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Perturbed weighted trapezoid inequalities for convex functions with applications

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ABSTRACT

We consider trapezoid type inequalities for twice differentiable convex functions, perturbed by a non-negative weight. Applications on a normed space $(X, \|\cdot\|)$ are considered, by establishing bounds for the term

$$\frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\|x\|^p + \|y\|^p}{2}\right] - \int_0^1 \|(1-t)x+ty\|^p\,dt, \quad x, y \in X,$$

which can be seen as a combination of both the midpoint and the trapezoid p-norm (with $2 \leq p < \infty$) inequalities.

RESUMEN

Consideramos desigualdades de tipo trapezoidal para funciones convexas dos veces diferenciables, perturbadas por un peso no-negativo. Se consideran aplicaciones en un espacio normado $(X, \|\cdot\|)$, estableciendo cotas para el término

$$\frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\|x\|^p + \|y\|^p}{2}\right] - \int_0^1 \|(1-t)x+ty\|^p\,dt, \quad x, y \in X,$$

que se puede ver como una combinación de las desigualdades de punto medio y trapezoidal para las p-normas (con $2 \leq p < \infty$).

Keywords and Phrases: Trapezoid inequality, midpoint inequaliy, Ostrowski's inequality, Čebyšev's inequality, norm inequality, semi-inner product.

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1 Introduction

The following inequality, which is known in the literature as the Hermite-Hadamard inequality, holds for any convex function f defined on $\mathbb{R}$ and all $a, b \in \mathbb{R}$:

$$f\left(\frac{a+b}{2}\right)(b-a) \leq \int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}(b-a). \tag{1.1}$$

Let $a, b \in \mathbb{R}$ with $a < b$, $f\colon [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (a,b)$. Then the following inequality, known as the Ostrowski inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M, \tag{1.2}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant. Note that when f is convex and $x = (a+b)/2$, the Ostrowski inequality (1.2) provides a sharp bound for the midpoint difference

$$\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a), \tag{1.3}$$

in view of the middle and the left-hand terms of (1.1). The following result provides some sharp bounds for the midpoint difference (*cf.* [5, Corollary 2.3]). We note the use of the notation $f'_{\pm}$ to denote the right-hand and left-hand derivatives of f, which exist for any convex function f.

Proposition 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then we have the inequality*

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a)^2 \leq \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a) \\ &\leq \frac{1}{8}\left[f'_-(b) - f'_+(a)\right](b-a)^2. \end{aligned} \tag{1.4}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

In what follows, a similar result provides some sharp bounds for the trapezoid difference (*cf.* [6, Corollary 2.3]):

$$\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt, \tag{1.5}$$

in view of the middle and the right-hand terms of (1.1).

Proposition 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then we have the inequality*

$$0\leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a)^{2}\leq \frac{f(a)+f(b)}{2}(b-a)-\int_{a}^{b}f(t)\,dt \qquad (1.6)$$
$$\leq \frac{1}{8}\left[f'_{-}(b)-f'_{+}(a)\right](b-a)^{2}.$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

There are many results in the literature which provide bounds for both midpoint and trapezoids differences. We refer the readers to the survey paper [9].

Let X be a real linear space, $x,\,y\in X$, $x\neq y$ and let $[x,y]:=\{(1-\lambda)x+\lambda y,\ \lambda\in[0,1]\}$ be the *segment* generated by x and y. We consider the function $f:[x,y]\to\mathbb{R}$ and the associated function $g(x,y):[0,1]\to\mathbb{R}$,

$$g(x,y)(t):=f[(1-t)x+ty],\quad t\in[0,1].$$

It is well known that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy the following properties:

(i) $g'_{\pm}(x,y)(s)=(\triangledown_{\pm}f[(1-s)x+sy])(y-x)$, $s\in[0,1)$;

(ii) $g'_{+}(x,y)(0)=(\triangledown_{+}f(x))(y-x)$;

(iii) $g'_{-}(x,y)(1)=(\triangledown_{-}f(y))(y-x)$;

where $(\triangledown_{\pm}f(x))(y)$ are the *Gâteaux lateral derivatives, i.e.*

$$(\triangledown_{\pm}f(x))(y):=\lim_{h\to0^{\pm}}\left[\frac{f(x+hy)-f(x)}{h}\right],$$

for $x,y\in X$.

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x\in X$, is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s:=(\triangledown_{+}f_0(y))(x)=\lim_{t\to0^{+}}\left[\frac{\|y+tx\|^2-\|y\|^2}{2t}\right]$;

(v) $\langle x,y\rangle_i:=(\triangledown_{-}f_0(y))(x)=\lim_{t\to0^{-}}\left[\frac{\|y+tx\|^2-\|y\|^2}{2t}\right]$;

for any $x,y\in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

In [14] Kikianty *et al.* obtained, among others, the following *midpoint p-norm inequalities*:

$$0 \le \int_0^1 \|(1-t)x+ty\|^p dt - \left\|\frac{x+y}{2}\right\|^p \tag{1.7}$$
$$\le p\|y-x\| \begin{cases} \frac{1}{4}\max\{\|x\|^{p-1},\|y\|^{p-1}\}, \\ \frac{1}{2(q'+1)^{\frac{1}{q'}}}\left(\frac{\|x\|^{q(p-1)}+\|y\|^{q(p-1)}}{2}\right)^{\frac{1}{q}}, \quad q>1,\ \frac{1}{q}+\frac{1}{q'}=1; \\ \frac{1}{4}(\|x\|^{p-1}+\|y\|^{p-1}), \end{cases}$$

that hold for any $x, y \in X$. The constants in the first and second cases of (1.7) are sharp. Furthermore, in [13], the following *trapezoid p-norm inequalities* are obtained:

$$\begin{aligned} 0 &\le \frac{1}{8}p\left\|\frac{x+y}{2}\right\|^{p-2}\left[\left\langle y-x,\frac{x+y}{2}\right\rangle_s - \left\langle y-x,\frac{x+y}{2}\right\rangle_i\right] \\ &\le \frac{\|y\|^p+\|x\|^p}{2} - \int_0^1 \|(1-t)\,x+ty\|^p\,dt \\ &\le \frac{1}{8}p\left[\|y\|^{p-2}\langle y-x,y\rangle_i - \|x\|^{p-2}\langle y-x,x\rangle_s\right] \end{aligned} \tag{1.8}$$

that hold for any $x, y \in X$ whenever $p \ge 2$; otherwise, they hold for linearly independent $x, y \in X$. The constant $\frac{1}{8}$ is best in (1.8).

In this paper, we provide bounds for the term

$$\frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\|x\|^p+\|y\|^p}{2}\right] - \int_0^1 \|(1-t)x+ty\|^p\,dt$$

which can be seen as a combination of both the midpoint and the trapezoid p-norm inequalities. This is done via a series of results on twice differentiable convex functions and we take integrals with respect to a weight function as outlined in Section 2.

2 Main results

Let φ be a twice differentiable convex function on $[0,1]$, w integrable and non-negative on $[0,1]$, and $\lambda \in (0,1)$. In this section, we establish bounds for the following

$$\begin{aligned} &\left(\int_\lambda^1 w(s)\,ds\right)\varphi(1) + \left(\int_0^\lambda w(s)\,ds\right)\varphi(0) - \int_0^1 w(t)\,\varphi(t)\,dt \\ &- \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (1-t)\,w(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda t w(t)\,dt, \end{aligned}$$

using Ostrowski's and Čebyšev's inequalities (*cf.* Propositions 2.1 and 2.4 below).

Recall the following inequality by Ostrowski [16], which was proven in 1970.

Proposition 2.1 (Ostrowski). *Let h be integrable and $n \leq h \leq N$ for some constants n, N on $[a,b]$, while g is absolutely continuous and its derivative is essentially bounded. Then,*

$$\left|\frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt-\frac{1}{b-a}\int_a^b g(t)\,dt\frac{1}{b-a}\int_a^b h(t)\,dt\right|\leq\frac{1}{8}(b-a)(N-n)\|g'\|_\infty. \tag{2.1}$$

The constant $\frac{1}{8}$ is best possible in the general case.

We derive the first set of inequalities.

Theorem 2.2. *Let φ be a twice differentiable convex function on $[0,1]$, w integrable and non-negative on $[0,1]$ and $\lambda \in (0,1)$. Then*

$$\begin{aligned}0 &\leq \left(\int_\lambda^1 w(s)\,ds\right)\varphi(1)+\left(\int_0^\lambda w(s)\,ds\right)\varphi(0)-\int_0^1 w(t)\,\varphi(t)\,dt\\&\quad-\frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1(1-t)\,w(t)\,dt+\frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda tw(t)\,dt\\&\leq\frac{1}{8}\left[(1-\lambda)^2\left(\int_\lambda^1 w(s)\,ds\right)+\lambda^2\left(\int_0^\lambda w(s)\,ds\right)\right]\|\varphi''\|_{\infty,[0,1]}.\end{aligned} \tag{2.2}$$

Proof. Let $\lambda \in [0,1]$. By using integration by parts, we have

$$\begin{aligned}\int_0^\lambda\left(\int_t^\lambda w(s)\,ds\right)\varphi'(t)\,dt &= \left(\int_t^\lambda w(s)\,ds\right)\varphi(t)\Big|_0^\lambda+\int_0^\lambda w(t)\,\varphi(t)\,dt\\&=\int_0^\lambda w(t)\,\varphi(t)\,dt-\left(\int_0^\lambda w(s)\,ds\right)\varphi(0)\end{aligned}$$

and

$$\begin{aligned}\int_\lambda^1\left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt &= \left(\int_\lambda^t w(s)\,ds\right)\varphi(t)\Big|_\lambda^1-\int_\lambda^1 w(t)\,\varphi(t)\,dt\\&=\left(\int_\lambda^1 w(s)\,ds\right)\varphi(1)-\int_\lambda^1 w(t)\,\varphi(t)\,dt.\end{aligned}$$

Then we have the following identity of interest

$$\begin{aligned}\int_0^1\left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt &= \int_\lambda^1\left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt-\int_0^\lambda\left(\int_t^\lambda w(s)\,ds\right)\varphi'(t)\,dt\\&=\left(\int_\lambda^1 w(s)\,ds\right)\varphi(1)+\left(\int_0^\lambda w(s)\,ds\right)\varphi(0)-\int_0^1 w(t)\,\varphi(t)\,dt\end{aligned} \tag{2.3}$$

for $\lambda \in [0,1]$. If we use (2.1) for $h(t)=\int_\lambda^t w(s)\,ds$ and $g(t)=\varphi'(t)$ on the interval $[\lambda,1]$, then we get

$$\begin{aligned} 0 &\le \int_\lambda^1 \left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 \left(\int_\lambda^t w(s)\,ds\right)dt \\ &= \int_\lambda^1 \left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (1-t)w(t)dt \\ &\le \frac{1}{8}(1-\lambda)^2\left(\int_\lambda^1 w(s)\,ds\right)\|\varphi''\|_{\infty,[\lambda,1]}. \end{aligned}$$

We also have, again using (2.1) for $h(t) = -\int_t^\lambda w(s)\,ds$ and $g(t) = \varphi'(t)$ on the interval $[0,\lambda]$, that

$$\begin{aligned} 0 &\le -\int_0^\lambda \left(\int_t^\lambda w(s)\,ds\right)\varphi'(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda \left(\int_t^\lambda w(s)ds\right)dt \\ &= -\int_0^\lambda \left(\int_t^\lambda w(s)\,ds\right)\varphi'(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda tw(t)\,dt \le \frac{1}{8}\lambda^2\left(\int_0^\lambda w(s)\,ds\right)\|\varphi''\|_{\infty,[0,\lambda]}. \end{aligned}$$

If we add these inequalities, then we get

$$\begin{aligned} 0 &\le \int_\lambda^1 \left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt - \int_0^\lambda \left(\int_t^\lambda w(s)\,ds\right)\varphi'(t)\,dt \\ &- \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (1-t)\,w(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda tw(t)\,dt \\ &\le \frac{1}{8}(1-\lambda)^2\left(\int_\lambda^1 w(s)\,ds\right)\|\varphi''\|_{\infty,[\lambda,1]} + \frac{1}{8}\lambda^2\left(\int_0^\lambda w(s)\,ds\right)\|\varphi''\|_{\infty,[0,\lambda]} \\ &\le \frac{1}{8}\left[(1-\lambda)^2\left(\int_\lambda^1 w(s)\,ds\right) + \lambda^2\left(\int_0^\lambda w(s)\,ds\right)\right]\|\varphi''\|_{\infty,[0,1]} \end{aligned}$$

and by (2.3) we obtain (2.2). □

When $\lambda = 1/2$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *With the assumptions of Theorem 2.2, we have*

$$\begin{aligned} 0 &\le \left(\int_{\frac{1}{2}}^1 w(s)\,ds\right)\varphi(1) + \left(\int_0^{\frac{1}{2}} w(s)\,ds\right)\varphi(0) - \int_0^1 w(t)\,\varphi(t)\,dt \\ &- 2\left[\left[\varphi(1) - \varphi\left(\frac{1}{2}\right)\right]\int_{\frac{1}{2}}^1 (1-t)\,w(t)\,dt - \left[\varphi\left(\frac{1}{2}\right) - \varphi(0)\right]\int_0^{\frac{1}{2}} tw(t)\,dt\right] \\ &\le \frac{1}{32}\left(\int_0^1 w(s)\,ds\right)\|\varphi''\|_{\infty,[0,1]}. \end{aligned} \tag{2.4}$$

The following result obtained by Čebyšev in 1882, [2]. For a function φ with a bounded derivative, we use the following notation

$$\|\varphi'\|_\infty = \sup_{t\in[a,b]} |\varphi'(t)|.$$

Proposition 2.4. *Let g, h be differentiable functions such that g', h' exist and are continuous on $[a,b]$. Then,*

$$\left|\frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt - \frac{1}{b-a}\int_a^b g(t)\,dt\frac{1}{b-a}\int_a^b h(t)\,dt\right| \le \frac{1}{12}(b-a)^2\|h'\|_\infty\|g'\|_\infty. \tag{2.5}$$

The constant $\frac{1}{12}$ cannot be improved in the general case.

We now derive the second set of inequalities.

Theorem 2.5. *Let φ be a twice differentiable convex function on $[0,1]$, w bounded and non-negative on $[0,1]$ and $\lambda \in (0,1)$. Then*

$$\begin{aligned} 0 &\le \left(\int_\lambda^1 w(s)\,ds\right)\varphi(1) + \left(\int_0^\lambda w(s)\,ds\right)\varphi(0) - \int_0^1 w(t)\,\varphi(t)\,dt \\ &\quad - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (1-t)\,w(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda tw(t)\,dt \\ &\le \frac{1}{12}\left[(1-\lambda)^3\|w\|_{\infty,[\lambda,1]} + \lambda^3\|w\|_{\infty,[0,\lambda]}\right]\|\varphi''\|_{\infty,[0,1]} \\ &\le \frac{1}{12}\left[\frac{1}{4} + 3\left(\lambda - \frac{1}{2}\right)^2\right]\|w\|_{\infty,[0,1]}\|\varphi''\|_{\infty,[0,1]}. \end{aligned} \tag{2.6}$$

Proof. If we use (2.5) for $h(t) = \int_\lambda^t w(s)\,ds$ and $g(t) = \varphi'(t)$ on the interval $[\lambda,1]$, then we get

$$\begin{aligned} 0 &\le \int_\lambda^1 \left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1\left(\int_\lambda^t w(s)\,ds\right)dt \\ &= \int_\lambda^1 \left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (1-t)w(t)\,dt \\ &\le \frac{1}{12}(1-\lambda)^3\sup_{s\in[\lambda,1]} w(s)\,\|\varphi''\|_{\infty,[\lambda,1]}. \end{aligned}$$

Again, by (2.5) for $h(t) = -\int_t^\lambda w(s)\,ds$ and $g(t) = \varphi'(t)$ on the interval $[0,\lambda]$, we get

$$\begin{aligned} 0 &\le -\int_0^\lambda \left(\int_t^\lambda w(s)\,ds\right)\varphi'(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda\left(\int_t^\lambda w(s)\,ds\right)dt \\ &= -\int_0^\lambda \left(\int_t^\lambda w(s)\,ds\right)\varphi'(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda tw(t)\,dt \\ &\le \frac{1}{12}\lambda^3\sup_{s\in[0,\lambda]} w(s)\,\|\varphi''\|_{\infty,[0,\lambda]}. \end{aligned}$$

If we add these inequalities, then we get

$$\begin{aligned}0 &\le \int_{\lambda}^{1}\left(\int_{\lambda}^{t} w(s)\,ds\right)\varphi'(t)\,dt - \int_{0}^{\lambda}\left(\int_{t}^{\lambda} w(s)\,ds\right)\varphi'(t)\,dt\\ &\quad - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_{\lambda}^{1}(1-t)\,w(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_{0}^{\lambda} tw(t)\,dt\\ &\le \frac{1}{12}\left[(1-\lambda)^3 \sup_{s\in[\lambda,1]} w(s)\left\|\varphi''\right\|_{\infty,[\lambda,1]} + \lambda^3 \sup_{s\in[0,\lambda]} w(s)\left\|\varphi''\right\|_{\infty,[0,\lambda]}\right],\end{aligned}$$

which proves (2.6). □

When $\lambda = 1/2$ in Theorem 2.5, we have the following corollary.

Corollary 2.6. *With the assumptions of Theorem 2.5, we have*

$$\begin{aligned}0 &\le \left(\int_{\frac{1}{2}}^{1} w(s)\,ds\right)\varphi(1) + \left(\int_{0}^{\frac{1}{2}} w(s)\,ds\right)\varphi(0) - \int_{0}^{1} w(t)\,\varphi(t)\,dt\\ &\quad - 2\left[\left[\varphi(1)-\varphi\left(\frac{1}{2}\right)\right]\int_{\frac{1}{2}}^{1}(1-t)\,w(t)\,dt - \left[\varphi\left(\frac{1}{2}\right)-\varphi(0)\right]\int_{0}^{\frac{1}{2}} tw(t)\,dt\right]\\ &\le \frac{1}{96}\left[\|w\|_{\infty,[\frac{1}{2},1]} + \|w\|_{\infty,[0,\frac{1}{2}]}\right]\left\|\varphi''\right\|_{\infty,[0,1]} \le \frac{1}{48}\|w\|_{\infty,[0,1]}\left\|\varphi''\right\|_{\infty,[0,1]}.\end{aligned} \tag{2.7}$$

3 Symmetrical weight functions

Simpler forms of the inequalities in Theorems 2.2 and 2.5 (also, Corollaries 2.3 and 2.6) are obtained when we consider the case that the weight w is symmetrical on $[0,1]$. Assume that w is symmetrical on $[0,1]$. Then,

$$\int_{\frac{1}{2}}^{1}(1-t)\,w(t)\,dt = \int_{0}^{\frac{1}{2}} tw(t)\,dt.$$

By assuming the symmetry of w on $[0,1]$ in Corollary 2.3, we have

$$\begin{aligned}0 &\le \frac{\varphi(1)+\varphi(0)}{2}\left(\int_{0}^{1} w(s)\,ds\right) - \int_{0}^{1} w(t)\,\varphi(t)\,dt\\ &\quad - 4\left[\frac{\varphi(1)+\varphi(0)}{2} - \varphi\left(\frac{1}{2}\right)\right]\int_{0}^{\frac{1}{2}} tw(t)\,dt \le \frac{1}{32}\left(\int_{0}^{1} w(s)\,ds\right)\left\|\varphi''\right\|_{\infty,[0,1]},\end{aligned} \tag{3.1}$$

and similarly in Corollary 2.6, we get

$$\begin{aligned}0 &\le \frac{\varphi(1)+\varphi(0)}{2}\left(\int_{0}^{1} w(s)\,ds\right) - \int_{0}^{1} w(t)\,\varphi(t)\,dt\\ &\quad - 4\left[\frac{\varphi(1)+\varphi(0)}{2} - \varphi\left(\frac{1}{2}\right)\right]\int_{0}^{\frac{1}{2}} tw(t)\,dt \le \frac{1}{48}\|w\|_{\infty,[0,1/2]}\left\|\varphi''\right\|_{\infty,[0,1]}.\end{aligned} \tag{3.2}$$

We give now some examples for simple symmetrical weights.

Example 3.1. *First, consider the weight* $w(t) = \left|t - \frac{1}{2}\right|$, $t \in [0,1]$. *Observe that*

$$\int_{\frac{1}{2}}^{1} (1-t)\left(t - \frac{1}{2}\right) dt = \frac{1}{48}, \quad \int_{0}^{\frac{1}{2}} t\left(\frac{1}{2} - t\right) dt = \frac{1}{48}$$

and

$$\int_{\frac{1}{2}}^{1} \left(t - \frac{1}{2}\right) dt = \int_{0}^{\frac{1}{2}} \left(\frac{1}{2} - t\right) dt = \frac{1}{8}.$$

From (3.1) *we obtain*

$$0 \le \frac{1}{12}\left[\varphi(1) + \varphi(0) + \varphi\left(\frac{1}{2}\right)\right] - \int_0^1 \left|t - \frac{1}{2}\right| \varphi(t)\, dt \le \frac{1}{128}\left\|\varphi''\right\|_{\infty,[0,1]}, \tag{3.3}$$

while from (3.2)

$$0 \le \frac{1}{12}\left[\varphi(1) + \varphi(0) + \varphi\left(\frac{1}{2}\right)\right] - \int_0^1 \left|t - \frac{1}{2}\right| \varphi(t)\, dt \le \frac{1}{96}\left\|\varphi''\right\|_{\infty,[0,1]} \tag{3.4}$$

which is not as good as (3.3).

In the above example, we choose a weight function w for which the bound obtained from Corollary 2.3 (and thus Theorem 2.2) is better than that obtained from Corollary 2.6 (and thus Theorem 2.5). Is this always the case? In what follows, we choose a weight function for which we obtain identical bounds.

Example 3.2. *Consider the weight* $w(t) = t(1-t)$, $t \in [0,1]$. *Observe that*

$$\int_{\frac{1}{2}}^{1} (1-t)^2\, t\, dt = \frac{5}{192}, \quad \int_{0}^{\frac{1}{2}} t^2 (1-t)\, dt = \frac{5}{192}$$

and

$$\int_{\frac{1}{2}}^{1} t(1-t)\, dt = \int_{0}^{\frac{1}{2}} t(1-t)\, dt = \frac{1}{12}.$$

From (3.1) *we get*

$$0 \le \frac{1}{96}\left[3\left[\varphi(1) + \varphi(0)\right] + 10\varphi\left(\frac{1}{2}\right)\right] - \int_0^1 t(1-t)\varphi(t)\, dt \le \frac{1}{192}\left\|\varphi''\right\|_{\infty,[0,1]}, \tag{3.5}$$

while from (3.2)

$$0 \le \frac{1}{96}\left[3\left[\varphi(1) + \varphi(0)\right] + 10\varphi\left(\frac{1}{2}\right)\right] - \int_0^1 t(1-t)\varphi(t)\, dt \le \frac{1}{192}\left\|\varphi''\right\|_{\infty,[0,1]}, \tag{3.6}$$

which is the same as (3.5).

In other cases, the bound obtained from Theorem 2.2 is better than that of Theorem 2.5, as outlined in the next two examples.

Example 3.3. *If $w \equiv 1$ in Theorem 2.2, then*

$$0 \leq \frac{1}{2}\left[\varphi(\lambda) + (1-\lambda)\varphi(1) + \lambda\varphi(0)\right] - \int_0^1 \varphi(t)\,dt \leq \frac{1}{8}\left[(1-\lambda)^3 + \lambda^3\right]\|\varphi''\|_{\infty,[0,1]} \tag{3.7}$$

for all $\lambda \in (0,1)$. Since

$$\lambda^3 + (1-\lambda)^3 = \frac{1}{4} + 3\left(\lambda - \frac{1}{2}\right)^2,$$

then (3.7) can be written as

$$0 \leq \frac{1}{2}\left[\varphi(\lambda) + (1-\lambda)\varphi(1) + \lambda\varphi(0)\right] - \int_0^1 \varphi(t)\,dt \leq \frac{1}{8}\left[\frac{1}{4} + 3\left(\lambda - \frac{1}{2}\right)^2\right]\|\varphi''\|_{\infty,[0,1]}. \tag{3.8}$$

In particular, we derive the inequality

$$0 \leq \frac{1}{2}\left[\frac{\varphi(1) + \varphi(0)}{2} + \varphi\left(\frac{1}{2}\right)\right] - \int_0^1 \varphi(t)\,dt \leq \frac{1}{32}\|\varphi''\|_{\infty,[0,1]}. \tag{3.9}$$

Example 3.4. *If $w \equiv 1$ in Theorem 2.5, then*

$$0 \leq \frac{1}{2}\left[\varphi(\lambda) + (1-\lambda)\varphi(1) + \lambda\varphi(0)\right] - \int_0^1 \varphi(t)\,dt \leq \frac{1}{12}\left[\frac{1}{4} + 3\left(\lambda - \frac{1}{2}\right)^2\right]\|\varphi''\|_{\infty,[0,1]} \tag{3.10}$$

for all $\lambda \in (0,1)$. In particular, we have

$$0 \leq \frac{1}{2}\left[\frac{\varphi(1) + \varphi(0)}{2} + \varphi\left(\frac{1}{2}\right)\right] - \int_0^1 \varphi(t)\,dt \leq \frac{1}{48}\|\varphi''\|_{\infty,[0,1]}. \tag{3.11}$$

These inequalities are better than the ones in Example 3.3.

4 Applications for norms

We assume that $(X, \|\cdot\|)$ is a real normed space throughout the sequel.

4.1 Smoothness of the norms and semi-inner products

The terminologies, definitions, and results in this subsection follow those of [7]. Let $x, y \in X$ with $x \neq 0$, then the following limits exist

$$\lim_{t \to 0^{\pm}} \frac{\|x + ty\|^2 - \|x\|^2}{2t}.$$

The mapping $[\cdot,\cdot] : X \times X \to \mathbb{R}$ given by

$$[y, x] := \lim_{t\to 0^+} \frac{\|x + ty\|^2 - \|x\|^2}{2t},$$

is called the T-semi-inner-product.

Definition 4.1. *The T-semi-inner-product $[\cdot,\cdot]$ is said to be continuous on X if*

$$\lim_{t\to 0} [y, x + ty] = [y, x], \quad \text{for all } x, y \in X.$$

Proposition 4.2. *The normed space X is smooth if and only if*

$$\lim_{t\to 0^+} \frac{\|x + ty\| - \|x\|}{t} = \lim_{t\to 0^-} \frac{\|x + ty\| - \|x\|}{t}$$

for all $x, y \in X$ with $x \neq 0$.

Proposition 4.3. *The normed space X is smooth if and only if the T-semi-inner-product is continuous.*

Definition 4.4. *A smooth normed space $(X, \|\cdot\|)$ is of (D)-type if the following limit*

$$\lim_{t\to 0} \frac{[y, x + ty] - [y, x]}{t}$$

exists for all $x, y \in X$, in which case the above limit is denoted as $[y, x]'$.

Every inner product space is a smooth normed space of (D)-type. Every ℓ^p space is a smooth normed space of (D)-type when $p \geq 2$.

Proposition 4.5. *Let $(X, \|\cdot\|)$ be a smooth normed space of (D)-type and $x, y \in X$. Then, the mapping $\varphi_{x,y} : \mathbb{R} \to \mathbb{R}$ given by*

$$\varphi_{x,y}(t) := \|x + ty\|^2$$

is twice differentiable on $\mathbb{R}$,

$$\varphi'_{x,y}(t) = 2 [y, x + ty], \quad \varphi''_{x,y}(t) = 2 [y, x + ty]', \quad \text{for all } t \in \mathbb{R},$$

and $\varphi''_{x,y}$ is non-negative on $\mathbb{R}$.

Definition 4.6. *A smooth normed space of (D)-type is of (BD)-type if there exists a real number $k \geq 1$ such that*

$$[y, x]' \leq k^2 \|y\|^2, \quad \text{for all } x, y \in X. \tag{4.1}$$

The least number k such that (4.1) holds will be called the boundedness modulus of $[\cdot,\cdot]'$ and we denote such a number by k_0.

Example 4.7. *Every inner product space is of* (BD)*-type. In fact,* X *is an inner product space if and only if its boundedness modulus* k_0 *is exactly 1. For all* $x, y \in X$*, we have* $[y, x]' = \|y\|^2$.

Example 4.8. *Every* ℓ^p *space is a smooth normed space of* (BD)*-type when* $p \geq 2$*. In particular, for all* $x, y \in \ell^p$*,* $x \neq 0$*, we have*

$$[y, x]' \leq (4k + 1) \|y\|^2$$

with $k = (p - 2)/2$*.*

4.2 Convex functions on normed spaces

Let $(X, \|\cdot\|)$ be a smooth normed space of (D)-type and $x, y \in X$. Let $f_{x,y} \colon \mathbb{R} \to \mathbb{R}$ be given by

$$f_{x,y}(t) := \|(1 - t)x + ty\|^2 = \|x + t(y - x)\|^2 .$$

By Proposition 4.5, f is convex and twice differentiable on $\mathbb{R}$, and

$$f'_{x,y}(t) = 2 [y - x, (1 - t)x + ty] , \quad \text{and} \quad f''_{x,y}(t) := 2 [y - x, (1 - t)x + ty]' ,$$

for all $t \in \mathbb{R}$.

Let $(X, \|\cdot\|)$ be a smooth normed space of (D)-type, $x, y \in X$, and $1 \leq p < \infty$. Let $g_{x,y,p} \colon \mathbb{R} \to \mathbb{R}$ be given by

$$g_{x,y,p}(t) := \|(1 - t)x + ty\|^p = \left(\|(1 - t)x + ty\|^2\right)^{\frac{p}{2}} .$$

Then, for all $t \in \mathbb{R}$, we have

$$g'_{x,y,p}(t) = \frac{p}{2} \left(\|(1 - t)x + ty\|^2\right)^{\frac{p}{2}-1} \frac{d}{dt} \|(1 - t)x + ty\|^2 = p \|(1 - t)x + ty\|^{p-2} [y - x, (1 - t)x + ty] ,$$

and

$$\begin{aligned} g''_{x,y,p}(t) &= p \left[[y - x, (1 - t)x + ty] \frac{d}{dt} \|(1 - t)x + ty\|^{p-2} + (\|(1 - t)x + ty\|)^{p-2} \frac{d}{dt} [y - x, (1 - t)x + ty] \right] \\ &= p \left[(p - 2) \|(1 - t)x + ty\|^{p-4} [y - x, (1 - t)x + ty]^2 + (\|(1 - t)x + ty\|)^{p-2} [y - x, (1 - t)x + ty]' \right] \\ &= p \|(1 - t)x + ty\|^{p-4} \left[(p - 2) [y - x, (1 - t)x + ty]^2 + (\|(1 - t)x + ty\|)^2 [y - x, (1 - t)x + ty]' \right] . \end{aligned}$$

Note that since $[y - x, (1 - \cdot)x + \cdot y]'$ is non-negative, then $g''_{x,y,p}$ is also non-negative and thus $g_{x,y,p}$ is convex. If we assume further that X is (BD)-smooth with constant $k \geq 1$, then, for all $t \in \mathbb{R}$, we have

$$\begin{aligned} g''_{x,y,p}(t) &= p \|(1 - t)x + ty\|^{p-4} \left[(p - 2) [y - x, (1 - t)x + ty]^2 + (\|(1 - t)x + ty\|)^2 [y - x, (1 - t)x + ty]' \right] \\ &\leq p \|(1 - t)x + ty\|^{p-4} \left[(p - 2) \|y - x\|^2 \|(1 - t)x + ty\|^2 + k^2 (\|(1 - t)x + ty\|)^2 \|y - x\|^2 \right] \end{aligned}$$

$$= p(p-2+k^2)\left\|(1-t)x+ty\right\|^{p-2}\left\|y-x\right\|^2.$$

Consequently,

$$\left\|g''_{x,y,p}\right\|_{\infty,[0,1]} \le p(p-2+k^2)\left\|y-x\right\|^2 \max\left\{\left\|x\right\|^{p-2}, \left\|y\right\|^{p-2}\right\}. \tag{4.2}$$

Remark 4.9. *Recall that when $X = \ell^p$ with $p \ge 2$, we have that*

$$[y,x]'_p \le (4k+1)\left\|y\right\|_p^2,$$

with $k = (p-2)/2$, that is

$$[y,x]'_p \le (2p-3)\left\|y\right\|_p^2,$$

for all $x, y \in \ell^p$ with $x \ne 0$. We use the subscripts p in the notation for the norms and semi-inner products here to highlight the fact that we consider the special case of ℓ^p spaces. Therefore (4.2) becomes

$$\begin{aligned}\left\|g''_{x,y,p}\right\|_{\infty,[0,1]} &\le p(p-2+(2p-3)^2)\left\|y-x\right\|_p^2 \max\left\{\left\|x\right\|_p^{p-2}, \left\|y\right\|_p^{p-2}\right\}\\ &= (4p^3-11p^2+7p)\left\|y-x\right\|_p^2 \max\left\{\left\|x\right\|_p^{p-2}, \left\|y\right\|_p^{p-2}\right\}.\end{aligned} \tag{4.3}$$

4.3 Application of Theorem 2.2

Let w be a non-negative, bounded, integrable weight on $[0,1]$ and $\lambda \in (0,1)$. Then, applying Theorem 2.2 to the function $g_{x,y,p}$, we have

$$\begin{aligned}0 &\le \left(\int_\lambda^1 w(s)\,ds\right)\left\|y\right\|^p + \left(\int_0^\lambda w(s)\,ds\right)\left\|x\right\|^p - \int_0^1 w(t)\left\|(1-t)x+ty\right\|^p dt\\ &- \frac{\left\|y\right\|^p - \left\|(1-\lambda)x+\lambda y\right\|^p}{1-\lambda}\int_\lambda^1 (1-t)\,w(t)\,dt + \frac{\left\|(1-\lambda)x+\lambda y\right\|^p - \left\|x\right\|^p}{\lambda}\int_0^\lambda t w(t)\,dt\\ &\le \frac{1}{8}\left[(1-\lambda)^2\left(\int_\lambda^1 w(s)\,ds\right) + \lambda^2\left(\int_0^\lambda w(s)\,ds\right)\right]\left\|g''_{x,y,p}\right\|_{\infty,[0,1]}.\end{aligned}$$

When the weight w is symmetrical on $[0,1]$ and $\lambda = 1/2$, we have

$$\begin{aligned}0 &\le \frac{\left\|x\right\|^p + \left\|y\right\|^p}{2}\left(\int_0^1 w(s)\,ds\right) - \int_0^1 w(t)\left\|(1-t)x+ty\right\|^p dt\\ &- 4\left[\frac{\left\|x\right\|^p + \left\|y\right\|^p}{2} - \left\|\frac{x+y}{2}\right\|^p\right]\int_0^{\frac{1}{2}} t w(t)\,dt \le \frac{1}{32}\left(\int_0^1 w(s)\,ds\right)\left\|g''_{x,y,p}\right\|_{\infty,[0,1]}.\end{aligned}$$

We obtain a simple inequality when $w \equiv 1$ and we assume further that X is (BD)-smooth

$$0 \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\left\|x\right\|^p + \left\|y\right\|^p}{2}\right] - \int_0^1 \left\|(1-t)x+ty\right\|^p dt \le \frac{1}{32}\left\|g''_{x,y,p}\right\|_{\infty,[0,1]}$$

$$\leq \frac{1}{32} p(p-2+k^2) \|y-x\|^2 \max\left\{\|x\|^{p-2}, \|y\|^{p-2}\right\}.$$

4.4 Application of Theorem 2.5

Let w be a non-negative, bounded, integrable weight on $[0,1]$ and $\lambda \in (0,1)$. Then, applying Theorem 2.5 to the function $g_{x,y,p}$, we have

$$\begin{aligned}
0 &\leq \left(\int_\lambda^1 w(s)\,ds\right)\|y\|^p + \left(\int_0^\lambda w(s)\,ds\right)\|x\|^p - \int_0^1 w(t)\,\|(1-t)x+ty\|^p\,dt \\
&- \frac{\|y\|^p - \|(1-\lambda)x+\lambda y\|^p}{1-\lambda}\int_\lambda^1 (1-t)\,w(t)\,dt + \frac{\|(1-\lambda)x+\lambda y\|^p - \|x\|^p}{\lambda}\int_0^\lambda t w(t)\,dt \\
&\leq \frac{1}{12}\left[(1-\lambda)^3\|w\|_{\infty,[\lambda,1]} + \lambda^3\|w\|_{\infty,[0,\lambda]}\right]\left\|g''_{x,y,p}\right\|_{\infty,[0,1]} \\
&\leq \frac{1}{12}\left[\frac{1}{4} + 3\left(\lambda - \frac{1}{2}\right)^2\right]\|w\|_{\infty,[0,1]}\left\|g''_{x,y,p}\right\|_{\infty,[0,1]}.
\end{aligned}$$

When the weight w is symmetrical on $[0,1]$ and $\lambda = 1/2$, we have

$$\begin{aligned}
0 &\leq \frac{\|x\|^p + \|y\|^p}{2}\left(\int_0^1 w(s)\,ds\right) - \int_0^1 w(t)\,\|(1-t)x+ty\|^p\,dt \\
&- 4\left[\frac{\|x\|^p + \|y\|^p}{2} - \left\|\frac{x+y}{2}\right\|^p\right]\int_0^{\frac{1}{2}} t w(t)\,dt \leq \frac{1}{48}\|w\|_{\infty,[0,1/2]}\left\|g''_{x,y,p}\right\|_{\infty,[0,1]}.
\end{aligned}$$

We obtain a simple inequality when $w \equiv 1$ and we assume further that X is (BD)-smooth

$$\begin{aligned}
0 &\leq \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\|x\|^p + \|y\|^p}{2}\right] - \int_0^1 \|(1-t)x+ty\|^p\,dt \leq \frac{1}{48}\left\|g''_{x,y,p}\right\|_{\infty,[0,1]} \\
&\leq \frac{1}{48} p(p-2+k^2)\|y-x\|^2 \max\left\{\|x\|^{p-2}, \|y\|^{p-2}\right\}.
\end{aligned}$$

4.5 Case of inner product spaces

In the case that $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, we have

$$\begin{aligned}
g''_{x,y,p}(t) &= p\,\|(1-t)x+ty\|^{p-4}\left[(p-2)\,\langle y-x, (1-t)x+ty\rangle^2 + (\|(1-t)x+ty\|)^2\,\langle y-x, (1-t)x+ty\rangle'\right] \\
&\leq p\,\|(1-t)x+ty\|^{p-4}\left[(p-2)\,\|y-x\|^2\,\|(1-t)x+ty\|^2 + \|(1-t)x+ty\|^2\,\|y-x\|^2\right] \\
&= p(p-1)\,\|(1-t)x+ty\|^{p-2}\,\|y-x\|^2,
\end{aligned}$$

and specifically when $p = 2$,

$$g''_{x,y,2}(t) \leq 2\,\|y-x\|^2,$$

We obtain simple inequalities when $w \equiv 1$, from (3.9) and (3.11),

$$0 \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\|x\|^p + \|y\|^p}{2}\right] - \int_0^1 \|(1-t)x + ty\|^p \, dt \le \frac{1}{32} p(p-1)\|y-x\|^2 \max\left\{\|x\|^{p-2}, \|y\|^{p-2}\right\} \tag{4.4}$$

and

$$0 \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\|x\|^p + \|y\|^p}{2}\right] - \int_0^1 \|(1-t)x + ty\|^p \, dt \le \frac{1}{48} p(p-1)\|y-x\|^2 \max\left\{\|x\|^{p-2}, \|y\|^{p-2}\right\}. \tag{4.5}$$

In particular, when $p = 2$

$$0 \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^2 + \frac{\|x\|^2 + \|y\|^2}{2}\right] - \int_0^1 \|(1-t)x + ty\|^2 \, dt \le \frac{1}{16}\|y-x\|^2 \tag{4.6}$$

and

$$0 \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^2 + \frac{\|x\|^2 + \|y\|^2}{2}\right] - \int_0^1 \|(1-t)x + ty\|^2 \, dt \le \frac{1}{24}\|y-x\|^2. \tag{4.7}$$

This last inequality is, in fact, an equality, since

$$\begin{aligned}\frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^2 + \frac{\|x\|^2 + \|y\|^2}{2}\right] - \int_0^1 \|(1-t)x + ty\|^2 \, dt &= \frac{1}{2}\left[\frac{1}{4}(\|x\|^2 + \|y\|^2 + 2\langle x, y\rangle) + \frac{\|x\|^2 + \|y\|^2}{2}\right] \\ &\quad - \int_0^1 [(1-t)^2\|x\|^2 + 2t(1-t)\langle x, y\rangle + t^2\|y\|^2] dt \\ &= \frac{1}{8}(3\|x\|^2 + 3\|y\|^2 + 2\langle x, y\rangle) - \frac{1}{3}(\|x\|^2 + \|y\|^2 + \langle x, y\rangle) \\ &= \frac{1}{24}(\|x\|^2 + \|y\|^2 - 2\langle x, y\rangle) = \frac{1}{24}\|y-x\|^2.\end{aligned}$$

The above shows that (4.5) is sharp, and consequently (2.6), (2.7), (3.2), (3.10), and (3.11), are also sharp.

We again consider $p = 2$, and we further consider the weight $w(t) = \left|t - \frac{1}{2}\right|$, $t \in [0, 1]$, as in Example 3.1, then (3.3) becomes

$$0 \le \frac{1}{12}\left[\|x\|^2 + \|y\|^2 + \left\|\frac{x+y}{2}\right\|^2\right] - \int_0^1 \left|t - \frac{1}{2}\right| \|(1-t)x + ty\|^2 \, dt \le \frac{1}{64}\|y-x\|^2.$$

We conjecture that the above bound is not sharp.

We again consider $p = 2$ with the weight $w(t) = t(1-t)$, $t \in [0, 1]$, as in Example 3.2, then (3.5) becomes

$$0 \le \frac{1}{96}\left[3(\|x\|^2 + \|y\|^2) + 10\left\|\frac{x+y}{2}\right\|^2\right] - \int_0^1 t(1-t)\|(1-t)x + ty\|^2 \, dt \le \frac{1}{96}\|y-x\|^2.$$

We conjecture that the above bound is not sharp.

References

[1] P. Cerone and S. S. Dragomir, "A refinement of the Grüss inequality and applications," *Tamkang J. Math.*, vol. 38, no. 1, pp. 37–49, 2007.

[2] P. Chebyshev, "Sur les expressions approximatives des intégrales définies par les autres prises entre les mêmes limites," *Proc. Math. Soc. Charkov*, vol. 2, pp. 93–98, 1882.

[3] X.-L. Cheng and J. Sun, "A note on the perturbed trapezoid inequality," *JIPAM. J. Inequal. Pure Appl. Math.*, vol. 3, no. 2, 2002, Art. ID 29.

[4] I. Cioranescu, *Geometry of Banach spaces, duality mappings and nonlinear problems*, ser. Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, 1990, vol. 62, doi: 10.1007/978-94-009-2121-4.

[5] S. S. Dragomir, "An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products," *JIPAM. J. Inequal. Pure Appl. Math.*, vol. 3, no. 2, 2002, Art. ID 31.

[6] S. S. Dragomir, "An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products," *JIPAM. J. Inequal. Pure Appl. Math.*, vol. 3, no. 3, 2002, Art. ID 35.

[7] S. S. Dragomir, "Smooth normed spaces of (BD)-type," *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, vol. 39, no. 1, pp. 1–15, 1992.

[8] S. S. Dragomir, *Semi-inner products and applications.* Nova Science Publishers, Inc., Hauppauge, NY, 2004.

[9] S. S. Dragomir, "Ostrowski type inequalities for Lebesgue integral: a survey of recent results," *Aust. J. Math. Anal. Appl.*, vol. 14, no. 1, 2017, Art. ID 1.

[10] L. Fejér, "Über die *Fourier*sche Reihe, II," *Mat. Természett. Értes.*, vol. 24, pp. 369–390, 1906.

[11] J. R. Giles, "Classes of semi-inner-product spaces," *Trans. Amer. Math. Soc.*, vol. 129, pp. 436–446, 1967, doi: 10.2307/1994599.

[12] G. Grüss, "Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx$," *Math. Z.*, vol. 39, no. 1, pp. 215–226, 1935, doi: 10.1007/BF01201355.

[13] E. Kikianty, S. S. Dragomir, and P. Cerone, "Sharp inequalities of Ostrowski type for convex functions defined on linear spaces and application," *Comput. Math. Appl.*, vol. 56, no. 9, pp. 2235–2246, 2008, doi: 10.1016/j.camwa.2008.03.059.

[14] E. Kikianty, S. S. Dragomir, and P. Cerone, “Ostrowski type inequality for absolutely continuous functions on segments in linear spaces,” *Bull. Korean Math. Soc.*, vol. 45, no. 4, pp. 763–780, 2008, doi: 10.4134/BKMS.2008.45.4.763.

[15] G. Lumer, “Semi-inner-product spaces,” *Trans. Amer. Math. Soc.*, vol. 100, pp. 29–43, 1961, doi: 10.2307/1993352.

[16] A. M. Ostrowski, “On an integral inequality,” *Aequationes Math.*, vol. 4, pp. 358–373, 1970, doi: 10.1007/BF01844168.