{4 \leq j \leq n} x_j \\
&\kappa(p_2) = p_1 \vee p_3 \vee \bigvee_{4 \leq j \leq n} x_j \\
&\kappa(p_3) = p_1 \vee x_3 \vee \bigvee_{5 \leq j \leq n} x_j \\
&\kappa(p_4) = p_1 \vee x_4 \vee \bigvee_{6 \leq j \leq n} x_j \\
&\ldots \\
&\kappa(p_{n-2}) = p_1 \vee x_{n-2} \vee x_n \\
&\kappa(p_{n-1}) = p_1 \vee x_{n-1}
\end{aligned}$$

Now we appeal to two lemmas from [18], the second one slightly enhanced.

Lemma 4.9. *Let* $\mathbf{L}$ *be a finite lattice. Then* $\mathbf{L}$ *satisfies* $(\mathrm{SD}_\wedge)$ *if and only if* $\kappa(a)$ *exists for each* $a \in \mathrm{J}(\mathbf{L})$.

Lemma 4.10. *Let* $\mathbf{L}$ *be a finite lattice that satisfies* $(\mathrm{SD}_\wedge)$. *The following are equivalent.*

(1) $\mathbf{L}$ *satisfies* $(\mathrm{SD}_\vee)$.

(2) *There do not exist* a, $b \in \mathrm{J}(\mathbf{L})$ *such that* $a \, B \, b \, B \, a$.

(3) *There do not exist* a, $b \in \mathrm{J}(\mathbf{L})$ *such that* $a \neq b$ *and* $\kappa(a) = \kappa(b)$.

Proof. The equivalence of (1) and (2) is Theorem 8 of [18].

The definition of $a \, B \, b$ is equivalent to $a \neq b$, $b_* \leq \kappa(a)$, $b \nleq \kappa(a)$. Thus $a \, B \, b$ implies $\kappa(a) \leq \kappa(b)$ in a meet semidistributive lattice (though not conversely). If $a \, B \, b \, B \, a$, then $a \neq b$ and $\kappa(a) = \kappa(b)$.

Finally, assume $a \neq b$ and $\kappa(a) = \kappa(b) = m$, say. Then $a \vee m = m^* = b \vee m > (a \wedge b) \vee m$, since $a \wedge b \leq a_*$ or $a \wedge b \leq b_*$. This is a failure of $(\mathrm{SD}_\vee)$. □

We have just checked that $\kappa(a)$ exists for each $a \in \mathrm{J}(\mathbf{X}_n)$, with the values given above. By Lemma 4.9, $\mathbf{X}_n$ satisfies $(\mathrm{SD}_\wedge)$. Moreover, it is straightforward to check that the values of $\kappa(a)$ are all distinct, so $\mathbf{X}_m$ satisfies $(\mathrm{SD}_\vee)$ by Lemma 4.10. Thus $\mathbf{X}_n$ is semidistributive.

This completes the proof of Theorem 4.8. □

Some comments on the differences between representing congruence lattices of bounded versus unbounded lattices are in order. The problems are twofold.

First, while Con $\mathbf{L} \cong \mathcal{O}(\mathrm{J}(\mathbf{L}), \overline{D})$ holds for all finite lattices, we would like to use the order induced by the E-relation. However, Con $\mathbf{L} \cong \mathcal{O}(\mathrm{J}(\mathbf{L}), \overline{E})$ holds for all bounded finite lattices, does not hold for all finite join semidistributive lattices, and it is unknown whether the E-relation suffices for finite semidistributive lattices. See the discussion in Section 6.6 of [2].

The second difficulty is that unbounded finite semidistributive lattices contain D-cycles, making the order on $\mathrm{J}(\mathbf{L})$ a proper quasi-order rather than a partial order. In that case it is necessary to work with $\overline{D}$-equivalence classes of join irreducibles. Little is known about the structure of unbounded finite semidistributive lattices, except that they fail Whitman's condition (W) [16]. The examples used above, from [14] and [18], may be the only examples in the literature. W. Geyer constructed others using formal concept analysis in connection with [9], but they may not have been published. Our general construction was modeled on [18].

5 A sufficient condition

If behooves us then to find *sufficient* conditions for a finite distributive lattice to be the congruence lattice of a finite semidistributive lattice.

Theorem 5.1. *Let* $\mathbf{P}$ *be a finite ordered set satisfying*

($\diamondsuit$) $\mathbf{P}$ *is a tree, i.e., no element has more than one lower cover,*

($\clubsuit$) *every non-maximal element in* $\mathbf{P}$ *has at least two upper covers.*

Then $\mathcal{O}(\mathbf{P})$ *is isomorphic to the congruence lattice of a finite bounded (and in particular semidistributive) lattice.*

In fact, the condition ($\diamondsuit$) that $\mathbf{P}$ be a tree is much stronger than needed for the construction to work, and is just the simplest way to guarantee that the technical condition of Theorem 5.11 holds.

Here is a sketch of our itinerary. We are given the ordered set $\mathbf{P} = (P, \leq)$. Define a new ordered set $\overline{\mathbf{P}} = (P, \sqsubseteq)$ with the same base set but a different order, described below. In fact, it will have the property that $x \sqsubseteq y$ implies $x \geq y$. The lattice $\mathbf{M}$ that we construct with Con $\mathbf{M} \cong \mathcal{O}(\mathbf{P})$ will be the lattice of closed ideals of a closure operator on $\overline{\mathbf{P}}$. The join irreducible elements of $\mathbf{M}$ will be the principal ideals $\downarrow_{\sqsubseteq} u$ with $u \in P$.

When there is any chance of confusion, we write either $(\mathbf{P}, \leq)$ or $(\overline{\mathbf{P}}, \sqsubseteq)$. The base set of both is P, and $\mathbf{P}$ by itself means $(\mathbf{P}, \leq)$.

The order $\sqsubseteq$ uses a function $\dagger : P \to P$ so that if x is not maximal in $(\mathbf{P}, \leq)$, then $x_\dagger$ is the unique lower cover of x in $(\overline{\mathbf{P}}, \sqsubseteq)$. This will imply that if $\downarrow_\sqsubseteq x$ is not an atom of $\mathbf{M}$, then $\downarrow_\sqsubseteq x_\dagger$ is its unique lower cover in $\mathbf{M}$, whence $\downarrow_\sqsubseteq x$ is join irreducible. The basic idea is to define the join on $\mathbf{M}$ so that $x \sqsubseteq x_\dagger \vee y$ whenever $x_\dagger \neq y \succ x$ in $(\mathbf{P}, \leq)$. There is a slight complication: if $x \sqsubseteq x_\dagger \vee y$ is a minimal nontrivial join cover and $y \sqsubseteq y_\dagger \vee z$, then meet semidistributivity implies $x \sqsubseteq x_\dagger \vee z$. The recursive definitions in the construction are a way of addressing this difficulty. With that guide, let us proceed.

Assume that $\mathbf{P}$ satisfies (♣). For each non-maximal $p \in P$, choose an element $p_\dagger \succ p$ in $(\mathbf{P}, \leq)$. If p is maximal, then $p_\dagger$ is undefined. Let $p_\dagger \sqsubset p$, and take the reflexive, transitive closure of $\sqsubset$ as the order $\sqsubseteq$ on P. (There will in general be many options for the $\dagger$-function, but choose one.)

Let us consider the order $\sqsubseteq$ on the elements of P. The ideal of $(\overline{\mathbf{P}}, \sqsubseteq)$ generated by an element $u \in P$ is $\downarrow_\sqsubseteq u = \{u, u_\dagger, u_{\dagger\dagger}, \dots\}$. Use $u_{(k)}$ to denote $u_{\dagger\cdots\dagger}$ with k daggers.

Lemma 5.2. *The order $\sqsubseteq$ on $\overline{\mathbf{P}}$ satisfies the following.*

(1) $u \sqsubseteq v$ *if and only if* $u = v_{(k)}$ *for some* $k \geq 0$.

(2) $u \sqsubseteq v$ *implies* $u \geq v$.

(3) $\downarrow_\sqsubseteq v$ *is a chain.*

(4) $\uparrow_\sqsubseteq u$ *is a tree.*

The proofs are straightforward. Note that (3) and (4) are equivalent in any ordered set.

Next, for each $x \in P$, we partition P into subsets $K(x)$ and $L(x) = P \setminus K(x)$. This is done recursively on the depth of x in $(\mathbf{P}, \leq)$. If x is a maximal element, then

$$K(x) = \{z \in P : x \not\sqsubseteq z\}$$
$$L(x) = \{z \in P : x \sqsubseteq z\} = \uparrow_\sqsubseteq x\,.$$

If x is not maximal in $(\mathbf{P}, \leq)$ and $K(u)$, $L(u)$ are defined for all $u > x$, set

$$K(x) = \bigcap_{x_\dagger \neq y \succ x} K(y) \ \cap \ \{z \in P : x \not\sqsubseteq z\}$$
$$L(x) = \bigcup_{x_\dagger \neq y \succ x} L(y) \ \cup \ \{z \in P : x \sqsubseteq z\}\,.$$

By induction on the depth of x in $(\mathbf{P}, \leq)$, and using DeMorgan's laws, one can show that $P = K(x) \,\dot\cup\, L(x)$ for all $x \in P$.

Lemma 5.3. *For $x \in P$,*

(1) $x \in L(x)$*, whence* $x \notin K(x)$*,*

(2) *if* $x_\dagger \neq y \succ x$*, then* $y \in L(x)$*,*

(3) *if* $z \sqsubseteq u \in K(x)$*, then* $z \in K(x)$*,*

(4) *if* $t \sqsupseteq v \in L(x)$*, then* $t \in L(x)$*.*

The last pair says that $K(x)$ is an order ideal in $(\overline{\mathbf{P}}, \sqsubseteq)$ and $L(x)$ is an order filter. Item (3) requires an easy induction, and (4) follows by complementation.

Let us describe $L(x)$ and $K(x)$ more completely. Recursively define subsets $L^*(x) \subseteq P$ for $x \in P$ by

$$L^*(x) = \begin{cases} \{x\} & \text{if } x \text{ is maximal in } (\mathbf{P}, \leq), \\ \{x\} \cup \bigcup_{x_\dagger \neq y \succ x} L^*(y) & \text{otherwise.} \end{cases}$$

Lemma 5.4. *For all $x \in P$,*

(1) $L^*(x)$ *is contained in* $\uparrow_\leq x$*,*

(2) $u \in L(x)$ *if and only if* $u \sqsupseteq v$ *for some* $v \in L^*(x)$*.*

The proofs are straightforward induction using the definitions of $L(x)$ and $L^*(x)$.

Now to describe $K(x)$.

Lemma 5.5. *For each $x \in P$,*

$$K(x) = P \setminus \bigcup_{u \in L^*(x)} \uparrow_\sqsubseteq u$$

Proof. If x is maximal, $K(x) = P \setminus \uparrow_\sqsubseteq x$. So assume the statement holds for all $u > x$. Then

$$\begin{aligned} K(x) &= \bigcap_{x_\dagger \neq y \succ x} K(y) \ \cap \ \{z \in P : x \not\sqsubseteq z\} \\ &= \bigcap_{x_\dagger \neq y \succ x} \left(P \setminus \bigcup_{u \in L^*(y)} \uparrow_\sqsubseteq u \right) \ \cap \ (P \setminus \uparrow_\sqsubseteq x) \\ &= \bigcap_{u \in L^*(x)} (P \setminus \uparrow_\sqsubseteq u) \\ &= P \setminus \bigcup_{u \in L^*(x)} \uparrow_\sqsubseteq u \end{aligned}$$

by DeMorgan's laws. □

Now we make additional assumptions about $(\mathbf{P}, \leq)$ and $\dagger$:

($\heartsuit$) if $x_\dagger \neq y \succ x$ in $(\mathbf{P}, \leq)$, then

(a) $x_\dagger \in K(y)$,

(b) $y \in K(x_\dagger)$,

(c) $y_\dagger \in K(x)$.

The condition looks mysterious, so some discussion is in order.

Long aside on ($\heartsuit$).

The first observation is straight from the definitions, using (a), but important.

Lemma 5.6. *If ($\heartsuit$) holds and $x \in P$ is not maximal in $(\mathbf{P}, \leq)$, then $x_\dagger \in K(x)$.*

Consequently, condition (c) is equivalent to

(c′) if both y and y' satisfy $x_\dagger \neq u \succ x$, then $y_\dagger \in K(y')$.

Corollary 5.7. *Assume ($\heartsuit$) holds and $x \in P$ is not maximal in $(\mathbf{P}, \leq)$. If $w \in P$ satisfies $w \sqsupseteq x_\dagger$ and $w \not\sqsupseteq x$, then $w \in K(x)$.*

Proof. If $w \notin K(x)$ then $w \in L(x)$, which means that $w \sqsupseteq t$ for some $t \in L^*(x)$. Remember that $\downarrow_\sqsubseteq w$ is a chain, so $x_\dagger = w_{(i)}$ and $t = w_{(j)}$ for some pair i, j. But $w \not\sqsupseteq x$, so $t \in L^*(y)$ for some y with $x_\dagger \neq y \succ x$. This implies $t > x$. Hence $i < j$, making $x_\dagger \sqsupset t$, which is a contradiction since $x_\dagger \in K(x)$ and $t \in L(x)$. ☐

How could ($\heartsuit$) fail? Consider $x_\dagger \neq y \succ x$ in $(\mathbf{P}, \leq)$, and for (iii) also $x_\dagger \neq y' \succ x$. Here are some failures of (a), (b), and (c′) respectively.

(i) If $y_\dagger \neq x_{(k)} \succ y$ for some $k > 0$, then $x_{(k)} \in L(y)$, whence $x_\dagger \in L(y)$ since $x_\dagger \sqsupseteq x_{(k)}$.

(ii) If $x_{\dagger\dagger} \neq y_{(\ell)} \succ x_\dagger$ for some $\ell > 0$, then $y_{(\ell)} \in L(x_\dagger)$, whence $y \in L(x_\dagger)$ since $y \sqsupseteq y_{(\ell)}$.

(iii) If $y'_\dagger \neq y_{(m)} \succ y'$ for some $m > 0$, then $y_{(m)} \in L(y')$, whence $y_\dagger \in L(y')$ since $y_\dagger \sqsupseteq y_{(m)}$, contra (c′).

Figures 6 and 7 illustrate these situations. Figure 6 shows the conditions (i)–(iii) prohibited by ($\heartsuit$), while Figure 7 indicates the exceptions allowed. Solid black lines are covers, solid red lines are covers of the form $u_\dagger \succ u$, and dashed red lines indicate sequences of covers from u to $u_{(k)}$ with $k \geq 1$.

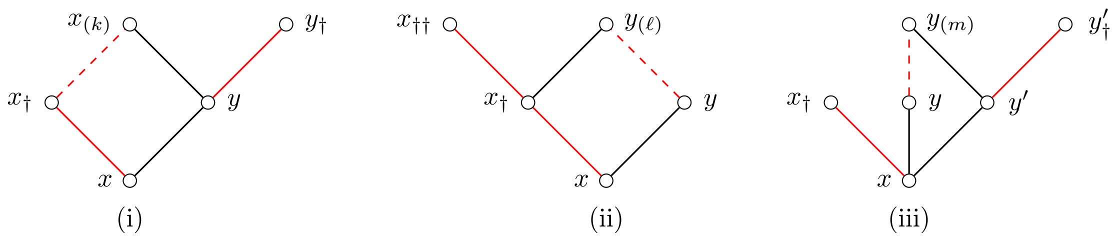

Figure 6: Configurations prohibited in $(\mathbf{P}, \leq)$ by $(\heartsuit)$

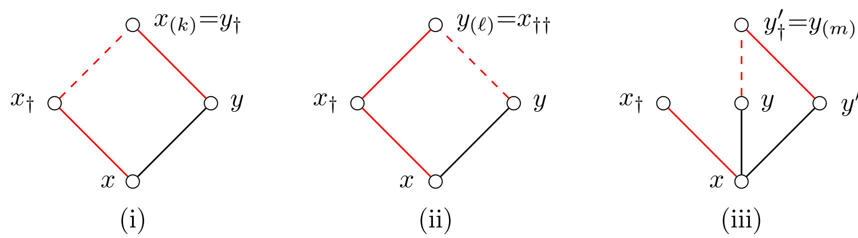

Figure 7: Exceptions allowed by $(\heartsuit)$

The failures of $(\heartsuit)$ in (i)–(iii) were *direct*, in that they used $x_\dagger$ and y. The next type of failures are *once removed*, using a cover z of one of those elements. Again let $x_\dagger \neq y \succ x$.

(iv) If $x_{(k)} \succ z \succ y$ for some $k > 0$, with $x_{(k)} \neq z_\dagger$ and $z \neq y_\dagger$, then $x_{(k)} \in L(y)$, whence $x_\dagger \in L(y)$.

(v) If $y_{(\ell)} \succ z \succ x_\dagger$ for some $\ell > 0$, with $y_{(\ell)} \neq z_\dagger$ and $z \neq x_{\dagger\dagger}$, then $y_{(\ell)} \in L(x_\dagger)$, whence $y \in L(x_\dagger)$.

(vi) If $y_{(m)} \succ z \succ y'$ for some $m > 0$, with $y_{(m)} \neq z_\dagger$ and $z \neq y'_\dagger$, then $y_{(m)} \in L(y')$, whence $y \in L(y')$, contra (c′).

Cases (iv)–(vi) are illustrated in Figure 8.

Continuing in this manner, we arrive at the following characterization.

Theorem 5.8. *An ordered set* $\mathbf{P}$ *with a* $\dagger$*-function satisfies* $(\heartsuit)$ *if and only if there do not exist* k, $\ell \geq 2$ *and elements* u, x *and covering chains*

$$u = c_0 \succ c_1 \succ \cdots \succ c_{k-1} \succ x$$
$$u = d_0 \succ d_1 \succ \cdots \succ d_{\ell-1} \succ x$$

with $c_{i-1} = c_{i\dagger}$ *for* $1 \leq i \leq k-1$ *and* $d_{j-1} \neq d_{j\dagger}$ *for* $1 \leq j \leq \ell - 1$.

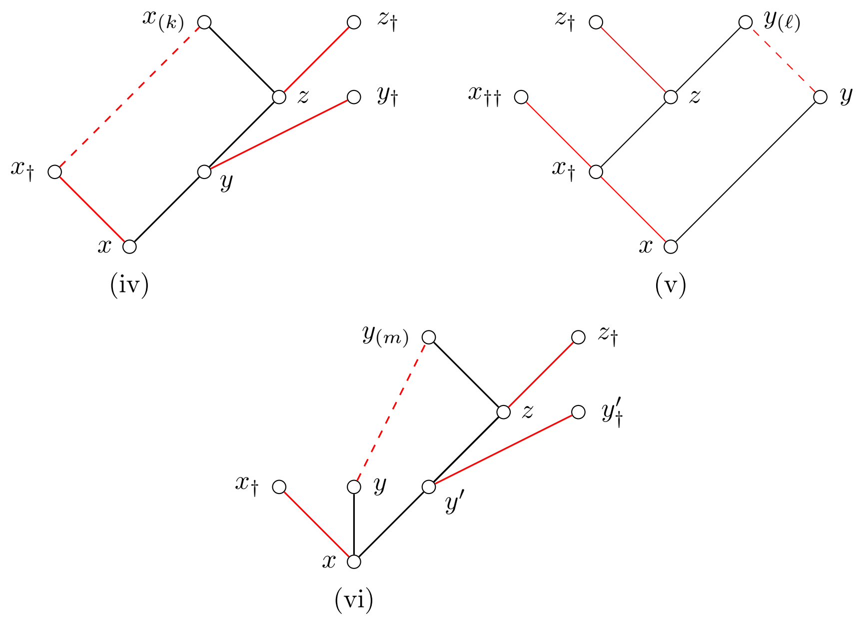

Figure 8: More configurations prohibited in $(\mathbf{P}, \leq)$ by $(\heartsuit)$, cases (iv)–(vi)

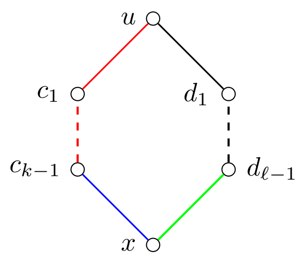

Figure 9: Prohibited configuration from Theorem 5.8. The blue and green edges can be red or black, but not both red.

The forbidden configuration is illustrated in Figure 9 where again red edges indicate $c_{i-1} = c_{i\dagger}$ and black edges indicate $d_{j-1} \neq d_{j\dagger}$. The blue and green edges can be either, except they cannot both be red, *i.e.*, we can have $x_\dagger = c_{k-1}$ or $x_\dagger = d_{\ell-1}$ or neither, but not both.

This must be balanced with the requirement that $x_\dagger$ be defined for every non-maximal $x \in P$. As an immediate consequence of Theorem 5.8, we see that there is a †-function satisfying $(\heartsuit)$ whenever

- $\mathbf{P}$ is a tree (the condition $(\diamondsuit)$ of Theorem 5.1), or
- the height of $\mathbf{P}$ is at most 2, *i.e.*, $\mathbf{P}$ contains no 3-element chain.

To find a more general sufficient condition for $\mathbf{P}$, satisfying $(\clubsuit)$, to admit a †-function satisfying $(\heartsuit)$, we imagine that † is given, and color the edges (covers) of the form $(c, c_\dagger)$ of $\mathbf{P}$ red, the

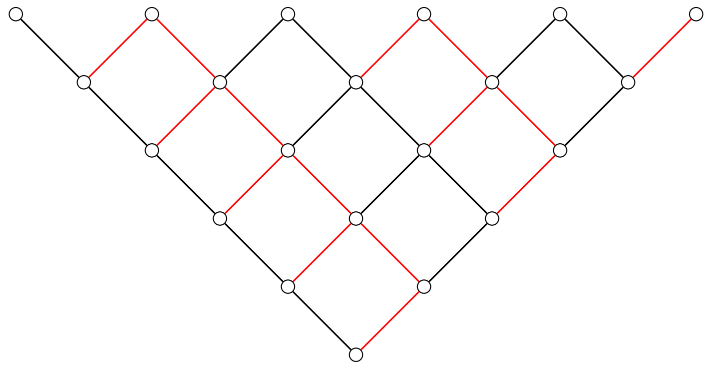

Figure 10: An ordered set with an edge-coloring that satisfies the conditions of Theorem 5.9, and hence $\mathcal{O}(\mathbf{P})$ is representable by Theorem 5.11.

remaining edges black. Classify the non-minimal vertices of $\mathbf{P}$ thusly.

- An element of P is a Λ-*node* if it has ≥ 2 lower covers.
- An element with 1 lower cover is an S-*node*.
- s is a *red* Λ-*node* if all its lower covering edges are red.
- t is a *black* Λ-*node* if all its lower covering edges are black.
- u is a *red* S-*node* if its unique lower covering edge is red.
- v is a *black* S-*node* if its unique lower covering edge is black.
- w is a *mixed node* if it is a Λ-node with both red and black lower covers.

Theorem 5.9. *Let* $\mathbf{P}$ *be a finite ordered set that satisfies* (♣). *Suppose there is a coloring of the edges of* $\mathbf{P}$ *such that*

(i) $\mathbf{P}$ *has no mixed nodes,*

(ii) every non-maximal node has exactly 1 red upper cover and ≥ 1 *black upper covers.*

For non-maximal elements $x \in P$*, define* $x_\dagger$ *to be the red upper cover of* x*. Then* $\mathbf{P}$ *with the function* $\dagger$ *satisfies* (♡).

For the configuration of Theorem 5.8 cannot occur, as every Λ-node is either red or black. Item (ii) guarantees that there is a unique choice for $x_\dagger$. Examples are given in Figures 10 and 12.

Conjecture: *If* $\mathbf{P}$ *is planar, then it has a coloring satisfying the conditions of* Theorem 5.9.

The ordered sets $\mathbf{P}$ for which $\mathcal{O}(\mathbf{P})$ is known not to be representable as the congruence lattice of a finite semidistributive or bounded lattice are all excluded by the condition (♣). It takes a

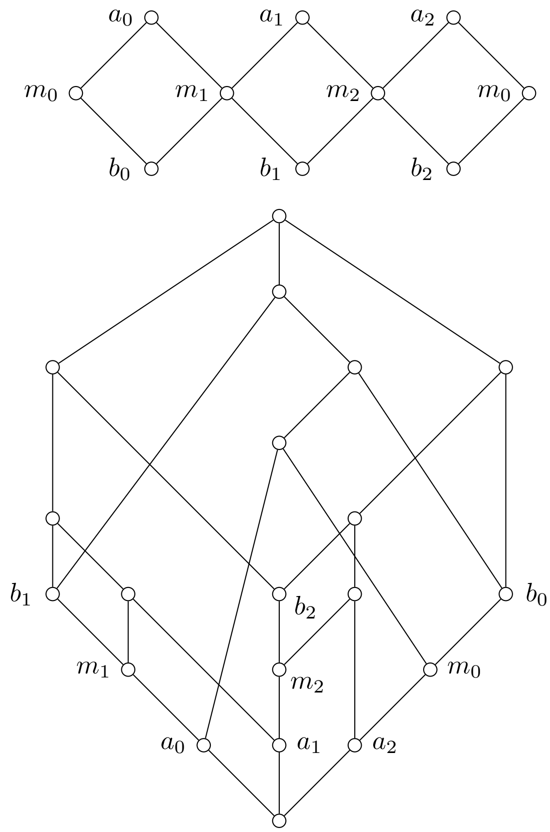

Figure 11: The ordered set $\mathbf{Q}$ at the top that satisfies (♣) but has no †-function satisfying (♡). Note that $\mathbf{Q}$ is a torus: m_0 is depicted twice. Nonetheless, $\mathcal{O}(\mathbf{Q}) \cong \operatorname{Con} \mathbf{K}$ for the bounded lattice at the bottom.

little effort to find an ordered set $\mathbf{Q}$ that satisfies (♣) but fails (♡). Nonetheless, they exist, and the ordered set $\mathbf{Q}$ at the top of Figure 11 gives one such. By circular symmetry we may assume $m_{0\dagger} = a_0$. To avoid the configuration of Figure 9, that implies $m_{1\dagger} = a_0$. Hence $m_{1\dagger} \neq a_1$, whence $m_{2\dagger} \neq a_1$. That in turn leads to $m_{2\dagger} = a_2$ and $m_{0\dagger} = a_2$, a contradiction.

Even though Theorem 5.11 does not apply, $\mathcal{O}(\mathbf{Q})$ is the congruence lattice of a finite bounded lattice. The lattice $\mathbf{K}$ at the bottom of Figure 11 was obtained from $\mathbf{B}_3$ by two sets of doubling three intervals, so it is bounded. The minimal nontrivial join covers in $\mathbf{K}$ are

$$m_i \leq a_i \vee a_{i+2}$$
$$b_i \leq m_i \vee a_{i+1}$$
$$b_i \leq m_{i+1} \vee a_{i+2}$$

where the subscripts are taken modulo 3. Thus $(\mathrm{J}(\mathbf{K}), \overline{D}) \cong \mathbf{Q}$.

The argument against the ordered set in Figure 11 satisfying (♡) depended on having an odd

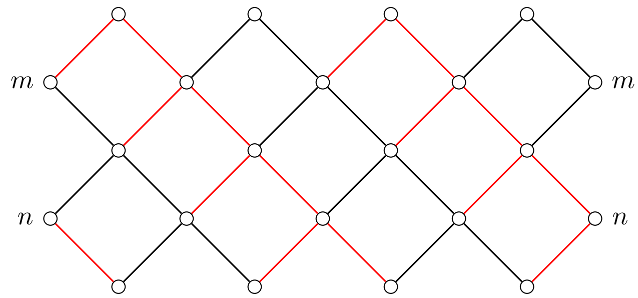

Figure 12: An ordered set that is a torus and has an edge-coloring satisfying the conditions of Theorem 5.9.

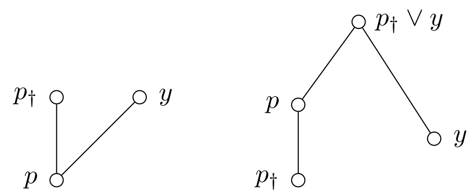

Figure 13: Illustrating a basic closure rule: on left $p_\dagger$, $y \succ p$ in $\mathbf{P}$, on right $p \sqsubseteq p_\dagger \vee y$ in $\mathbf{M}$.

number of squares across the top row. With an even number, there is no problem satisfying the conditions of Theorem 5.9, and the pattern can be extended downward as well, as in Figure 12.

Finally we are in position to construct the lattice $\mathbf{M}$. Assume that $(\mathbf{P}, \leq)$ satisfies (♣) and that the †-function has been chosen to satisfy (♡). Form the ordered set $\overline{\mathbf{P}} = (P, \sqsubseteq)$ with $u \sqsubseteq v$ iff $u = v_{(k)}$ for some $k \geq 0$. Then define closure rules on P by setting $p \in \gamma(\{y\})$ if $p \sqsubseteq y$, and

$$p \in \gamma(\{p_\dagger, u\})$$

for each non-$\sqsubseteq$-minimal $p \in P$ and every $u \in L(p)$. With a slight abuse of notation, it is convenient to think of γ as a join operation and write the closure rule as

$$p \sqsubseteq p_\dagger \vee u$$

for each $u \in L(p)$. The condition (♣) makes this not vacuous. Let $\mathbf{M}$ be the lattice of γ-closed order ideals of $(\overline{\mathbf{P}}, \sqsubseteq)$, *i.e.*, subsets closed under joins and downward containment $\sqsupseteq$.

The closure rule $p \sqsubseteq p_\dagger \vee y$ when $p_\dagger \neq y \succ p$ in $(\mathbf{P}, \leq)$ is illustrated in Figure 13. In general there will be other closure rules: if $p_\dagger \neq y \succ p$ and $y_\dagger \neq z \succ y$, then we also have $p \sqsubseteq p_\dagger \vee z$, etc.

Lemma 5.10. *Let $\mathbf{M}$ be the lattice constructed above. Then the order $\sqsubseteq$ on P and the set of closure rules $x \sqsubseteq x_\dagger \vee u$ with $u \in L^*(x)$ are a basis for $\mathbf{M}$. Moreover, the join irreducible elements of $\mathbf{M}$ are exactly the ideals $\downarrow_\sqsubseteq u$ for $u \in P$.*

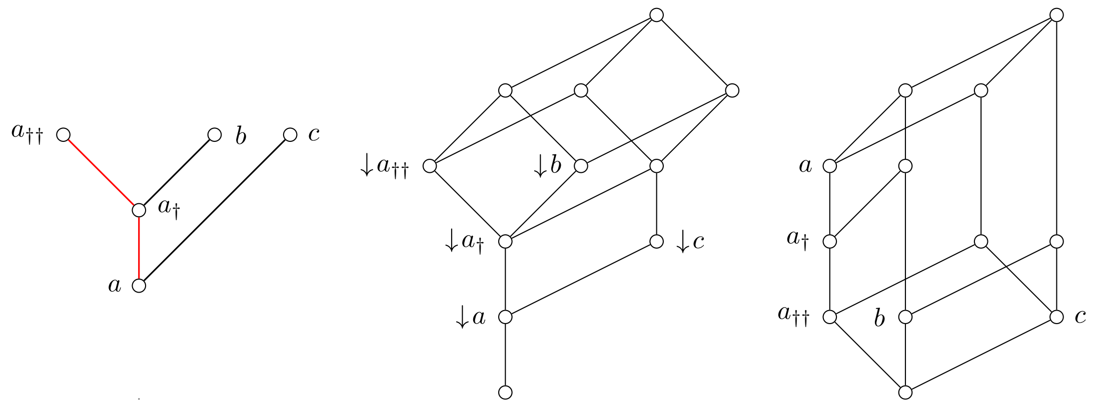

Figure 14: Representing a small distributive lattice as Con **M** with **M** semidistributive: **P**, $\mathcal{O}(\mathbf{P})$, **M**.

Proof. The first part follows from Lemma 5.4.

Clearly every ideal in **M** is the join of the principal ideals $\downarrow_{\sqsubseteq} u$ that it contains. Note that for any $u \in P$, $\downarrow_{\sqsubseteq} u$ is closed with respect to the join operation in **M**, since $\downarrow_{\sqsubseteq} u$ is a chain. Thus you can identify u with the ideal $\downarrow_{\sqsubseteq} u$, as usual, and observe that $u_\dagger$ is the unique lower cover of u in **M**. In particular, each $u \in P$ is join irreducible in **M**. (This is slightly more subtle than it appears. If we had $p_\dagger \sqsubseteq u$ and $p \not\sqsubseteq u$, then $u \in K(p)$ by Corollary 5.7. That implies $\downarrow_{\sqsubseteq} u \subseteq K(p)$, so no closure rule can apply in $\downarrow_{\sqsubseteq} u$.) □

Now we can state the stronger version of Theorem 5.1.

Theorem 5.11. *Let* **P** *be a finite ordered set with a* †*-function satisfying* (♡) *and* (♣)*. Then* $\mathcal{O}(\mathbf{P})$ *is isomorphic to the congruence lattice of a finite bounded (and hence semidistributive) lattice.*

Figure 14 provides an example of the construction, giving **P**, $\mathcal{O}(\mathbf{P})$, and **M**. The defining relations for **M** are $a \sqsupset a_\dagger \sqsupset a_{\dagger\dagger}$, $a \sqsubseteq a_\dagger \vee c$, and $a_\dagger \sqsubseteq a_{\dagger\dagger} \vee b$. It is straightforward to verify that Con $\mathbf{M} \cong \mathcal{O}(\mathbf{P})$.

Figure 15 provides another example of the construction. The defining relations for **M** are $a \sqsupset a_\dagger$, $b \sqsupset b_\dagger$, $a \sqsubseteq a_\dagger \vee b_\dagger$, and $b \sqsubseteq a_\dagger \vee b_\dagger$.

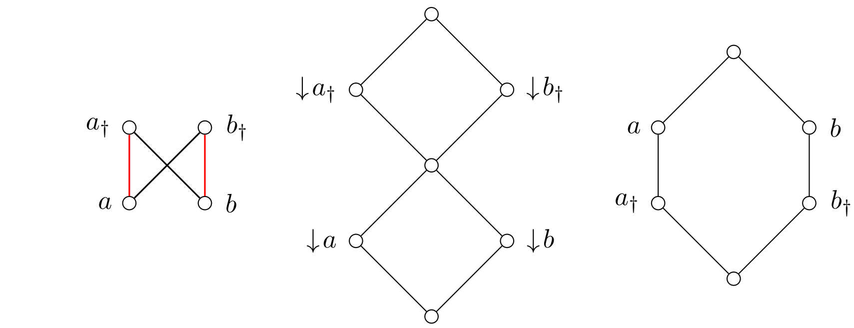

Figure 15: Representing a small distributive lattice as Con **M** with **M** semidistributive: **P**, $\mathcal{O}(\mathbf{P})$, **M**.

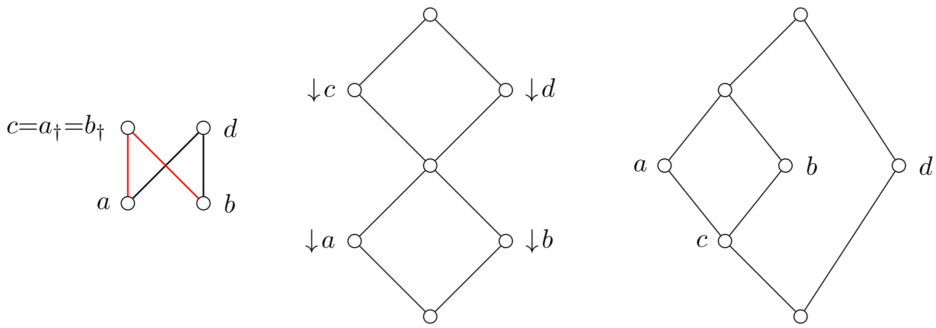

Figure 16: The same **P** as Figure 15 with a different †-function: **P**, $\mathcal{O}(\mathbf{P})$, $\mathbf{M}'$.

Figure 16 has the same ordered set **P** as Figure 15, with a different †-function. Thus $\mathcal{O}(\mathbf{P})$ remains the same, but the closure rules for $\mathbf{M}'$ are $c \sqsubseteq a$, b and a, $b \sqsubseteq c \vee d$.

Now let us return to the business of proving that the construction works, *i.e.*, produces a bounded lattice **M** with Con $\mathbf{M} \cong \mathcal{O}(\mathbf{P})$ when the two conditions (♡) and (♣) are satisfied.

Lemma 5.12. *Let $Q \subseteq P$. The join $\bigvee Q$ in **M** is obtained by:*

(1) *for each $q \in Q$, add $\downarrow_{\sqsubseteq} q$ to obtain Q_1;*

(2) *recursively, if $x_\dagger$, $u \in Q_j$ with $u \in L^*(x)$, let $Q_{j+1} = Q_j \cup \{x\}$.*

If Q_m denotes the end result of applying (2) *as long as possible, then Q_m is a closed ideal of $(\overline{\mathbf{P}}, \sqsubseteq)$, and hence $\bigvee Q = Q_m$. In particular, one need not go back to* (1).

The crucial observation here is that when one adds x to Q_j in step (2), we already have $\downarrow_{\sqsubseteq} x_\dagger \subseteq Q_j$.

Lemma 5.13. *Let $x \in P$. Then every join cover $x \sqsubseteq \bigvee Q$ refines to a join cover $x \sqsubseteq \bigvee R$ with $R \ll Q$ and $R \subseteq \uparrow_{\leq} x$. Thus every minimal nontrivial join cover of x is contained in $\uparrow_{\leq} x$.*

Proof. Suppose $x \in Q_{m'}$ with $m' \leq m$ from Lemma 5.12. If $m' = 1$ then $x \sqsubseteq q$ for some $q \in Q$; note that implies $x \geq q$ in $(\mathbf{P}, \leq)$, *i.e.*, the trivial cover $\{x\}$ refines Q. So assume $m' > 1$. Then there exists $u \in L^*(x)$ such that $x_\dagger$, $u \in Q_{m'-1}$. Note $x \leq x_\dagger$, $x \leq u$, and both $x_\dagger$, $u \leq \bigvee Q$. By induction, there exist $R_1 \subseteq \uparrow_\leq x_\dagger$ with $R_1 \ll Q$ and $x_\dagger \sqsubseteq \bigvee R_1$, and $R_2 \subseteq \uparrow_\leq u$ with $R_2 \ll Q$ and $u \sqsubseteq \bigvee R_2$. Then $R_1 \cup R_2 \ll Q$, $R_1 \cup R_2 \subseteq (\uparrow_\leq x_\dagger) \cup (\uparrow_\leq u) \subseteq \uparrow_\leq x$, and

$$x \sqsubseteq x_\dagger \vee u \sqsubseteq \bigvee R_1 \vee \bigvee R_2$$

as desired. □

Lemma 5.14. *For each $x \in P$, $K(x)$ is a closed ideal of $(\overline{\mathbf{P}}, \sqsubseteq)$.*

Proof. Since maximal elements of $(\mathbf{P}, \leq)$ are join prime in $\mathbf{M}$, this certainly holds for them. So assume x is not maximal and that $K(y)$ is a closed ideal for every $y > x$. Recall that

$$K(x) = \bigcap_{x_\dagger \neq y \succ x} K(y) \cap S_x$$

where $S_x = \{z \in P : x \not\sqsubseteq z\}$.

Suppose $K(x)$ is not a closed ideal. Now $K(x)$ is an ideal with respect to $\sqsubseteq$ by Lemma 5.3(3). Assume it is not join-closed. Let $u \sqsubseteq u_\dagger \vee v$ be the first instance where a basic closure rule applies, *i.e.*, $u \notin K(x)$ but $u_\dagger$, $v \in K(x)$ and $v \in L^*(u)$. (We can do this because $K(x)$ is $\sqsupseteq$-closed.) Then, since each $K(y)$ is closed, we must have $u \notin S_x$ and $u_\dagger$, $v \in S_x$. Now $u_\dagger \in S_x$ means $u_{\dagger(k)} \neq x$ for all $k \geq 0$. But that implies $u_{(k+1)} \neq x$ for all $k \geq 0$. Meanwhile $u \notin S_x$ says $u_{(\ell)} = x$ for some $\ell \geq 0$. This only makes sense if $\ell = 0$, *i.e.*, $u = x$. But then $v \in L^*(x) \subseteq L(x)$, whence $v \notin K(x)$, a contradiction. □

Lemma 5.15. *If $x_\dagger \neq y \succ x$ in $(\mathbf{P}, \leq)$, then $x \sqsubseteq x_\dagger \vee y$ is a minimal nontrivial join cover in $\mathbf{M}$. Hence $x \, D \, x_\dagger$ and $x \, D \, y$.*

Proof. Let $x \in P$, so $\downarrow_\sqsubseteq x \in \mathbf{M}$. We have $x_{\dagger\dagger} \in K(x_\dagger)$ by Lemma 5.6, while $y \in K(x_\dagger)$ by $(\heartsuit)$(b). Thus $x_{\dagger\dagger} \vee y \sqsubseteq \bigvee K(x_\dagger) = K(x_\dagger)$ using Lemma 5.14, while $x \notin K(x_\dagger)$ since $x \sqsupset x_\dagger$. Therefore $x \not\sqsubseteq x_{\dagger\dagger} \vee y$.

Similarly, $x_\dagger \in K(x)$ by Lemma 5.6, while $y_\dagger \in K(x)$ by $(\heartsuit)$(c). Thus $x_\dagger \vee y_\dagger \sqsubseteq \bigvee K(x) = K(x)$, while $x \notin K(x)$. Hence $x \not\sqsubseteq x_\dagger \vee y_\dagger$. □

Lemma 5.13 does not tell us exactly which join covers are minimal. It is often the case in semidistributive lattices that compounding the defining join covers produces more minimal nontrivial join covers (though not *doubly* minimal join covers!). However, we know the following.

(i) The join irreducible elements of $\mathbf{M}$ are exactly the ideals $\downarrow_{\sqsubseteq} x$ for $x \in P$ (Lemma 5.10).

(ii) If $z \succ p$ in $(\mathbf{P}, \leq)$ then $x \, D \, z$ (Lemma 5.15).

(iii) If $p \sqsubseteq \bigvee Q$ is a minimal nontrivial join cover in $\mathbf{M}$, then $p < q$ in $(\mathbf{P}, \leq)$ for each $q \in Q$ (Lemma 5.13).

Consequently, the dependency relation D on $\mathbf{M}$ satisfies $\prec_{\mathbf{P}} \subseteq D \subseteq \leq_{\mathbf{P}}$ and we get $\operatorname{Con} \mathbf{M} \cong \mathcal{O}(\mathbf{P})$. Moreover, in view of (iii), there can be no D-cycles. Thus $\mathbf{M}$ is lower bounded, and hence join semidistributive

(It is interesting to note how the construction fails on the ordered set $\mathbf{Y}$, which fails (♣). On the other hand, in nature the defining closure operators need not use only covers.)

Now let us prove that $\mathbf{M}$ is meet semidistributive by showing that $\kappa(x)$ exists for each $x \in P$. It is useful to have a slightly enhanced technical version of Lemma 4.9.

Lemma 5.16. *In a finite lattice* $\mathbf{L}$*, the following conditions are equivalent (to* $(\mathrm{SD}_\wedge)$*).*

(1) *For all* $x \in \mathrm{J}(\mathbf{L})$ *there exists* $\kappa(x)$ *such that* $x \nleq \kappa(x)$ *and for all* $u \in L$*,* $x \nleq x_* \vee u$ *implies* $u \leq \kappa(x)$*.*

(2) *For all* $x \in \mathrm{J}(\mathbf{L})$ *there exists* $\kappa(x)$ *such that* $x \nleq \kappa(x)$ *and for all join irreducible elements* $w \in \mathrm{J}(\mathbf{L})$*,* $x \nleq x_* \vee w$ *implies* $w \leq \kappa(x)$*.*

Condition (1) is a traditional equivalent to meet semidistributivity, and (2) allows us to check it at join irreducibles only.

Proof. Clearly (1) implies (2). Conversely, assume that $\mathbf{L}$ satisfies (2) and that $x \nleq x_* \vee u$ for some $u \in L$. Let $u = \bigvee u_i$ with each $u_i \in \mathrm{J}(\mathbf{L})$. Since $u_i \leq u$ we have $x \nleq x_* \vee u_i$ for all i, whence $u_i \leq \kappa(x)$ by (2). Thus $u = \bigvee u_i \leq \kappa(x)$ as well. □

For each $x \in P$ we claim that $K(x) \subseteq P$ has these properties.

(a) $K(x)$ is a closed ideal of $(P, \sqsubseteq)$, *i.e.*, $\bigvee K(x) = K(x)$,

(b) $x_\dagger \in K(x)$,

(c) $x \notin K(x)$,

(d) for all $u \in P$ we have $x \not\sqsubseteq x_\dagger \vee u$ if and only if $u \in K(x)$.

Indeed, (a) is Lemma 5.13, (b) is Lemma 5.6, and (c) is Lemma 5.3(1). For (d), if $u \in L(x)$ then $x \sqsubseteq x_\dagger \vee u$ by the definition of join in $\mathbf{M}$. If $u \in K(x)$, though, then $x_\dagger \vee u \in K(x)$ by (a) and (b), while $x \notin K(x)$. Thus we cannot have $x \sqsubseteq x_\dagger \vee u$ if $u \in K(x)$.

We conclude by Lemma 5.16 that $\mathbf{M}$ is meet semidistributive. Moreover, since $\mathbf{M}$ is lower bounded and semidistributive, it is also upper bounded by Theorem 4.1.

Thus $\mathbf{M}$ as constructed is a finite bounded lattice with Con $\mathbf{M} \cong \mathcal{O}(\mathbf{P}, \leq)$, completing the proof of Theorem 5.1.

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