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On the approximation of the δ-shell interaction for the 3-D Dirac operator.

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ABSTRACT

We consider the three-dimensional Dirac operator coupled with a combination of electrostatic and Lorentz scalar δ-shell interactions. We approximate this operator with general local interactions V. Without any hypotheses of smallness on the potential V, we investigate convergence in the strong resolvent sense to the Dirac Hamiltonian coupled with a δ-shell potential supported on Σ, a bounded smooth surface. However, the coupling constant depends nonlinearly on the potential V.

RESUMEN

Consideramos el operador de Dirac tridimensional acoplado con una combinación de interacciones electrostáticas y δ-cáscara escalar de Lorentz. Aproximamos este operador con interacciones locales generales V. Sin ninguna hipótesis en la pequeñez del potencial V, investigamos la convergencia en el sentido resolvente fuerte del Hamiltoniano de Dirac acoplado con un potencial δ-cáscara soportado en Σ, una superficie suave acotada. Sin embargo, la constante de acoplamiento depende no-linealmente del potencial V.

Keywords and Phrases: Dirac operators, self-adjoint operators, shell interactions, non critical and non-confining interaction strengths, approximations.

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1 Introduction

Dirac Hamiltonians of the type $D_m + V$, where D_m is the free Dirac operator and V represents a suitable perturbation, are used in many problems where the implications of special relativity play an important role. This is the case, for example, in the description of elementary particles such as quarks, or in the analysis of graphene, which is used in research for batteries, water filters, or photovoltaic cells. For these problems, mathematical investigations are still in their infancy. The current study focuses on analyzing the three-dimensional Dirac operator with a singular interaction on a closed surface Σ.

Mathematically, the Hamiltonian of interest is formally represented as

$$D_{\eta,\tau} = D_m + B_{\eta,\tau}\delta_\Sigma = D_m + \big(\eta\,\mathbb{I}_4 + \tau\beta\big)\delta_\Sigma, \tag{1.1}$$

where $B_{\eta,\tau} := \big(\eta\,\mathbb{I}_4+\tau\beta\big)$ is a combination of *electrostatic* and *Lorentz scalar* potentials of strengths η and τ, respectively. Physically, the Hamiltonian $D_{\eta,\tau}$ is used as an idealized model for Dirac operators with strongly localized electric and massive potential near an interface Σ (*e.g.*, an annulus), *i.e.*, it replaces a Hamiltonian of the form

$$\mathbb{H}_{\tilde\eta,\tilde\tau} = D_m + \big(\tilde\eta\,\mathbb{I}_4 + \tilde\tau\beta\big)\mathfrak{B}_\Sigma, \tag{1.2}$$

where $\mathfrak{B}_\Sigma$ is a regular potential localized in a thin layer containing the interface Σ.

The operators $D_{\eta,\tau}$ have been studied in detail recently. The initial direct study on the spectral analysis of the Hamiltonian $D_{\eta,\tau}$ dates back to Ref. [9], in which the authors treated all self-adjoint realizations for spherical surfaces. Besides, they also noted that a shell can confine a particle under the coupling constants assumption: $\eta^2 - \tau^2 = -4$, a phenomenon known in physics as the *confinement case*, which indicates the stability of a particle (for example, an electron) within its initial region during time evolution. In other words, if the particle is confined within a region $\Omega \subset \mathbb{R}^3$ at time $t = 0$, it cannot cross the boundary $\partial\Omega$ and enter the region $\mathbb{R}^3\setminus\overline{\Omega}$ for all subsequent times $t > 0$. Mathematically, this implies that the Dirac operator under consideration can be decomposed into a direct sum of two Dirac operators acting on Ω and $\mathbb{R}^3\setminus\overline{\Omega}$, respectively, each with appropriate boundary conditions. Subsequently, spectral analyses involving Schrödinger operators coupled to δ-shell interactions have developed considerably, while research into the spectral aspects of δ-shell interactions associated with Dirac operators were comparatively inactive. However, in 2014, a resurgence in the spectral study of δ-shell interactions of Dirac operators occurred in [1], where the authors developed a new technique to characterize the self-adjointness of the free Dirac operator coupled to a δ-shell potential. In a special case, they treated pure electrostatic δ-shell interactions (*i.e.*, $\tau = 0$) supported on the boundary of a bounded regular domain and proved that the perturbed operator is self-adjoint. The same authors continued their investigation into the spectral analysis of the electrostatic case, exploring the existence of a point spectrum and associated issues in works such as [2] and [3].

The approximation of the Dirac operator $D_{\eta,\tau}$ by Dirac operators with regular potentials with shrinking support (*i.e.*, of the form (1.2)) provides a justification of the considered idealized model. In the one-dimensional framework, the analysis is carried out in [17], where Šeba showed that convergence is true in the norm resolvent sense. Subsequently, Hughes and Tušek established strong resolvent convergence and norm resolvent convergence for Dirac operators with general point interactions in [11, 12] and [20], respectively. In the two-dimensional case, [8, Section 8] addressed the approximation of Dirac operators with electrostatic, Lorentz scalar, and magnetic δ-shell potentials on closed and bounded curves. A related problem was also considered in [7] for a straight line scenario. More precisely, taking parameters $(\tilde{\eta}, \tilde{\tau}) \in \mathbb{R}^2$ in (1.2) and a potential $\mathfrak{V}^{\varepsilon}_{\Sigma}$ that converges to δ_Σ when ε tends to 0 (in the sense of distributions), then $D_m + \big(\tilde{\eta}\, \mathbb{I}_4 + \tilde{\tau}\beta\big)\mathfrak{V}^{\varepsilon}_{\Sigma}$ converges to the Dirac operator $D_{\eta,\tau}$ with different coupling constants $(\eta, \tau) \in \mathbb{R}^2$ that depend nonlinearly on the potential $\mathfrak{V}^{\varepsilon}_{\Sigma}$. This dependence has been observed in the one-dimensional case, for example [17, 20], and in higher dimensional cases, see [8, 15].

In the three-dimensional case, the situation seems to be even more complex, as recently shown in [15]. There, too, the authors were able to show convergence in the strong resolvent sense in the non-confining case, however, a smallness assumption on the potential $\mathfrak{V}^{\varepsilon}_{\Sigma}$ was required to achieve such a result. On the other hand, this assumption unfortunately prevents us from obtaining an approximation of the operator $D_{\eta,\tau}$ with the physically or mathematically more relevant parameters η and τ. Recognizing this limitation, the authors of the recent paper [4] delved into and verified the approximation problem for two- and three-dimensional Dirac operators with δ-shell potential in the norm resolvent sense. Without the smallness assumption on the potential $\mathfrak{V}^{\varepsilon}_{\Sigma}$ no results could be obtained here either. Finally, in [14], the authors of [15] treated the approximation of the operator (1.2) in the case of the sphere without assuming any hypothesis of smallness on the potential.

The primary aim of our work is to extend the approximation result explored in [8, Section 8] to the three-dimensional case. We seek to verify whether the methodologies employed in the two-dimensional context allow us to establish a comparable approximation in terms of strong resolvent. Specifically, we aim to achieve this in the non-critical and non-confinement cases (*i.e.*, when $\eta^2 - \tau^2 \neq \pm 4$) without relying on the smallness assumption as stipulated in [15].

Organization of the paper. The present paper is structured as follows. We start with Section 2, where we define the free Dirac operators D_m and the model to be studied in our paper by introducing the family $\{\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\}_\varepsilon$, which is the approximate Dirac operators family of operator $D_{\eta,\tau}$. We also discuss our main results by establishing Theorem 2.2. Moreover, in this section we give some geometric aspects characterizing the surface Σ, as well as some spectral properties of the

Dirac operator coupled with the δ-shell interaction presented in Lemma 2.6. Section 3 is devoted to the proof of Theorem 2.2, which approximates the Dirac operator with δ-shell interaction by sequences of Dirac operators with regular potentials at the appropriate scale in the strong resolvent sense.

2 Model and main results

First, let me define the free Dirac operator and describe some of its properties. Given $m > 0$, the free Dirac operator D_m on $\mathbb{R}^3$ is defined by

$$D_m := -i\alpha \cdot \nabla + m\beta,$$

where

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \text{for } k = 1, 2, 3, \quad \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad \mathbb{I}_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

is the family of Dirac and Pauli matrices satisfying the anticommutation relations:

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk}\mathbb{I}_4, \quad \{\alpha_j, \beta\} = 0, \quad \text{and} \quad \beta^2 = \mathbb{I}_4, \quad j, k \in \{1, 2, 3\}, \tag{2.1}$$

where $\{\cdot, \cdot\}$ is the anticommutator bracket. We use the notation $\alpha \cdot x = \sum_{j=1}^{3} \alpha_j x_j$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We recall that $(D_m, \mathrm{dom}(D_m))$ is self-adjoint (see, *e.g.*, [18, Subsection 1.4]), and that

$$\mathrm{Sp}(D_m) = \mathrm{Sp}_{\mathrm{ess}}(D_m) = (-\infty, -m] \cup [m, +\infty).$$

Throughout this paper, for $\Omega \subset \mathbb{R}^3$ a C^∞-smooth bounded domain with boundary $\Sigma := \partial\Omega$, we refer to $H^1(\Omega, \mathbb{C}^4) := H^1(\Omega)^4$ as the first order Sobolev space

$$H^1(\Omega)^4 = \{\varphi \in L^2(\Omega, \mathbb{C}^4) : \text{ there exists } \tilde{\varphi} \in H^1(\mathbb{R}^3)^4 \text{ such that } \tilde{\varphi}|_\Omega = \varphi\}.$$

We denote by $H^{1/2}(\Sigma, \mathbb{C}^4) := H^{1/2}(\Sigma)^4$ the Sobolev space of order $1/2$ along the boundary Σ, and by $t_\Sigma : H^1(\Omega)^4 \to H^{1/2}(\Sigma)^4$ the classical trace operator. The surface Σ divides the Euclidean space into the disjoint union $\mathbb{R}^3 = \Omega_+ \cup \Sigma \cup \Omega_-$, where $\Omega_+ := \Omega$ is a bounded domain and $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega_+}$. We denote by ν and dS the unit outward pointing normal to Ω and the surface measure on Σ, respectively. We also denote by $f_\pm := f \downharpoonright \Omega_\pm$ the restriction of f in $\Omega_\pm$, for all $\mathbb{C}^4$–valued functions f defined on $\mathbb{R}^3$. Then, we define the distribution $\delta_\Sigma f$ by

$$\langle \delta_\Sigma f,\, g\rangle := \frac{1}{2}\int_\Sigma (t_\Sigma f_+ + t_\Sigma f_-)\, g \,\mathrm{dS}, \qquad \text{for any test function } g \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4).$$

Finally, we define the Dirac operator coupled with a combination of electrostatic and Lorentz scalar δ-shell interactions of strengths η and τ, respectively, which we will denote $D_{\eta,\tau}$ in what follows.

Definition 2.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\Sigma = \partial\Omega$. Let $(\eta,\tau) \in \mathbb{R}^2$. Then, $D_{\eta,\tau} = D_m + B_{\eta,\tau}\delta_\Sigma := D_m + (\eta\mathbb{I}_4 + \tau\beta)\delta_\Sigma$ acting in $L^2(\mathbb{R}^3)^4$ is defined as follows:*

$$\begin{aligned} D_{\eta,\tau} f &= D_m f_+ \oplus D_m f_-,\\ \forall f \in \operatorname{dom}(D_{\eta,\tau}) &:= \{f = f_+ \oplus f_- \in H^1(\Omega)^4 \oplus H^1(\mathbb{R}^3 \setminus \overline{\Omega})^4 :\\ &\qquad \textit{the transmission condition (T.C) below holds in } H^{1/2}(\Sigma)^4\}. \end{aligned}$$

Transmission condition:

$$i\alpha\cdot\nu(t_\Sigma f_+ - t_\Sigma f_-) + \frac{1}{2}(\eta\,\mathbb{I}_4 + \tau\beta)(t_\Sigma f_+ + t_\Sigma f_-) = 0, \tag{2.2}$$

where ν is the outward pointing normal to Ω.

Recall that for $\eta^2 - \tau^2 \neq 4$, the Dirac operator $(D_{\eta,\tau}, \operatorname{dom}(D_{\eta,\tau}))$ is self-adjoint and verifies the following assertions (see, *e.g.*, [6, Theorem 3.4, 4.1])

(i) $\mathrm{Sp}_{\mathrm{ess}}(D_{\eta,\tau}) = (-\infty, m] \cup [m, +\infty)$.

(ii) $\mathrm{Sp}_{\mathrm{dis}}(D_{\eta,\tau}) \cap (-m, m)$ is finite.

Now, we explicitly construct regular symmetric potentials $V_{\tilde\eta,\tilde\tau,\varepsilon} \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times4})$ supported on a tubular ε-neighbourhood of Σ and such that

$$V_{\tilde\eta,\tilde\tau,\varepsilon} \xrightarrow[\varepsilon\to 0]{} (\tilde\eta\,\mathbb{I}_4 + \tilde\tau\beta)\delta_\Sigma \quad \text{in the sense of distributions.}$$

To explicitly describe the approximate potentials $V_{\tilde\eta,\tilde\tau,\varepsilon}$, we will introduce some additional notations. For $\gamma > 0$, we define $\Sigma_\gamma := \{x \in \mathbb{R}^3, \operatorname{dist}(x,\Sigma) < \gamma\}$ a tubular neighborhood of Σ with width γ. For $\gamma > 0$ small enough, Σ_γ is parametrized in a similar way as in [5, 15], given by

$$\Sigma_\gamma = \{x_\Sigma + p\nu(x_\Sigma),\, x_\Sigma \in \Sigma \quad \text{and} \quad p \in (-\gamma,\gamma)\}. \tag{2.3}$$

For $0 < \varepsilon < \gamma$, let $h_\varepsilon(p) := \frac{1}{\varepsilon} h\left(\frac{p}{\varepsilon}\right)$, for all $p \in \mathbb{R}$, with the function h verifies the following

$$h \in L^\infty(\mathbb{R},\mathbb{R}), \quad \operatorname{supp} h \subset (-1,1) \text{ and } \int_{-1}^{1} h(t)\,\mathrm{d}t = 1.$$

Thus, we have:

$$\operatorname{supp} h_\varepsilon \subset (-\varepsilon, \varepsilon), \ \int_{-\varepsilon}^{\varepsilon} h_\varepsilon(t)\,\mathrm{d}t = 1, \ \text{and} \ \lim_{\varepsilon\to 0} h_\varepsilon = \delta_0 \ \text{in the sense of distributions,} \tag{2.4}$$

where δ_0 is the Dirac δ-function supported at the origin. Finally, for any $\varepsilon \in (0, \gamma)$, we define the symmetric approximate potentials $V_{\tilde{\eta},\tilde{\tau},\varepsilon} \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4})$, as follows:

$$V_{\tilde{\eta},\tilde{\tau},\varepsilon}(x) := \begin{cases} B_{\tilde{\eta},\tilde{\tau}} h_\varepsilon(p), & \text{if} \quad x = x_\Sigma + p\nu(x_\Sigma) \in \Sigma_\varepsilon, \\ 0, & \text{if} \quad x \in \mathbb{R}^3 \setminus \Sigma_\varepsilon. \end{cases} \tag{2.5}$$

It is easy to see that $\lim_{\varepsilon\to 0} V_{\tilde{\eta},\tilde{\tau},\varepsilon} = B_{\tilde{\eta},\tilde{\tau}}\delta_\Sigma$, in $\mathcal{D}'(\mathbb{R}^3, \mathbb{C}^{4\times 4})$. For $0 < \varepsilon < \gamma$, we define the family of Dirac operators $\{\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\}_\varepsilon$ as follows:

$$\begin{aligned} &\operatorname{dom}(\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}) := \operatorname{dom}(D_m) = H^1(\mathbb{R}^3)^4, \\ &\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\psi = D_m\psi + V_{\tilde{\eta},\tilde{\tau},\varepsilon}\psi, \quad \text{for all } \psi \in \operatorname{dom}(\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}). \end{aligned} \tag{2.6}$$

The main purpose of the present manuscript is to study the strong resolvent limit of $\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}$ at $\varepsilon \to 0$. The following theorem is the main result of this paper.

Theorem 2.2. *Let $(\tilde{\eta}, \tilde{\tau}) \in \mathbb{R}^2$ such that $\tilde{d} := \tilde{\eta}^2 - \tilde{\tau}^2$. Let $(\eta, \tau) \in \mathbb{R}^2$ be defined as follows:*

- *if $\tilde{d} < 0$, then $(\eta, \tau) = \dfrac{\tanh\left(\sqrt{-\tilde{d}}/2\right)}{\sqrt{-\tilde{d}}/2}(\tilde{\eta}, \tilde{\tau})$,*
- *if $\tilde{d} = 0$, then $(\eta, \tau) = (\tilde{\eta}, \tilde{\tau})$,*
- *if $\tilde{d} > 0$ such that $d \neq (2k+1)^2\pi^2$, $k \in \mathbb{N} \cup \{0\}$, then $(\eta, \tau) = \dfrac{\tan\left(\sqrt{\tilde{d}}/2\right)}{\sqrt{\tilde{d}}/2}(\tilde{\eta}, \tilde{\tau})$.*

Now, let $\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}$ be defined as in (2.6) and $D_{\eta,\tau}$ as in Definition 2.1. Then,

$$\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon} \xrightarrow[\varepsilon\to 0]{} D_{\eta,\tau} \quad \textit{in the strong resolvent sense.} \tag{2.7}$$

Remark 2.3. *We mention that in this work we find approximations by regular potentials in the sense of strong resolvent for the Dirac operator with δ-shell potentials $\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}$ in the non-critical case (i.e., when $d \neq 4$) and the non-confining case, (i.e., when $d \neq -4$) everywhere on Σ. This is what we will show in the proof of Theorem 2.2.*

Now, we will introduce some notations and geometrical aspects which we will use in the rest of the paper.

2.1 Notations and geometric aspects

Let Σ be parametrized by the family $\{\phi_j, U_j, V_j,\}_{j\in J}$ with J a finite set, $U_j \subset \mathbb{R}^2$, $V_j \subset \mathbb{R}^3$, $\Sigma \subset \bigcup_{j\in J} V_j$ and $\phi_j(U_j) = V_j \cap \Sigma \subset \Sigma \subset \mathbb{R}^3$ for all $j \in J$. We set $s = \phi_j^{-1}(x_\Sigma)$ for any $x_\Sigma \in \Sigma$.

Definition 2.4 (Weingarten map). *For $x_\Sigma = \phi_j(s) \in \Sigma \cap V_j$ with $s \in U_j$, the Weingarten map (arising from the second fundamental form) is defined as the following linear operator*

$$\begin{aligned} W_{x_\Sigma} := W(x_\Sigma) : T_{x_\Sigma} &\to T_{x_\Sigma} \\ \partial_i \phi_j(s) &\mapsto W(x_\Sigma)[\partial_i\phi_j](s) := -\partial_i \nu(\phi_j(s)), \end{aligned}$$

where T_{x_Σ} denotes the tangent space of Σ on x_Σ and $\{\partial_i\phi_j(s)\}_{i=1,2}$ are the basis vectors of T_{x_Σ}.

Proposition 2.5 ([19, Chapter 9 (Theorem 2), 12 (Theorem 2)]). *Let Σ be an $n-$surface in $\mathbb{R}^{n+1}$, oriented by the unit normal vector field ν, and let $x \in \Sigma$. The principal curvatures of Σ at x (i.e., the eigenvalues $k_1(x), \dots, k_n(x)$ of the Weingarten map W_x) are uniformly bounded on Σ.*

2.1.1 Tubular neighborhood of Σ

Recall that for $\Omega \subset \mathbb{R}^3$ a bounded domain with smooth boundary Σ parametrized by $\phi \in \{\phi_j\}_{j\in J}$. Let $\{\phi, U_\phi, V_\phi\}$ belong to $\{\phi_j, U_j, V_j\}_{j\in J}$ and set $\nu_\phi = \nu \circ \phi : U_\phi \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, with ν the outward pointing unit normal to Ω.

For $\gamma > 0$, Σ_γ (2.3) is a tubular neighborhood of Σ with width γ. We define the diffeomorphism Φ_ϕ as follows:

$$\begin{aligned} \Phi_\phi : U_\phi \times (-\gamma, \gamma) &\longrightarrow \mathbb{R}^3 \\ (s,p) &\longmapsto \Phi_\phi(s,p) = \phi(s) + p\nu(\phi(s)). \end{aligned}$$

For sufficiently small γ, Φ_ϕ is a smooth parametrization of Σ_γ. Moreover, the matrix of the differential $\mathrm{d}\Phi_\phi$ of Φ_ϕ in the canonical basis of $\mathbb{R}^3$ is given by

$$\mathrm{d}\Phi_\phi(s,p) = \Big(\partial_1\phi(s) + p\,\mathrm{d}\nu(\partial_1\phi)(s) \quad \partial_2\phi(s) + p\,\mathrm{d}\nu(\partial_2\phi)(s) \quad \nu_\phi(s)\Big). \tag{2.8}$$

Thus, the differential on U_ϕ and the differential on $(-\gamma, \gamma)$ of Φ_ϕ are respectively given by

$$\begin{aligned} &\mathrm{d}_s\Phi_\phi(s,p) = \partial_i\phi_j(s) - pW(x_\Sigma)\partial_i\phi_j(s) \quad \text{ for } i = 1,2 \text{ and } x_\Sigma = \phi(s) \in \Sigma, \\ &\mathrm{d}_p\Phi_\phi(s,p) = \nu_\phi(s), \end{aligned} \tag{2.9}$$

where $\partial_i\phi$, ν_ϕ should be understood as column vectors, and $W(x_\Sigma)$ is the Weingarten map defined in Definition 2.4. Next, we define

$$\begin{aligned}\mathscr{P}_\phi &:= \left(\Phi_\phi^{-1}\right)_1 : \Sigma_\gamma \longrightarrow U_\phi \subset \mathbb{R}^2; \quad \mathscr{P}_\phi\big(\phi(s)+p\nu(\phi(s))\big) = s \in \mathbb{R}^2,\\ \mathscr{P}_\perp &:= \left(\Phi_\phi^{-1}\right)_2 : \Sigma_\gamma \longrightarrow (-\gamma,\gamma); \quad \mathscr{P}_\perp\big(\phi(s)+p\nu(\phi(s))\big) = p.\end{aligned} \tag{2.10}$$

Using the inverse function theorem and equation (2.8), for $x = \phi(s)+p\nu(\phi(s)) \in \Sigma_\gamma$, we obtain the following differential

$$\nabla \mathscr{P}_\phi(x) = \left(J_{\Phi_\phi^{-1}}\right)_1 := \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} J_{\Phi_\phi^{-1}} \quad \text{and} \quad \nabla \mathscr{P}_\perp(x) = \nu_\phi(s), \tag{2.11}$$

with $J_{\Phi_\phi^{-1}}$ the Jacobian matrix of the diffeomorphism Φ_ϕ^{-1} given by the following formula:

$$J_{\Phi_\phi^{-1}} = \frac{1}{\det(J_{\Phi_\phi})} \times \operatorname{Adj}(J_{\Phi_\phi}).$$

Here $\operatorname{Adj}(J_{\Phi_\phi})$ is expressed in terms of the partial derivatives of ϕ, J_{Φ_ϕ} is the Jacobian matrix of the diffeomorphism Φ_ϕ and $\det(J_{\Phi_\phi}) = 1 + p\kappa_1 + p^2\kappa_2$ (see, for example [13, Lemma 2.3 (1)]), where κ_1 and κ_2 depend on the principal curvatures $k_1,\dots,k_n$ of Σ.

2.2 Preparations for the proof

Before presenting the tools for the proof of Theorem 2.2, we state several properties satisfied by the operator $D_{\eta,\tau}$, which appeared in almost the same form in several papers, for example, [8, Section 5] and [6, Section 3].

Lemma 2.6. *Let $(\eta,\tau)\in\mathbb{R}^2$, and let $D_{\eta,\tau}$ be defined as in Definition 2.1. Then, the following hold:*

(i) If $\eta^2-\tau^2 \neq -4$, there exists an invertible matrix $R_{\eta,\tau}$ such that a function $f = f_+ \oplus f_- \in H^1(\Omega_+)^4 \oplus H^1(\Omega_-)^4$ belongs to $\operatorname{dom}(D_{\eta,\tau})$ if and only if $t_\Sigma f_+ = R_{\eta,\tau} t_\Sigma f_-$, with $R_{\eta,\tau}$ given by

$$R_{\eta,\tau} := \left(\mathbb{I}_4 - \frac{i\alpha\cdot\nu}{2}(\eta\,\mathbb{I}_4 + \tau\beta)\right)^{-1}\left(\mathbb{I}_4 + \frac{i\alpha\cdot\nu}{2}(\eta\,\mathbb{I}_4+\tau\beta)\right). \tag{2.12}$$

(ii) If $\eta^2-\tau^2 = -4$, then a function $f = f_+\oplus f_- \in H^1(\Omega_+)^4 \oplus H^1(\Omega_-)^4$ belongs to $\operatorname{dom}(D_{\eta,\tau})$ if and only if

$$\left(\mathbb{I}_4 - \frac{i\alpha\cdot\nu}{2}(\eta\,\mathbb{I}_4 + \beta\tau)\right)t_\Sigma f_+ = 0 \quad \textit{and} \quad \left(\mathbb{I}_4 + \frac{i\alpha\cdot\nu}{2}(\eta\,\mathbb{I}_4+\beta\tau)\right)t_\Sigma f_- = 0.$$

Proof. Let us show (i). Using the transmission condition in equation (2.2), we find that for all $f = f_+ \oplus f_- \in \mathrm{dom}(D_{\eta,\tau})$,

$$\left(i\alpha \cdot \nu + \frac{1}{2}(\eta \mathbb{I}_4 + \tau\beta)\right) t_\Sigma f_+ = \left(i\alpha \cdot \nu - \frac{1}{2}(\eta \mathbb{I}_4 + \tau\beta)\right) t_\Sigma f_-.$$

Thanks to properties in (2.1) and the fact that $(i\alpha \cdot \nu)^{-1} = -i\alpha \cdot \nu$, we have

$$\big(\mathbb{I}_4 - M\big) t_\Sigma f_+ = \big(\mathbb{I}_4 + M\big) t_\Sigma f_-, \tag{2.13}$$

with M a 4×4 matrix having the following form

$$M = \frac{i\alpha \cdot \nu}{2}(\eta\, \mathbb{I}_4 + \beta\tau),$$

thus (2.12) is established.

Now, using the anticommutation relations from (2.1), we have:

$$M^2 = -\frac{d}{4}\mathbb{I}_4 \quad \text{and} \quad (\mathbb{I}_4 - M)(\mathbb{I}_4 + M) = \frac{4+d}{4}\mathbb{I}_4,$$

where $d := \eta^2 - \tau^2$. When $d \neq -4$, then $\mathbb{I}_4 - M$ is invertible with $(\mathbb{I}_4 - M)^{-1} = \dfrac{4}{4+d}(\mathbb{I}_4 + M)$. Consequently, using (2.13) we obtain that $t_\Sigma f_+ = R_{\eta,\tau} t_\Sigma f_-$, where $R_{\eta,\tau}$ has the explicit form

$$R_{\eta,\tau} = \frac{4}{4+d}\left(\frac{4-d}{4}\mathbb{I}_4 + i\alpha \cdot \nu(\eta \mathbb{I}_4 + \tau\beta)\right). \tag{2.14}$$

For assertion (ii), we multiply (2.13) by $(\mathbb{I}_4 \pm M)$, giving

$$(\mathbb{I}_4 + M)^2 t_\Sigma f_- = 0 \quad \text{and} \quad (\mathbb{I}_4 - M)^2 t_\Sigma f_+ = 0.$$

Moreover, we mention that in the case $d = -4$, we have $(\mathbb{I}_4 \pm M)^2 = 2(\mathbb{I}_4 \pm M)$. This completes the proof of Lemma 2.6. □

3 Proof of Theorem 2.2

Proof. Following the ideas in [8, Section 8], the key step in proving Theorem 2.2 is to establish the convergence (2.7) in the strong graph limit sense. Let $\{\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\}_{\varepsilon \in (0,\gamma)}$ and $D_{\eta,\tau}$ be as defined in (2.6) and Definition 2.1, respectively. Since the singular interactions $V_{\tilde{\eta},\tilde{\tau},\varepsilon}$ are bounded and symmetric, the Kato-Rellich theorem implies that the operators $\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}$ are self-adjoint in $L^2(\mathbb{R}^3, \mathbb{C}^4)$. Moreover, we know that $D_{\eta,\tau}$ is self-adjoint, with $\mathrm{dom}(D_{\eta,\tau}) \subset H^1(\mathbb{R}^3 \setminus \Sigma)^4$. Thus, the convergence of $\{\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\}_{\varepsilon \in (0,\gamma)}$ to $D_{\eta,\tau}$ in the strong resolvent sense as $\varepsilon \to 0$ holds if and only if it converges in the strong graph limit sense, as shown in [16, Theorem VIII.26]. This means we must show the following:

For $\psi \in \operatorname{dom}(D_{\eta,\tau})$, there is a family of vectors $\{\psi_\varepsilon\}_{\varepsilon\in(0,\gamma)} \subset H^1(\mathbb{R}^3)^4$ such that

$$\text{(a)} \lim_{\varepsilon\to 0} \psi_\varepsilon = \psi \quad \text{and} \quad \text{(b)} \lim_{\varepsilon\to 0} \mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon}\psi_\varepsilon = D_{\eta,\tau}\psi \quad \text{in } L^2(\mathbb{R}^3)^4, \tag{3.1}$$

with $H^1(\mathbb{R}^3)^4 = \operatorname{dom}(\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon})$ for all $\varepsilon \in (0,\gamma)$.

Let $\psi \equiv \psi_+ \oplus \psi_- \in \operatorname{dom}(D_{\eta,\tau})$. From Theorem 2.2, we have that

$$\begin{aligned} d = \eta^2 - \tau^2 &= -4\tanh^2\left(\sqrt{-\tilde d}/2\right), \quad \text{if} \quad \tilde d < 0,\\ d = \eta^2 - \tau^2 &= 4\tan^2\left(\sqrt{\tilde d}/2\right), \quad \text{if} \quad \tilde d > 0,\\ d = \eta^2 - \tau^2 &= 0, \quad \text{if} \quad \tilde d = 0. \end{aligned} \tag{3.2}$$

In all cases, we have that $d > -4$ (in particular $d \neq -4$). Then, by Lemma 2.6 (i),

$$t_\Sigma\psi_+ = R_{\eta,\tau} t_\Sigma \psi_-,$$

where $R_{\eta,\tau}$ is given in (2.14). Moreover, using Definition 2.1, we obtain that $t_\Sigma\psi_\pm \in H^{1/2}(\Sigma)^4$.

Show that

$$e^{i\alpha\cdot\nu B_{\tilde\eta,\tilde\tau}} = R_{\eta,\tau}. \tag{3.3}$$

Recall the definition of the family $\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon}$ and the potential $V_{\tilde\eta,\tilde\tau,\varepsilon}$ defined in (2.6) and (2.5), respectively. We have that

$$(i\alpha\cdot\nu B_{\tilde\eta,\tilde\tau})^2 = (i\alpha\cdot\nu(\tilde\eta\mathbb{I}_4 + \tilde\tau\beta))^2 = -(\tilde\eta^2 - \tilde\tau^2) =: \tilde D^2, \quad \text{with } \tilde D = \sqrt{-(\tilde\eta^2-\tilde\tau^2)} = \sqrt{-\tilde d}.$$

Using this equality, we can write: $e^{i\alpha\cdot\nu B_{\tilde\eta,\tilde\tau}} = e^{-\tilde D}\Pi_- + e^{\tilde D}\Pi_+$, with $\pm\tilde D$ the eigenvalues of $i\alpha\cdot\nu B_{\tilde\eta,\tilde\tau}$, and $\Pi_\pm$ the eigenprojections are given by:

$$\Pi_\pm := \frac{1}{2}\left(\mathbb{I}_4 \pm \frac{i\alpha\cdot\nu B_{\tilde\eta,\tilde\tau}}{\tilde D}\right).$$

Therefore,

$$\begin{aligned} e^{(i\alpha\cdot\nu B_{\tilde\eta,\tilde\tau})} &= \left(\frac{e^{\tilde D} + e^{-\tilde D}}{2}\right)\mathbb{I}_4 + \frac{i\alpha\cdot\nu B_{\tilde\eta,\tilde\tau}}{\tilde D}\left(\frac{e^{\tilde D} - e^{-\tilde D}}{2}\right)\\ &= \cosh(\tilde D)\mathbb{I}_4 + \frac{\sinh(\tilde D)}{\tilde D}(i\alpha\cdot\nu(\tilde\eta\mathbb{I}_4 + \tilde\tau\beta)). \end{aligned}$$

Now, the idea is to show (3.3), *i.e.*, that it remains to show

$$\frac{4}{4+d}\left(\frac{4-d}{4}\mathbb{I}_4 + i\alpha\cdot\nu(\eta\mathbb{I}_4+\tau\beta)\right) = \cosh(\tilde{D})\mathbb{I}_4 + \frac{\sinh(\tilde{D})}{\tilde{D}}\left(i\alpha\cdot\nu(\tilde{\eta}\mathbb{I}_4+\tilde{\tau}\beta)\right). \tag{3.4}$$

To this end, set $\mathfrak{U} = \frac{4-d}{4+d} - \cosh(\tilde{D})$ and $\mathfrak{V} = \frac{4}{4+d} - \frac{\sinh(\tilde{D})}{\tilde{D}}$. If we apply (3.4) to the unit vector $e_1 = (1\ 0\ 0\ 0)^t$, and, since the matrices $\mathbb{I}_4$ and $\alpha\cdot\nu(\eta\mathbb{I}_2+\tau\beta)$ are linearly independent for $(\eta,\tau)\neq(0,0)$, then we find that $\mathfrak{U}=\mathfrak{V}=0$. Hence, (3.4) makes sense if and only if

$$\cosh(\tilde{D}) = \frac{4-d}{4+d} \quad \text{and} \quad \frac{\sinh(\tilde{D})}{\tilde{D}}(\tilde{\eta},\tilde{\tau}) = \frac{4}{4+d}(\eta,\tau).$$

Consequently, we have $R_{\eta,\tau} = e^{i\alpha\cdot\nu B_{\tilde{\eta},\tilde{\tau}}}$.

Dividing $\frac{\sinh(\tilde{D})}{\tilde{D}}$ by $(1+\cosh(\tilde{D}))$ we obtain $(\eta,\tau) = \frac{\sinh(\tilde{D})}{1+\cosh(\tilde{D})}\frac{1}{\tilde{D}/2}(\tilde{\eta},\tilde{\tau})$.

Now, applying the elementary identity $\tanh\left(\frac{\theta}{2}\right) = \frac{\sinh(\theta)}{1+\cosh(\theta)}$, for all $\theta\in\mathbb{C}\setminus\{i(2k+1)\pi,\ k\in\mathbb{Z}\}$. We conclude that

$$\frac{\tanh(\sqrt{-\tilde{d}}/2)}{\sqrt{-\tilde{d}}/2}(\tilde{\eta},\tilde{\tau}) = (\eta,\tau), \quad \text{if } \tilde{d}<0,$$

and so, for $\tilde{d}>0$ we apply the elementary identity $-i\tanh(i\theta)=\tan(\theta)$ for all $\theta\in\mathbb{C}\setminus\left\{\pi\left(k+\frac{1}{2}\right),\ k\in\mathbb{Z}\right\}$, and then we get that

$$\frac{\tanh\left(\sqrt{-\tilde{d}}/2\right)}{\sqrt{-\tilde{d}}/2} = \frac{\tan\left(\sqrt{\tilde{d}}/2\right)}{\sqrt{\tilde{d}}/2}.$$

Hence, for $\tilde{d}>0$ such that $\tilde{d}\neq(2k+1)^2\pi^2$, we obtain $(\eta,\tau) = \frac{\tan(\sqrt{\tilde{d}}/2)}{\sqrt{\tilde{d}}/2}(\tilde{\eta},\tilde{\tau})$. Consequently, the equality $e^{i\alpha\cdot\nu B_{\tilde{\eta},\tilde{\tau}}} = R_{\eta,\tau}$ is shown, with the following parameters satisfying:

- $\frac{\tanh(\sqrt{-\tilde{d}}/2)}{\sqrt{-\tilde{d}}/2}(\tilde{\eta},\tilde{\tau}) = (\eta,\tau)$, if $\tilde{d}<0$,
- $\frac{\tan(\sqrt{\tilde{d}}/2)}{\sqrt{\tilde{d}}/2}(\tilde{\eta},\tilde{\tau}) = (\eta,\tau)$, if $\tilde{d}>0$,
- $(\tilde{\eta},\tilde{\tau}) = (\eta,\tau)$, if $\tilde{d}=0$.

Moreover, the fact that $\int_{-\varepsilon}^{\varepsilon} h_\varepsilon(t)dt = 1$ (see, (2.4)) with the statement (3.3) make it possible to write

$$\exp\left[\left(-i\int_{-\varepsilon}^{0} h_\varepsilon(t)\,\mathrm{d}t\right)(\alpha\cdot\nu B_{\eta,\tau})\right]t_\Sigma\psi_+ = \exp\left[\left(i\int_{0}^{\varepsilon} h_\varepsilon(t)\,\mathrm{d}t\right)(\alpha\cdot\nu B_{\eta,\tau})\right]t_\Sigma\psi_-. \tag{3.5}$$

Remark 3.1. *We mention that, in the case where $\tilde{D} = 0$, the phenomenon of renormalization of the coupling constants does not arise. This was already observed in the one-dimensional setting in [20]. Indeed, using* (3.2) *and equation* (3.4)*, we find that* $(\tilde{\eta}, \tilde{\tau}) = (\eta, \tau)$*, where* $\dfrac{\sinh(\tilde{D})}{\tilde{D}}$ *is taken to be equal to* 1 *when* $\tilde{D} = 0$.

Construction of the family $\{\psi_\varepsilon\}_{\varepsilon\in(0,\gamma)}$. Proceeding as in the construction of [8, Section 8], one can construct the following family. The reader should look at that paper for the details. For all $0 < \varepsilon < \gamma$, we define the function $H_\varepsilon : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ as follows:

$$H_\varepsilon(p) := \begin{cases} \displaystyle\int_p^\varepsilon h_\varepsilon(t)\,\mathrm{d}t, & \text{if} \quad 0 < p < \varepsilon, \\ \displaystyle -\int_{-\varepsilon}^p h_\varepsilon(t)\,\mathrm{d}t, & \text{if} \quad -\varepsilon < p < 0, \\ 0, & \text{if} \quad |p| \geq \varepsilon. \end{cases}$$

Clearly, $H_\varepsilon \in L^\infty(\mathbb{R})$ and is supported in $(-\varepsilon, \varepsilon)$. Since $||H_\varepsilon||_{L^\infty} \leq ||h||_{L^1}$, we get that $\{H_\varepsilon\}_\varepsilon$ is bounded uniformly in ε. For all $\varepsilon \in (0, \gamma)$, the restrictions of H_ε to $\mathbb{R}_\pm$ are uniformly continuous, with finite limits at $p = 0$ exist, and are differentiable a.e., with a bounded derivative, since $h_\varepsilon \in L^\infty(\mathbb{R}, \mathbb{R})$. Using these functions, we set the matrix functions $\mathbb{U}_\varepsilon : \mathbb{R}^3 \setminus \Sigma \to \mathbb{C}^{4\times 4}$ such that

$$\mathbb{U}_\varepsilon(x) := \begin{cases} e^{(i\alpha\cdot\nu)B_{\tilde{\eta},\tilde{\tau}}H_\varepsilon(\mathscr{P}_\perp(x))}, & \text{if} \quad x \in \Sigma_\varepsilon \setminus \Sigma, \\ \mathbb{I}_4, & \text{if} \quad x \in \mathbb{R}^3 \setminus \Sigma_\varepsilon, \end{cases} \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4}), \tag{3.6}$$

where the mapping $\mathscr{P}_\perp$ is defined as in (2.10). The functions $\mathbb{U}_\varepsilon$ are bounded, uniformly in ε, and uniformly continuous in $\Omega_\pm$, with a jump discontinuity across Σ. Then, ψ_ε can be constructed by

$$\begin{aligned} &\psi_\varepsilon = \psi_{\varepsilon,+} \oplus \psi_{\varepsilon,-} := \mathbb{U}_\varepsilon \psi \in L^2(\mathbb{R}^3, \mathbb{C}^4), \quad \text{where} \quad \forall x_\Sigma \in \Sigma,\, y_\pm \in \Omega_\pm : \\ &\mathbb{U}_\varepsilon(x_\Sigma^-) := \lim_{y_- \to x_\Sigma} \mathbb{U}_\varepsilon(y_-) = \exp\left[i\left(\int_0^\varepsilon h_\varepsilon(t)\,\mathrm{d}t\right)(\alpha\cdot\nu(x_\Sigma))B_{\tilde{\eta},\tilde{\tau}}\right], \\ &\mathbb{U}_\varepsilon(x_\Sigma^+) := \lim_{y_+ \to x_\Sigma} \mathbb{U}_\varepsilon(y_+) = \exp\left[-i\left(\int_{-\varepsilon}^0 h_\varepsilon(t)\,\mathrm{d}t\right)(\alpha\cdot\nu(x_\Sigma))B_{\tilde{\eta},\tilde{\tau}}\right]. \end{aligned} \tag{3.7}$$

Since $\mathbb{U}_\varepsilon$ are bounded, uniformly in ε, using the construction of ψ_ε we get that $\psi_\varepsilon - \psi := (\mathbb{U}_\varepsilon - \mathbb{I}_4)\psi$. Then, by the dominated convergence theorem and the fact that $\operatorname{supp}(\mathbb{U}_\varepsilon - \mathbb{I}_4) \subset |\Sigma_\varepsilon|$ with $|\Sigma_\varepsilon| \to 0$ as $\varepsilon \to 0$, it is easy to show that

$$\psi_\varepsilon \xrightarrow[\varepsilon\to 0]{} \psi \quad \text{in } L^2(\mathbb{R}^3, \mathbb{C}^4). \tag{3.8}$$

This proves assertion (a).

Show that $\psi_\varepsilon \in \mathrm{dom}(\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}) = H^1(\mathbb{R}^3)^4$. This means that we must show, for all $0 < \varepsilon < \gamma$,

$$\text{(i)}\, \psi_{\varepsilon,\pm} \in H^1(\Omega_\pm)^4 \quad \text{and} \quad \text{(ii)}\, t_\Sigma \psi_{\varepsilon,+} = t_\Sigma \psi_{\varepsilon,-} \in H^{1/2}(\Sigma)^4.$$

Let us show point (i). By the construction of ψ_ε, we have $\psi_\varepsilon \in L^2(\mathbb{R}^3, \mathbb{C}^4)$. It remains to show that $\partial_j \mathbb{U}_\varepsilon \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4})$, for $j = 1, 2, 3$. To do so, recall the parametrization ϕ of Σ defined at the beginning of Subsection 2.1 and let $A \in C^\infty(\mathbb{R}^2, \mathbb{C}^{4\times 4})$ such that $A(s) := i\alpha \cdot \nu(\phi(s)) B_{\tilde{\eta},\tilde{\tau}}$, for $s = (s_1, s_2) \in U \subset \mathbb{R}^2$. Thus, the matrix functions $\mathbb{U}_\varepsilon$ in (3.6) can be written

$$\mathbb{U}_\varepsilon(x) = \begin{cases} e^{A(\mathscr{P}_\phi(x)) H_\varepsilon(\mathscr{P}_\perp(x))}, & \text{if} \quad x \in \Sigma_\varepsilon \setminus \Sigma, \\ \mathbb{I}_4, & \text{if} \quad x \in \mathbb{R}^3 \setminus \Sigma_\varepsilon, \end{cases} \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4}),$$

where $\mathscr{P}_\phi$ is defined as in (2.10).

For $j = 1, 2, 3$, we have $\operatorname{supp} \partial_j \mathbb{U}_\varepsilon \subset \Sigma_\varepsilon$. By the Wilcox formula as used in [8, Eq. 8.12], we obtain that

$$\partial_j \mathbb{U}_\varepsilon(x) = \int_0^1 \Big[\exp\Big(zA(\mathscr{P}_\phi(x)) H_\varepsilon(\mathscr{P}_\perp(x)) \Big) \partial_j \Big(A(\mathscr{P}_\phi(x)) H_\varepsilon(\mathscr{P}_\perp(x)) \Big) \times \\ \exp\Big((1-z) A(\mathscr{P}_\phi(x)) H_\varepsilon(\mathscr{P}_\perp(x)) \Big) \Big] \mathrm{d}z.$$

Based on the quantities (2.11), for $x = \phi(s) + p\nu(\phi(s)) \in \Sigma_\gamma$, and for $s = \mathscr{P}_\phi(x)$, $p = \mathscr{P}_\perp(x)$, with $\mathscr{P}_\phi(x)$ and $\mathscr{P}_\perp(x)$ the mappings introduced in (2.10), together with

$$\partial_j \Big(A(\mathscr{P}_\phi(x)) H_\varepsilon(\mathscr{P}_\perp(x)) \Big) = \partial_j \Big(A(\mathscr{P}_\phi(x) \Big) H_\varepsilon(p) - A(s) h_\varepsilon(p) (\nu_\phi(s))_j,$$

yields that $\partial_j \mathbb{U}_\varepsilon$ has the following form

$$\partial_j \mathbb{U}_\varepsilon(x) = -A(s) h_\varepsilon(p)(\nu_\phi(s))_j \mathbb{U}_\varepsilon(x) + \int_0^1 e^{zA(s)H_\varepsilon(p)} \Big[\partial_j \Big(A(\mathscr{P}_\phi(x) \Big) H_\varepsilon(p) \Big] e^{(1-z)A(s)H_\varepsilon(p)} \,\mathrm{d}z, \quad (3.9)$$

with

$$\partial_j \Big(A(\mathscr{P}_\phi(x) \Big) = \sum_{k=1}^{2} \frac{\partial A(s)}{\partial s_k} (J_{\Phi_\phi^{-1}})_{kj},$$

where $(J_{\Phi_\phi^{-1}})_{kj}$ is the coefficient of the k-th row and j-th column of the matrix $\Big(J_{\Phi_\phi^{-1}} \Big)$ given in (2.11).

We denote by $\mathbb{E}_{\varepsilon,j}$ the second term of the right-hand side of the equality (3.9), *i.e.*,

$$\mathbb{E}_{\varepsilon,j} = \int_0^1 e^{zA(s)H_\varepsilon(p)} \Big[\partial_s A(s) \times \sum_{k=1}^{2} \frac{\partial A(s)}{\partial s_k} (J_{\Phi_\phi^{-1}})_{kj} \times H_\varepsilon(p) \Big] e^{(1-z)A(s)H_\varepsilon(p)} \,\mathrm{d}z. \quad (3.10)$$

Thanks to Proposition 2.5, the matrix-valued functions $\mathbb{E}_{\varepsilon,j}$ are bounded, uniformly for $0<\varepsilon<\gamma$, and $\operatorname{supp}\mathbb{E}_{\varepsilon,j}\subset\Sigma_\varepsilon$. Moreover, we have $\mathbb{U}_\varepsilon$ and $\partial_j\mathbb{U}_\varepsilon\in L^\infty(\Omega_\pm,\mathbb{C}^{4\times4})$, and we deduce that for all $\psi_\pm\in H^1(\Omega_\pm)^4$ we have that $\psi_{\varepsilon,\pm}=\mathbb{U}_\varepsilon\psi_\pm\in H^1(\Omega_\pm)^4$ and statement (i) is verified.

Let us now check point (ii). Since $\psi_{\varepsilon,\pm}\in H^1(\Omega_\pm)^4$, we get that $t_\Sigma\psi_{\varepsilon,\pm}\in H^{1/2}(\Sigma)^4$. On the other hand, as $\mathbb{U}_\varepsilon$ is continuous in $\overline{\Omega_\pm}$, we get

$$t_\Sigma\psi_{\varepsilon,\pm}(x_\Sigma)=\mathbb{U}_\varepsilon(x_\Sigma^\pm)t_\Sigma\psi_\pm(x_\Sigma)\quad\text{ for a.e. } x_\Sigma\in\Sigma;$$

see [10, Chapter 4 (p.133)] and [8, Section 8] for a similar argument.

Consequently, (3.5) with (3.7) give us that $t_\Sigma\psi_{\varepsilon,+}=t_\Sigma\psi_{\varepsilon,-}\in H^{1/2}(\Sigma)^4$. With this, (ii) is valid and $\psi_\varepsilon\in\operatorname{dom}(\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon})$.

To complete the proof of Theorem 2.2, it remains to show the property (b), mentioned in (3.1). Since $(\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon}\psi_\varepsilon-D_{\eta,\tau}\psi)$ belongs to $L^2(\mathbb{R}^3,\mathbb{C}^4)$, it suffices to prove the following:

$$\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}-D_{\eta,\tau}\psi_\pm\xrightarrow[\varepsilon\to0]{}0\quad\text{in } L^2(\Omega_\pm,\mathbb{C}^4).\tag{3.11}$$

To do this, let $\psi\equiv\psi_+\oplus\psi_-\in\operatorname{dom}(D_{\eta,\tau})$ and $\psi_\varepsilon\equiv\psi_{\varepsilon,+}\oplus\psi_{\varepsilon,-}\in\operatorname{dom}(\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon})$. We have

$$\begin{aligned}\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}-D_{\eta,\tau}\psi_\pm&=-i\alpha\cdot\nabla\psi_{\varepsilon,\pm}+m\beta(\psi_{\varepsilon,\pm}-\psi_\pm)+V_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}+i\alpha\cdot\nabla\psi_\pm\\&=-i\alpha\cdot\nabla(\mathbb{U}_\varepsilon\psi_\pm)+i\alpha\cdot\nabla\psi_\pm+m\beta(\mathbb{U}_\varepsilon-\mathbb{I}_4)\psi_\pm+V_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}\\&=-i\sum_{j=1}^3\alpha_j\big[(\partial_j\mathbb{U}_\varepsilon)\psi_\pm+(\mathbb{U}_\varepsilon-\mathbb{I}_4)\partial_j\psi_\pm\big]+m\beta(\mathbb{U}_\varepsilon-\mathbb{I}_4)\psi_\pm+V_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}.\end{aligned}\tag{3.12}$$

Using the form of $\partial_j\mathbb{U}_\varepsilon$ given in (3.9), the quantity $-i\sum_{j=1}^3\alpha_j(\partial_j\mathbb{U}_\varepsilon)\psi_\pm$ yields

$$\begin{aligned}-i\sum_{j=1}^3\alpha_j(\partial_j\mathbb{U}_\varepsilon)\psi_\pm&=-i\sum_{j=1}^3\alpha_j\big[-i\alpha\cdot\nu V_{\tilde\eta,\tilde\tau,\varepsilon}\nu_j\mathbb{U}_\varepsilon\psi_\pm+\mathbb{E}_{\varepsilon,j}\psi_\pm\big]\\&=-(\alpha\cdot\nu)^2V_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}-i\sum_{j=1}^3\alpha_j\mathbb{E}_{\varepsilon,j}\psi_\pm=-V_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}+\mathbb{R}_\varepsilon\psi_\pm,\end{aligned}$$

where $\mathbb{E}_{\varepsilon,j}$ is given in (3.10) and $\mathbb{R}_\varepsilon=-i\sum_{j=1}^3\alpha_j\mathbb{E}_{\varepsilon,j}$, a matrix-valued function in $L^\infty(\mathbb{R}^3,\mathbb{C}^{4\times4})$, verifies the same property of $\mathbb{E}_{\varepsilon,j}$ for all $\varepsilon\in(0,\gamma)$. Thus, (3.12) becomes

$$\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}-D_{\eta,\tau}\psi_\pm=-i\sum_{j=1}^3\alpha_j\big[(\mathbb{U}_\varepsilon-\mathbb{I}_4)\partial_j\psi_\pm\big]+m\beta(\mathbb{U}_\varepsilon-\mathbb{I}_4)\psi_\pm+\mathbb{R}_\varepsilon\psi.$$

Since $\psi_\pm\in H^1(\Omega_\pm)^4$, $(\mathbb{U}_\varepsilon-\mathbb{I}_4)$ and $\mathbb{R}_\varepsilon$ are bounded, uniformly in $\varepsilon\in(0,\gamma)$ and supported in Σ_ε, and $|\Sigma_\varepsilon|$ tends to 0 as $\varepsilon\to0$. By the dominated convergence theorem, we conclude that

$$\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\psi_{\varepsilon,\pm} - D_{\eta,\tau}\psi_{\pm} \xrightarrow[\varepsilon\to 0]{} 0, \quad \text{holds in } L^2(\Omega_\pm, \mathbb{C}^4), \tag{3.13}$$

and this achieves the assertion (3.11).

Thus, both conditions mentioned in (3.1) $\big($*i.e.*, (a) and (b)$\big)$ of the convergence in the strong graph limit sense are proved (see, (3.8) and (3.13)). Hence, the family $\{\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\}_{\varepsilon\in(0,\gamma)}$ converges in the strong resolvent sense to $D_{\eta,\tau}$ as $\varepsilon \to 0$. The proof of the Theorem 2.2 is complete. □

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References

[1] N. Arrizabalaga, A. Mas, and L. Vega, "Shell interactions for Dirac operators," *J. Math. Pures Appl. (9)*, vol. 102, no. 4, pp. 617–639, 2014, doi: 10.1016/j.matpur.2013.12.00.

[2] N. Arrizabalaga, A. Mas, and L. Vega, "Shell interactions for Dirac operators: on the point spectrum and the confinement," *SIAM J. Math. Anal.*, vol. 47, no. 2, pp. 1044–1069, 2015, doi: 10.1137/14097759X.

[3] N. Arrizabalaga, A. Mas, and L. Vega, "An isoperimetric-type inequality for electrostatic shell interactions for Dirac operators," *Comm. Math. Phys.*, vol. 344, no. 2, pp. 483–505, 2016, doi: 10.1007/s00220-015-2481-y.

[4] J. Behrndt, M. Holzmann, and C. Stelzer, "Approximation of Dirac operators with δ-shell potentials in the norm resolvent sense," 2023, *arXiv:2308.13344*.

[5] J. Behrndt, P. Exner, M. Holzmann, and V. Lotoreichik, "Approximation of Schrödinger operators with δ-interactions supported on hypersurfaces," *Math. Nachr.*, vol. 290, no. 8-9, pp. 1215–1248, 2017, doi: 10.1002/mana.201500498.

[6] J. Behrndt, P. Exner, M. Holzmann, and V. Lotoreichik, "On Dirac operators in $\mathbb{R}^3$ with electrostatic and Lorentz scalar δ-shell interactions," *Quantum Stud. Math. Found.*, vol. 6, no. 3, pp. 295–314, 2019, doi: 10.1007/s40509-019-00186-6.

[7] J. Behrndt, M. Holzmann, and M. Tušek, "Two-dimensional Dirac operators with general δ-shell interactions supported on a straight line," *J. Phys. A*, vol. 56, no. 4, 2023, Art. ID 045201, doi: 10.1088/1751-8121/acafaf.

[8] B. Cassano, V. Lotoreichik, A. Mas, and M. Tušek, "General δ-shell interactions for the two-dimensional Dirac operator: self-adjointness and approximation," *Rev. Mat. Iberoam.*, vol. 39, no. 4, pp. 1443–1492, 2023, doi: 10.4171/rmi/1354.

[9] J. Dittrich, P. Exner, and P. Šeba, "Dirac operators with a spherically symmetric δ-shell interaction," *J. Math. Phys.*, vol. 30, no. 12, pp. 2875–2882, 1989, doi: 10.1063/1.528469.

[10] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, revised ed., ser. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015.

[11] R. J. Hughes, "Relativistic point interactions: approximation by smooth potentials," *Rep. Math. Phys.*, vol. 39, no. 3, pp. 425–432, 1997, doi: 10.1016/S0034-4877(97)89757-1.

[12] R. J. Hughes, "Finite-rank perturbations of the Dirac operator," *J. Math. Anal. Appl.*, vol. 238, no. 1, pp. 67–81, 1999, doi: 10.1006/jmaa.1999.6504.

[13] V. Lotoreichik and T. Ourmières-Bonafos, "Spectral asymptotics of the Dirac operator in a thin shell," 2023, *arXiv:2307.09033*.

[14] A. Mas and F. Pizzichillo, "The relativistic spherical δ-shell interaction in $\mathbb{R}^3$: spectrum and approximation," *J. Math. Phys.*, vol. 58, no. 8, 2017, Art. ID 082102, doi: 10.1063/1.5000381.

[15] A. Mas and F. Pizzichillo, "Klein's paradox and the relativistic δ-shell interaction in $\mathbb{R}^3$," *Anal. PDE*, vol. 11, no. 3, pp. 705–744, 2018, doi: 10.2140/apde.2018.11.705.

[16] M. Reed and B. Simon, *Methods of modern mathematical physics. I Functional analysis*, 2nd ed. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980.

[17] P. Šeba, "Klein's paradox and the relativistic point interaction," *Lett. Math. Phys.*, vol. 18, no. 1, pp. 77–86, 1989, doi: 10.1007/BF00397060.

[18] B. Thaller, *The Dirac equation*, ser. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992, doi: 10.1007/978-3-662-02753-0.

[19] J. A. Thorpe, *Elementary topics in differential geometry*, ser. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1979.

[20] M. Tušek, "Approximation of one-dimensional relativistic point interactions by regular potentials revised," *Lett. Math. Phys.*, vol. 110, no. 10, pp. 2585–2601, 2020, doi: 10.1007/s11005-020-01325-6.