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The metric dimension of cyclic hexagonal chain honeycomb triangular mesh and pencil graphs

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ABSTRACT

The metric dimension of a graph serves a fundamental role in organizing structures of varying dimensions and establishing their foundations through diverse perspectives. Studying symmetric network characteristics like connectedness, diameter, vertex centrality, and complexity depends heavily on the distance parameter. In this article, we explore the exact value for different hexagonal networks' metric dimensions, such as cyclic hexagonal chains, triangular honeycomb mesh, and pencil graphs.

RESUMEN

La dimensión métrica de un grafo cumple un rol fundamental para organizar estructuras de dimensiones variables y establecer sus fundamentos a través de perspectivas diversas. Estudiar características de redes simétricas como la conexidad, diámetro, centralidad de vértices y complejidad depende fuertemente del parámetro de distancia. En este artículo exploramos el valor exacto de la dimensión métrica de diferentes redes hexagonales, tales como cadenas hexagonales cíclicas, la malla triangular panal y grafos lápices.

Keywords and Phrases: Metric basis, metric dimension, cyclic hexagonal chain, triangular honeycomb mesh, pencil graph.

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1 Introduction

In the field of robotics, the metric dimension problem is important. A robot is an automated machine designed to move through space while avoiding obstacles. It does not understand either direction or visibility. However, it is presumable that it can detect the separation of a collection of landmarks. Evidently, the robot can establish its precise location in space if it is aware of the distances to a significant number of landmarks. In order to perform this, the idea of "landmarks in a graph" was created [12], and later it was expanded to the metric dimension in which networks are taken into consideration within the framework of the graph structure.

Finding a metric basis for the graph is the goal of the metric dimension problem in graph theory; the landmarks that make up a metric basis are known as landmarks, and the cardinality of a metric basis is referred to as the metric dimension of the graph. Harary and Melter [10] did the first investigation into the metric dimension problem. They provided a description of the trees' metric dimensions. Melter and Tomescu investigated the grid graphs' metric dimension problem [18]. For each arbitrary graph, the metric dimension problem is NP-complete [9]. Since then, a great deal of study has been conducted on this problem. In many fields of science and technology, the metric dimension has several uses. For grid graphs and trees, the metric dimension problem has been studied [12], hexagonal and honeycomb networks [15], silicate networks [16], torus networks [14], and enhanced hypercubes [17]. Metric dimension is used to address issues with robot navigation and pattern recognition [12], network discovery and validation [5], and issues with coin weighing and graph joins [20, 22].

In this paper, in Section 2, preliminaries and basis definitions are discussed. Section 3, deals with the metric dimension of the cyclic hexagonal chain, honeycomb triangular mesh, and pencil graph. Finally, the Significance and Contributions of the Results, concluding remarks and, open problem are given in Section 4 and Section 5 respectively.

2 Basis concepts

A finite simple connected graph $G = (V, E)$ is used in this paper, where V and E are the set of vertices and edges respectively. The distance between two vertices a and b in a graph G, denoted as $d(a, b)$, is defined as the minimum number of edges in any path from a to b. It is normal to have questions about the characterizations of graphs based on their metric dimension. Researchers are continuously interested in determining whether the metric dimension of a network family is constant, bounded, or unbounded. Consequently, there has been significant research focused on finding the metric dimension of networks, resulting in numerous findings. Examples of such findings include: Muhammad *et al.* [19] investigated the metric dimension of some chemical structures. Akhter and Farooq [3] investigated the metric dimension of the Indu-Bala product of graphs. The metric dimension of the subdivided honeycomb network and Aztec diamond network was determined by Xiujun *et al.* [23]. Ahmad *et al.* [1] found the metric dimension for benzenoid hammer graph. A bicyclic network's metric dimension was examined by Khan *et al.* [11]. Bokhary *et al.* [11] studied the metric dimension of the subdivision graph of a circulant network. Koam *et al.* [13] investigated the metric dimension and exchange property of nanotubes. Resolving sets have been discussed across the literature [2, 4, 8, 10, 18].

In this study, we obtain the metric dimension of specific planar architectures. To prove the main results we need the following.

Definition 2.1. *The diameter of a graph is the greatest distance between any pair of vertices, where the distance is defined as the length of the shortest path connecting them.*

Definition 2.2. *The metric basis or resolving set for a graph $G = (V, E)$, a resolving set of G is a subset of vertices $S \subseteq V$ such that every vertex $v \in V$ is uniquely determined by its distance vector to the vertices in S. For each vertex $v \in V$, its distance vector with respect to S is defined as $(d(v, s_1), d(v, s_2), \ldots, d(v, s_k))$, where $s_1, s_2, \ldots, s_k \in S$, and $d(v, s_i)$ is the shortest distance between v and s_i in the graph.*

The subset S is a metric basis if, for any two distinct vertices $u, v \in V$, their distance vectors relative to S are distinct, i.e.,

$$d(u, s_i) \neq d(v, s_i) \quad \text{for at least one } s_i \in S.$$

The cardinality of the metric basis or resolving set S is called the metric dimension of the graph and is denoted as $\dim(G)$.

Theorem 2.3 ([7])**.** *A simple connected graph G has a metric dimension 1 if and only if it is precisely identical in structure to the path graph P_n.*

Theorem 2.4 ([12])**.** *Suppose G is a graph with a minimum metric dimension of 2, and let $\{a, b\}$ be a subset of the vertices set V that forms a metric basis in B. In this context, the subsequent statements hold true:*

(a) Only one shortest route is possible between a and b.

(b) Each a and b has a maximum degree of three.

3 Main results

In this section, we determine the metric dimension of the cyclic hexagonal chain, honeycomb triangular mesh, and pencil graph.

3.1 Cyclic hexagonal chain

A catacondensed hexagonal structure known as a hexagonal chain has each hexagon being next to no more than two other hexagons. The graph representation of linear polyacene is a linear hexagonal chain, which is a hexagonal chain. A cyclic hexagonal chain is created when the ends of a linear hexagonal chain are bent to touch. The symbol H_n will be used to represent a cyclic hexagonal chain of dimension n respectively. We split the vertices of H_n as I and J, where I and J are the set of all vertices in the inner and outer cycle respectively. The cyclic hexagonal chain is symmetric in rotation and has $4n$ vertices in which $2n$ vertices are in each of the inner and outer cycles labeled as $I = \{i_1, i_2, i_3, \ldots, i_{2n}\}$ and $J = \{j_1, j_2, j_3, \ldots, j_{2n}\}$ in the clockwise direction respectively. For example, the labeling of a cyclic hexagonal chain of dimension n is given in Figure 1.

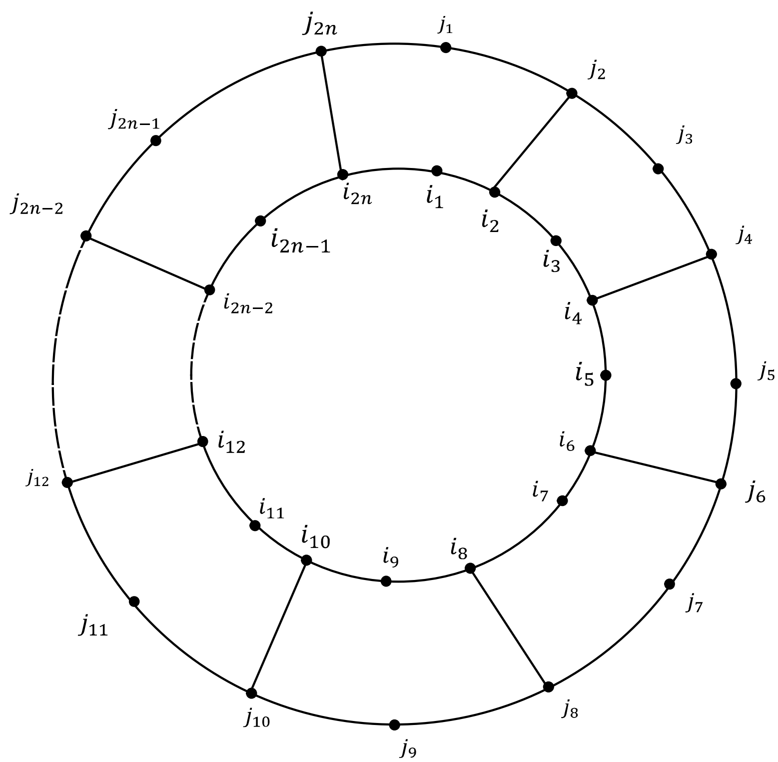

Figure 1: Labeling of cyclic hexagonal chain H_n

Theorem 3.1. *The metric dimension of the graph of the cyclic hexagonal chain H_n is more than 2 for $n \geq 2$.*

Proof. Based on Theorem 2.4, suppose that there exists a resolving set T with size 2. There are two cases for T.

Case 1. Suppose that $T = \{j_k, j_l\}$ for some k and l where $1 \leq k \leq n+1$ (by the symmetry of H_n, it is enough to consider the first half of the cycle). Then we have $r(i_{l+1}|T) = r(j_{l+2}|T) = (2, l-k+2)$.

Case 2. Suppose that $T = \{i_k, j_l\}$ for some $k, l = 1, 2, \ldots, n+1$. If $k = 1$, then we have $r(i_{k-1}|T) = r(i_{k+1}|T) = (1, 2)$. If $k < l$ (without loss of generality), then we have two subcases: if l is odd, then $r(i_l|T) = r(j_{l-1}|T) = (1, l-k)$, and if l is even, then $r(i_{k+2}|T) = r(j_{k+1}|T) = (2, l-k-1)$.

From these two cases, we find two vertices having the same representations. Therefore, T is not a resolving set of H_n, a contradiction. □

Theorem 3.2. *The metric dimension of the graph of the cyclic hexagonal chain H_n is 3 for $n \geq 2$.*

Proof. Let $T = \{j_1, j_2, j_{n+1}\}$ be a resolving set of H_n. To prove that T is a resolving set. It is enough to prove that all the vertices j_l, i_l $1 \leq l \leq 2n$ of H_n have unique representations with respect to T.

For $1 \leq l \leq 2n$, the representation j_l of H_n with respect to T is given as follows:

$$r(j_l|T) = \begin{cases} (l-1, 1, n), & \text{if } l = 1 \\ (l-1, l-2, n-l+1) & \text{if } 2 \leq l \leq n \\ (l-1, l-2, 0) & \text{if } l = n+1 \\ (2n-l+1, 2n-l+2, l-n-1) & \text{if } n+2 \leq l \leq 2n. \end{cases}$$

For $1 \leq l \leq 2n$, the representation of i_l of H_n with respect to T is given as follows:

$$r(i_l|T) = \begin{cases} (3, 2, n+1) & \text{if } l = 1 \\ (l, l-1, n+2-l) & \text{if } 2 \leq l \leq n \\ (n+1, n, 1) & \text{if } l = n+1 \\ (2n-l+2, 2n-l+3, l-n) & \text{if } n+2 \leq l \leq 2n. \end{cases}$$

We can see that each vertex of H_n has a distinct representation and satisfies the notion of a resolving set with regard to T. Hence $\dim(H_n) = 3$. □

3.2 Honeycomb triangular mesh

In this section, we show that the construction and the metric dimension of the honeycomb triangular mesh are discussed. Honeycomb triangular mesh is built recursively using hexagonal tessellations with three pendant edges. The honeycomb triangular mesh HTM_1 is a single vertex. The honeycomb triangular mesh HTM_2 is obtained by adding 3 pendant edges to HTM_1. In a similar manner, the n-dimensional honeycomb triangular mesh HTM_n is adding $(n-2)$ hexagons to the boundary of HTM_{n-1} with three pendent edges in the triangular form. The number of vertices, edges, faces, and diameter of HTM_n are n^2, $\frac{3(n^2-n)}{2}$, $\frac{n^2-3n+4}{2}$, and $(2n-2)$ respectively. A honeycomb triangular mesh HTM_1, HTM_2, HTM_3, and HTM_4 are shown in Figure 2.

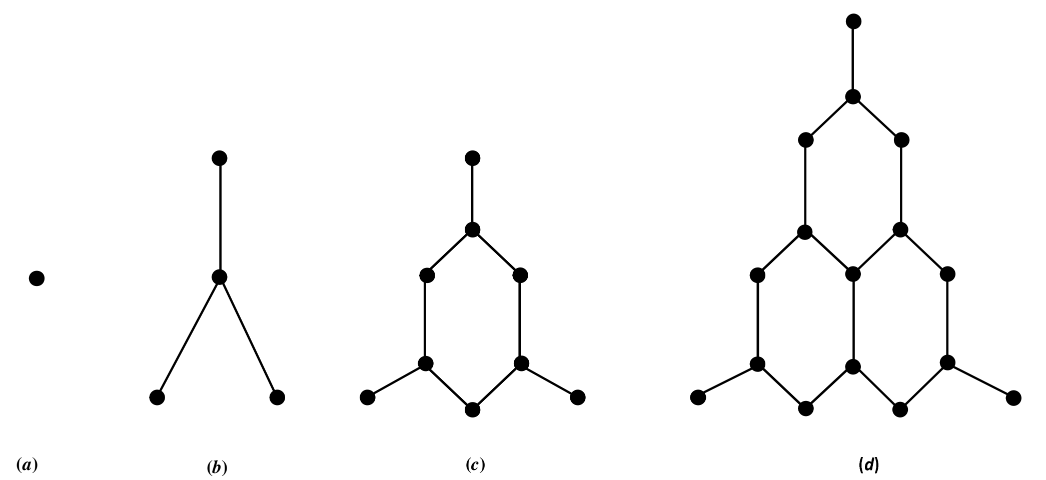

Figure 2: Honeycomb triangular mesh (a) HTM_1, (b) HTM_2, (c) HTM_3, and (d) HTM_4

The strip between two successive lines is marked in Honeycomb Triangular mesh is called the segments and it is denoted by S_L. The representation of any two points $p(l_1, m_1)$ and $q(l_2, m_2)$ in the honeycomb triangular mesh is defined by if $l_1 = l_2$, then p and q lies in the same segment, and if $l_1 \neq l_2$, then p and q are lies in the different segments. The distance between any two vertices $p(l_1, m_1)$ and $q(l_2, m_2)$ is non zero, when p and q lie in the same and different segments. We partition the vertices of HTM_n into n segments, namely S_1, S_2, S_3, …, S_n, and the segment representation of Honeycomb triangular mesh is shown in Figure 3.

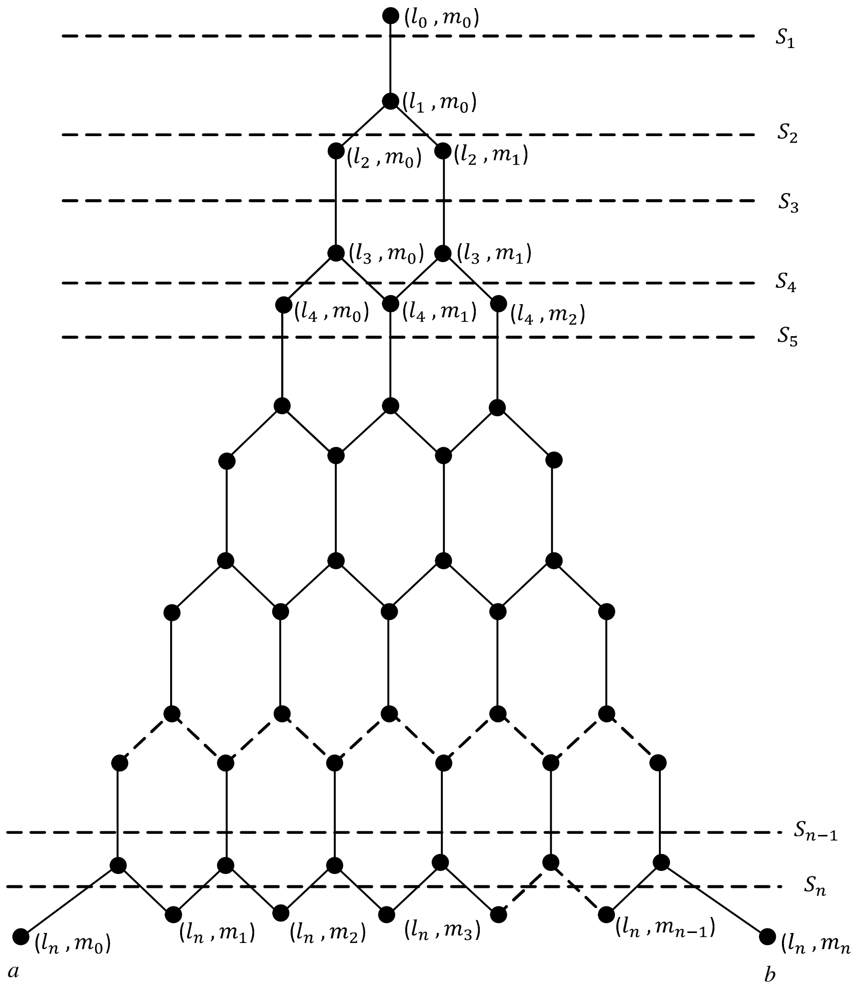

Figure 3: Honeycomb triangular mesh of dimension HTM_n with segments $S_1, S_2, \ldots, S_n$

Theorem 3.3. *The metric dimension of the graph of the honeycomb triangular mesh HTM_n is* 2 *for $n \geq 2$.*

Proof. Based on Theorem 2.3, we have $\dim(HTM_n) \geq 2$. Next, we will show that $\dim(HTM_n) \leq 2$.

Let $A = \{x : deg(x) = 1\}$ and $T = \{a, b\}$ where $a, b \in A$. We will show that T is a resolving set or not.

Now, we have the following cases.

Let $p = (l_1, m_1)$ and $q = (l_2, m_2)$ be any two distinct vertices in HTM_n.

Case 1: If $l_1 = l_2$ and $m_1 \neq m_2$, then p and q are resolved by either a or b. Suppose that, if $d(p, a) = d(q, b)$, then p and q are resolved by either a or b, *i.e.*, $d(p, a) \neq d(q, a)$ or $d(p, b) \neq d(q, b)$.

Case 2: If $l_1 \neq l_2$ and $m_1 = m_2$, then both p and q are resolved by a and b.

Case 3: If $l_1 \neq l_2$ and $m_1 \neq m_2$, then p and q are resolved by either a or b. Suppose that, if p and q are at equal distance to a, then p and q must be resolved by b, *i.e.*, if $d(p, a) = d(q, a)$, then $d(p, b) \neq d(q, b)$.

From the above cases if we take any two vertices in a honeycomb triangular mesh are resolved by a and b. Therefore $\dim(HTM_n) \geq 2$. Hence, $\dim(HTM_n) = 2$ □

3.3 Pencil graph

In this section, we determine the pencil graph's metric dimension. In 2015, Simamora and Salman [23] introduced and studied vertex rainbow connection numbers for a new cubic graph called pencil graph. Pencil graph are a specific type of graph in graph theory that consist of a central hub vertex connected to a set of outer vertices called spokes. Pencil graphs have applications in various areas, including network topology, and algorithm design.

Definition 3.4. *Suppose that n is a positive integer with $n \geq 2$. The graph PC_n is a pencil graph with $2n + 2$ vertices and the vertex and edge sets are as follows: $V(PC_n) = \{a\} \cup \{b\} \cup \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$ and $E(PC_n) = \{(ax_1), (ay_1), (ab), (bx_n), (by_n)\} \cup \{(x_i x_{i+1}), (y_i y_{i+1}) : 1 \leq i \leq n-1\} \cup \{(x_i y_i) : 1 \leq i \leq n\}$*

For $n \geq 2$, the pencil graph PC_n is a 3-regular graph with diameter $\lceil n/2 \rceil + 1$ and $3(n + 1)$.

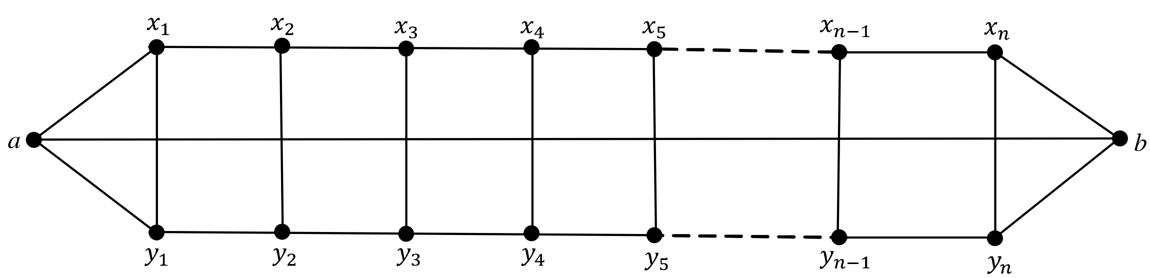

Figure 4: Labeling of pencil graph of dimension n

Theorem 3.5. *The metric dimension of the graph of the pencil graph PC_n, for $n \geq 1$ is*

$$\dim(G) = \begin{cases} 2 & \textit{if } n \textit{ is even} \\ 3 & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. **Case 1 (For n even):**

Let $T = \{a, x_{\frac{n}{2}}\}$ be a resolving set of PC_n. To prove that T is a resolving set, it is enough to prove all the vertices a, b, $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, y_3, \ldots, y_n$ of PC_n have distinct representations with respect to T.

The representation of a and b in PC_n with respect to T as $r(a|T) = (0, \frac{n}{2})$ and $r(b|T) = (1, \frac{n}{2} + 1)$.

For $1 \leq i \leq n$, the representation of x_i in PC_n with respect to T is given as follows:

$$r(x_i|T) = \begin{cases} (i, \frac{n-i}{2}) & \text{if } 1 \leq i \leq \frac{n}{2} \\ (n - i + 2, \frac{2i-n}{2}) & \text{if } \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

For $1 \leq i \leq n$, the representation of y_i in PC_n with respect to T is given as follows:

$$r(y_i|T) = \begin{cases} (i, \frac{n-2i+2}{2}) & \text{if } 1 \leq i \leq \frac{n}{2} \\ (n - i + 2, \frac{2i-n+2}{2}) & \text{if } \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

Since all vertices have distinct representations we obtain $\dim(PC_n) = 2$ in this case.

Case 2 (For n odd): Let $T = \{a, x_{\frac{n+1}{2}}, y_{\frac{n-1}{2}}\}$ be a resolving set of PC_n. To prove that T is a resolving set, it is enough to prove all the vertices a, b, $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, y_3, \ldots, y_n$ of PC_n have distinct representations with respect to T.

The representation of a and b in PC_n with respect to T as $r(a|T) = (0, \frac{n+1}{2}, \frac{n-1}{2})$ and $r(b|T) = (1, \frac{n+1}{2}, \frac{n+1}{2})$.

For $1 \leq i \leq n$, the representation of x_i in PC_n with respect to T as follows

$$r(x_i|T) = \begin{cases} (i, \frac{n+1-2i}{2}, \frac{n+1-2i}{2}) & \text{if } 1 \leq i \leq \frac{n-1}{2} \\ (i, 0, 2) & \text{if } i = \frac{n+1}{2} \\ (n - i + 2, \frac{2i-n-1}{2}, \frac{2i-n+3}{2}) & \text{if } \frac{n+3}{2} \leq i \leq n \end{cases}$$

For $1 \leq i \leq n$, the representation of y_i in PC_n with respect to T as follows

$$r(y_i|T) = \begin{cases} (i, \frac{n+3-2i}{2}, \frac{n-1-2i}{2}) & \text{if } 1 \leq i \leq \frac{n-1}{2} \\ (i, 1, 1) & \text{if } i = \frac{n+1}{2} \\ (n - i + 2, \frac{2i-n+1}{2}, \frac{2i-n+1}{2}) & \text{if } \frac{n+3}{2} \leq i \leq n \end{cases}$$

It is clear that every vertex of PC_n has a unique representation with repect to T. Therefore $\dim(PC_n) \leq 3$.

Next we show that $\dim(PC_n) \geq 3$. We suppose on contrary that $\dim(PC_n) = 2$. Now we have the following cases.

Subcase 2.1: For $1 \leq i \leq n$, let $T = \{a, b\}$ be a resolving set, then $r(x_i|T) = r(y_i|T)$, which is a contradiction to our assumption.

Subcase 2.2: Let $T = \{a, x_1\}$ be a resolving set, then $r(x_n|T) = r(y_n|T)$, which leads to a contradiction.

Subcase 2.3: For $2 \leq i \leq \frac{n+1}{2}$, let $T = \{a, x_i\}$ be a resolving set, then $r(x_{\frac{n+3}{2}}|T) = r(y_{\frac{n+1}{2}}|T)$, which is a contradiction to our assumption.

Subcase 2.4: For $\frac{n+3}{2} \leq i \leq n$, let $T = \{a, x_i\}$ be a resolving set, then $r(x_{\frac{n+1}{2}}|T) = r(y_{\frac{n+3}{2}}|T)$, which is a contradiction to our assumption.

Subcase 2.5: For $2 \leq i \leq \frac{n+1}{2}$, let $T = \{x_1, x_i\}$ be a resolving set, then $r(x_{i+1}|T) = r(y_i|T)$, which is a contradiction to our assumption.

Subcase 2.6: For $\frac{n+3}{2} \leq i \leq n$, let $T = \{x_1, x_i\}$ be a resolving set, then $r(x_{i+1}|T) = r(y_{i-1}|T)$, which is a contradiction to our assumption.

Subcase 2.7: Let $T = \{x_1, y_1\}$ be a resolving set, then $r(x_n|T) = r(y_n|T)$, which leads to a contradiction.

Subcase 2.8: For $2 \leq i \leq \frac{n+3}{2}$, let $T = \{x_1, y_i\}$ be a resolving set, then $r(x_{\frac{n+3}{2}}|T) = r(y_{\frac{n+5}{2}}|T)$, which is a contradiction to our assumption.

Subcase 2.9: For $\frac{n+5}{2} \leq i \leq n$, let $T = \{x_1, y_i\}$ be a resolving set, then $r(x_{\frac{n+3}{2}}|T) = r(y_{\frac{n+1}{2}}|T)$, which is a contradiction to our assumption.

By the symmetrical nature of the pencil graph the remaining possibility of resolving sets for $1 \leq i \leq n$, $T = \{\{a, b\}, \{a, y_i\}, \{b, x_i\}, \{b, y_i\}, \{y_i, x_i\}\}$ is ruled out. From all the above cases it is clear that $\dim(PC_n) \geq 3$. Hence $\dim(PC_n) = 3$. □

4 Significance and contributions of the results

This research offers valuable insights into the metric dimension of cyclic hexagonal chains, honeycomb triangular meshes, and pencil graphs, with direct applications in modern network design, particularly in the field of robot navigation for smart home environments. The primary significance and contributions are as follows:

Novel Metric Dimension Analysis: This study presents a detailed investigation of the metric dimension of three distinct graph structures: cyclic hexagonal chains, honeycomb triangular meshes, and pencil graphs. The results contribute to expanding the mathematical foundation of graph theory, particularly in relation to chemical, geometric, and computational networks.

Applications in Robot Navigation: By determining the metric dimension of these structures, the research provides optimized strategies for robot navigation. The results are critical for localization and pathfinding within networks like smart homes, where robots or autonomous agents need precise positioning with minimal sensors.

Insights for Chemical Graph Theory: Cyclic hexagonal chains represent fundamental structures in chemical graph theory, modeling molecular systems. Understanding their metric dimension helps chemists analyze molecular distances and design efficient chemical compounds or materials with predictable properties.

Optimizing Network Design: The honeycomb triangular mesh and pencil graphs offer useful models for wireless networks and sensor systems. Analyzing their metric dimension improves the efficiency of node placement and minimizes redundancy, supporting the development of more reliable and cost-effective communication networks.

Bridging Theory and Practical Applications: This work bridges theoretical graph metrics with real-world applications, especially in robot-assisted smart homes. The findings enable better design of indoor networks, where efficient navigation plays a critical role in tasks such as surveillance, cleaning, and elderly assistance.

Framework for Future Studies: The approach and results of this research provide a basis for future investigations into other graph families with similar structures. Researchers working on emerging technologies, such as smart cities or the Internet of Things (IoT), can build on the analytical methods presented here.

In summary, this study significantly advances the understanding of the metric dimension in three important graph classes, contributing to both theory and practice. It offers practical solutions for smart environments while enriching the field of graph theory with new perspectives and methods.

5 Concluding remarks

In this paper, we investigated the metric dimension of three significant graph structures: cyclic hexagonal chains, honeycomb triangular meshes, and pencil graphs. Metric dimensions of honeycomb networks and hexagonal-type derived networks have constant metric dimensions, according to research by Manuel *et al.* [15]. In this article, a different kind of honeycomb network known as a triangular honeycomb mesh was created and it was demonstrated that its metric dimension is 2. This research also looked at pencil graphs and the metric dimension of cyclic hexagonal chains. Further obtaining the metric dimensions for symmetric types of honeycomb and hexagonal networks is under investigation. Moreover, computing the metric dimension of the triangular honeycomb network is still an open problem.

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