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Steffensen-like method in Riemannian manifolds

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ABSTRACT

In this paper, we present semilocal convergence of Steffensen-like method for approximating zeros of a vector field in Riemannian manifolds. We establish the convergence of Steffensen-like method under Lipschitz continuity condition on first order covariant derivative of a vector field. Finally, two examples are given to show the application of our theorem.

RESUMEN

En este artículo, presentamos la convergencia semilocal del método de tipo Steffensen para aproximar los ceros de un campo de vectores en una variedad Riemanniana. Establecemos la convergencia del método de tipo Steffensen bajo la condición de continuidad Lipschitz de la derivada covariante de primer orden de un campo de vectores. Finalmente, damos dos ejemplos para mostrar la aplicabilidad de nuestro teorema.

Keywords and Phrases: Vector fields, Riemannian manifolds, Lipschitz condition, Steffensen-like method.

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1 Introduction

There are many problems in applied sciences and other including engineering, optimization, dynamic economic system, physics, biological problems which is formulated in an equation by using mathematical modeling to find the zeros of equations (see for example [7, 9, 10, 17, 19] and the references therein). To solve the nonlinear equations many types of iterative methods have been studied in Banach spaces. The most famous second order iterative method to solve a non-linear equation in Banach space is Newton's method. Recently, attention has been paid in studying iterative methods in Riemannian manifolds. There are many types of numerical methods that have been studied in manifolds which arise in many contexts. Some problems including eigenvalue problem, minimization problems with orthogonality constraints, optimization problems with equality constraints, invariant subspace computations (see for example [1–3, 6–8, 12–15, 21] and the references therein). To solve this problem, we have to find the zeros of a vector field in Riemannian manifolds. Generally convergence of iterative methods are usually centered on two types: semilocal and local convergence analysis. The convergence analysis which provides information around a solution and calculates the radius of convergence, it is local and when the convergence analysis provides information around an initial point, it is semilocal. The Steffensen-like method [5] which is second order method in Banach space is defined as:

$$\left.\begin{aligned} x_0 &\in \Omega, \\ y_{n-1} &= x_{n-1} - a\mathfrak{M}(x_{n-1}),\ a \in \mathbb{R}^+,\ n \in \mathbb{N}, \\ z_{n-1} &= x_{n-1} + b\mathfrak{M}(x_{n-1}),\ b \in \mathbb{R}^+, \\ x_n &= x_{n-1} - [y_{n-1}, z_{n-1}; \mathfrak{M}]^{-1}\mathfrak{M}(x_{n-1}), \end{aligned}\right\} \tag{1.1}$$

where $\mathfrak{M}$ is a nonlinear operator defined in an open convex subset Ω of a Banach space B into itself and $\mathfrak{M}$ is first Fréchet differentiable in Ω. The computational efficiency of Steffensen-like method is the same as Newton's method, when it is applied to find the solution of finite dimensional system of nonlinear equations. The convergence of this second order method in Banach space has been studied in [5]. As motivation, the numerical solution of the vector field

$$G(u_1, u_2, u_3) = (-u_2,\ u_1 - u_1u_3^2,\ u_1u_2u_3)$$

using Newton's and Euler-Chebyshev's method on $\mathbb{R}^3$ is difficult to find as the Jacobian is a non-invertible matrix at the point $(0, 0, -1)^T$, but using the algorithm given in [11] such singularity is found on the two-dimensional sphere $\mathbf{S}^2$. In this paper, we extend the method (1.1) to the case of equations in Riemannian manifolds to find the singular point of a vector field.

The paper is organized as follows: Section 2, contains all the necessary background on fundamental properties and notation of Riemannian manifolds. In Section 3, we present the semilocal convergence of Steffensen-like method under Lipschitz continuity condition on the first order covariant derivative of vector field. In Section 4, two examples are given to show the application of our theorem. Finally, in Section 5, some brief conclusions are given.

2 Preliminaries

In this section, we introduce some basic definitions and properties of Riemannian manifolds (for more details see [16, 18, 20]).

Let Q be a real n-dimensional Riemannian manifold, $\mathfrak{X}(Q)$ be a set of all vector fields of class C^∞ on Q, T_uQ be a tangent space of Q at u, and TQ be a tangent bundle defined as $TQ = \bigcup_{u\in Q} T_uQ$. Suppose Q is equipped with a Riemannian metric $\langle .,. \rangle$ with corresponding norm $\|\cdot\|$. The arc length of piecewise smooth curve $\psi : [0,1] \to Q$ joining u to v is defined by $l(\psi) = \int_0^1 \|\psi'(z)\| dz$ and the Riemannian distance joining u to v is defined by $d(u,v) = \inf_\psi l(\psi)$. Let $D(Q)$ be the ring of real-valued functions of class C^∞ defined on Q. An affine connection ∇ on Q is a map

$$\begin{array}{rcccl} \nabla & : & \mathfrak{X}(Q)\times\mathfrak{X}(Q) & \longrightarrow & \mathfrak{X}(Q) \\ & & (X,G) & \longmapsto & \nabla_X G \end{array}$$

which satisfies the properties

(i) $\nabla_{fX+gG}\mathfrak{V} = f\nabla_X\mathfrak{V} + g\nabla_G\mathfrak{V}$.

(ii) $\nabla_X(G+\mathfrak{V}) = \nabla_X G + \nabla_X\mathfrak{V}$.

(iii) $\nabla_X(fG) = f\nabla_X G + X(f)G$,

where $X, G, \mathfrak{V} \in \mathfrak{X}(Q)$ and $f, g \in D(Q)$. The covariant derivative of G determined by the connection ∇ defines at each point $u \in Q$ a linear application as

$$\begin{array}{rcccl} DG(u) & : & T_uQ & \longrightarrow & T_uQ \\ & & v & \longmapsto & DG(u)(v) = \nabla_X G(u), \end{array}$$

where $G \in \mathfrak{X}(Q)$ of class C^1 on Q and X is a vector field that satisfies $X(u) = v$. We define the open and closed geodesic ball with centre u and radius v respectively, as

$$V(u,v) = \{t \in Q : d(u,t) < v\} \quad \text{and} \quad V[u,v] = \{t \in Q : d(u,t) \leq v\}.$$

A parametrized curve $\psi : I \to Q$ is said to be a geodesic at $t_0 \in I$ if $\nabla_{\psi'(t)}\psi'(t) = 0$ in the point t_0. If ψ is a geodesic at t, for all $t \in I$, we say that ψ is a geodesic. If $[p,q] \subseteq I$, the restriction of ψ to $[a,b]$ is called a geodesic segment joining $\psi(p)$ to $\psi(q)$. By the Hopf-Rinow theorem, if Q is complete metric space then for any $u, t \in Q$ there exists a geodesic ψ, called minimizing geodesic joining u to t with

$$l(\psi) = d(u, t).$$

Also, if $v \in T_uQ$, then there exists a unique minimizing geodesic ψ such that $\psi(0) = u$ and $\psi'(0) = v$. The point $\psi(1)$ is called the image of v by the exponential map at u, *i.e.*

$$\exp_u : T_uQ \longrightarrow Q$$

such that $\exp_u(v) = \psi(1)$ and for all $p \in [0,1]$, $\psi(p) = \exp_u(pv)$. Let ψ be a piecewise smooth curve, then for any $x, y \in \mathbb{R}$, the parallel transport along ψ is a mapping from $T_{\psi(x)}Q$ to $T_{\psi(y)}Q$. It is denoted by $\mathbb{M}_{\psi,\cdot,\cdot}$ and given by

$$\begin{array}{rccc} \mathbb{M}_{\psi,x,y} & : & T_{\psi(x)}Q & \longrightarrow & T_{\psi(y)}Q \\ & & v & \longmapsto & V(\psi(y)), \end{array}$$

where V is the unique vector field along ψ which satisfies $\nabla_{\psi'(t)}V = 0$ and $V(\psi(x)) = v$. It is easy to show that $\mathbb{M}_{\psi,x,y}$ is linear and one-to-one. Therefore $\mathbb{M}_{\psi,x,y} : T_{\psi(x)}Q \to T_{\psi(y)}Q$ is an isomorphism and inverse of parallel transport is denoted by $\mathbb{M}_{\psi,y,x}$. Thus $\mathbb{M}_{\psi,x,y}$ is an isometry between $T_{\psi(x)}Q$ and $T_{\psi(y)}Q$. For $i \in \mathbb{N}$, we define $\mathbb{M}^i_{\psi}$ as

$$\mathbb{M}^i_{\psi,x,y} : (T_{\psi(x)}Q)^i \longrightarrow (T_{\psi(y)}Q)^i,$$

where

$$\mathbb{M}^i_{\psi,x,y}(u_1, u_2, \ldots, u_i) = (\mathbb{M}_{\psi,x,y}(u_1), \mathbb{M}_{\psi,x,y}(u_2), \ldots, \mathbb{M}_{\psi,x,y}(u_i)).$$

It has the important properties:

$$\mathbb{M}^{-1}_{\psi,y,x} = \mathbb{M}_{\psi,x,y}, \quad \mathbb{M}_{\psi,x,y} \circ \mathbb{M}_{\psi,y,z} = \mathbb{M}_{\psi,x,z}.$$

Definition 2.1. *Let $G \in \mathfrak{X}(Q)$ of class C^k on Q and $j \in \mathbb{N}$. The covariant derivative of order j of G is denoted by D^jG and defined as:*

$$D^jG : \underbrace{C^k(TQ) \times C^k(TQ) \times \cdots \times C^k(TQ)}_{j\text{-}times} \longrightarrow C^{k-j}(TQ),$$

where

$$\begin{aligned} D^jG(A_1, A_2, \ldots, A_{j-1}, A) = \nabla_A D^{j-1}G(A_1, A_2, \ldots, A_{j-1}) \\ - \sum_{i=1}^{j-1} D^{j-1}G(A_1, A_2, \ldots, \nabla_A A_i, \ldots, A_{j-1}) \end{aligned} \tag{2.1}$$

for all $A_1, A_2, \ldots, A_{j-1} \in C^k(TQ)$.

Definition 2.2. *Let $\mho \subseteq Q$ be an open convex set and $G \in \mathfrak{X}(Q)$. The covariant derivative $DG = \nabla_{(.)}G$ is Lipschitz with constant $\mathfrak{E} > 0$, if for any geodesic ψ and $x, y \in \mathbb{R}$ such that $\psi[x, y] \subseteq \mho$, and it holds the inequality*

$$\|\mathbb{M}_{\psi,y,x} DG(\psi(y))\mathbb{M}_{\psi,x,y} - DG(\psi(x))\| \leq \mathfrak{E} \int_x^y \|\psi'(t)\| dt,$$

and we write $DG \in Lip_{\mathfrak{E}}(\mho)$. If Q is finite dimensional Euclidean space, then it coincides with Lipschitz condition for $DG : Q \to Q$.

Definition 2.3. *Let $\mho \subseteq Q$, be an open convex set. Suppose ψ is a curve in Q, $[t, t+\delta e] \subset Dom(\psi)$ and $G \in \mathfrak{X}(Q)$ of class C^0 on Q. The divided difference of first order for G on the points $\psi(t)$ and $\psi(t + \delta e)$ in the direction $\psi'(t)$, is defined by*

$$[\psi(t + \delta x), \psi(t); G]\psi'(t) = \frac{1}{\delta e}(\mathbb{M}_{\psi,t+\delta e,t} G(\psi(t + \delta e)) - G(\psi(t))). \tag{2.2}$$

When Q is a Banach space, if ψ is the geodesic joining u_1 and u_2, such that

$$\psi(t) = u_1 + t(u_2 - u_1), \quad t \in \mathbb{R},$$

then from (2.2)*, we obtain*

$$[u_2, u_1; G](u_2 - u_1) = G(u_2) - G(u_1).$$

Also if $DG(u)$ exists, then $DG(u) = [u, u; G]$.

Proposition 2.4. *The covariant derivative of G in the direction of $\psi'(t)$ is defined as:*

$$\begin{aligned} DG(\psi(t))\psi'(t) &= \nabla_{\psi'(t)} G_{\psi(t)} \\ &= \lim_{\delta e \to 0} \frac{1}{\delta e}(\mathbb{M}_{\psi,t+\delta e,t} G(\psi(t + \delta e)) - G(\psi(t))), \end{aligned}$$

where ψ is a curve on Q and $G \in \mathfrak{X}(Q)$ of class C^1 on Q. If Q is finite dimensional Euclidean space, then it coincides with the directional derivative in finite dimensional Euclidean space.

Next, we will show Taylor-type expansions in Riemannian manifolds which will be used in the proof of the convergence of our iterative method.

Theorem 2.5. *Let ψ be a geodesic in Q and $G \in \mathfrak{X}(Q)$ of class C^1 on Q. Then*

$$\mathbb{M}_{\psi,t,0} G(\psi(t)) = G(\psi(0)) + \int_0^t \mathbb{M}_{\psi,e,0} DG(\psi(e))\psi'(e) de. \tag{2.3}$$

Proof. See [4]. □

3 Steffensen-like method in Riemannian manifolds

In this section, we will prove convergence and uniqueness of Steffensen-like method in Riemannian manifolds. The method (1.1) in Riemannian manifolds has the form

$$\left.\begin{aligned}
u_0 &\in \mho,\\
L_{n-1} &= -aG(u_{n-1}), \quad a \in \mathbb{R}^+,\ n \in \mathbb{N},\\
v_{n-1} &= \exp_{u_{n-1}}(L_{n-1}),\\
\psi_{n-1}(t) &= \exp_{u_{n-1}}(tL_{n-1}),\\
M_{n-1} &= bG(u_{n-1}), \quad b \in \mathbb{R}^+,\\
w_{n-1} &= \exp_{u_{n-1}}(M_{n-1}),\\
N_n &= -\mathbb{M}_{\psi_{n-1},1,0}[v_{n-1}, w_{n-1}; G]^{-1}\mathbb{M}_{\psi_{n-1},0,1}G(u_{n-1}),\\
u_n &= \exp_{u_{n-1}}(N_n).
\end{aligned}\right\} \tag{3.1}$$

Assume that $G(u)$ satisfies the following conditions:

(1) $\|G(u_0)\| \leq \xi$,

(2) $\|DG(u_0)^{-1}\| \leq \zeta_0$,

(3) $\|\mathbb{M}_{\phi,j,i}DG(\phi(j))\mathbb{M}_{\phi,i,j} - DG(\phi(i))\| \leq K\int_i^j \|\phi'(x)\|dx$, where ϕ is a geodesic such that $\phi[i,j] \subseteq \mho$.

Firstly, we shall show that a operator $[v_0, w_0; G]^{-1}$ is bounded. Let $I_{u_0} : T_{u_0}Q \to T_{u_0}Q$ be a identity operator, ψ_n and α_n be a family of minimizing geodesics such that $\psi_n(0) = u_n$, $\psi_n(1) = v_n$, $\alpha_n(0) = w_n$, $\alpha_n(1) = v_n$ for each $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned}
\|DG(u_0)^{-1}\mathbb{M}_{\psi_0,1,0}[v_0, w_0; G]\mathbb{M}_{\psi_0,0,1} - I_{u_0}\| &\leq \|DG(u_0)^{-1}\mathbb{M}_{\psi_0,1,0}([v_0, w_0; G] - DG(v_0))\mathbb{M}_{\psi_0,0,1}\|\\
&\quad + \|DG(u_0)^{-1}(\mathbb{M}_{\psi_0,1,0}DG(v_0)\mathbb{M}_{\psi_0,0,1} - DG(u_0))\|\\
&\leq \|DG(u_0)^{-1}\| \int_0^1 \|\mathbb{M}_{\alpha_0,1,0}DG(\alpha_0(t))\mathbb{M}_{\alpha_0,0,1} - DG(v_0)\|dt\\
&\quad + \|DG(u_0)^{-1}\|\|\mathbb{M}_{\psi_0,1,0}DG(v_0)\mathbb{M}_{\psi_0,0,1} - DG(u_0)\|\\
&\leq \zeta_0\Big(Kd(v_0, u_0) + \frac{K}{2}d(v_0, w_0)\Big) \leq \frac{(3a+b)}{2}K\zeta_0\xi,
\end{aligned}$$

if $(3a+b)K\xi\zeta_0 < 2$, then the operator $\mathbb{M}_{\psi_0,1,0}[v_0, w_0; G]\mathbb{M}_{\psi_0,0,1}$ is invertible and

$$\|[v_0, w_0; G]^{-1}\| \leq \frac{2\zeta_0}{2 - (3a+b)K\xi\zeta_0} = c.$$

Now, we define the polynomial

$$z(f) = \frac{L}{2}f^2 - \frac{f}{c} + \xi, \quad L = K\left(1 + \frac{3a+b}{c}\right), \quad f \in [0, f']. \tag{3.2}$$

Let $f^* = \dfrac{1-\sqrt{1-2L\xi c^2}}{Lc}$ and $f^{**} = \dfrac{1+\sqrt{1-2L\xi c^2}}{Lc}$ be two positive roots of $z(f)$ such that $0 < f^* \leq f^{**} < f'$ if $L\xi c^2 \leq \frac{1}{2}$. Also for all $n \geq 0$, define the sequences

$$\begin{aligned} f_{n+1} &= f_n - \frac{z(f_n)}{z'(f_n)}, \quad f_0 = 0, \\ \zeta_{n+1} &= \frac{\zeta_0}{1 - \zeta_0 K d(u_{n+1}, u_0)}. \end{aligned} \tag{3.3}$$

Before proving the convergence of iterative method firstly we will prove some lemmas which will be used to prove the theorem.

Lemma 3.1. *Let $G \in \mathfrak{X}(Q)$ of class C^1 on Q, then for any $n \in \mathbb{N}$, we have*

$$\begin{aligned} \mathbb{M}_{\phi,1,0}G(u_n) = \Big(\int_0^1 \big(\mathbb{M}_{\phi,1,0}DG(\phi(t))\mathbb{M}_{\phi,0,1} - DG(u_{n-1})\big)dt \\ + \big(DG(u_{n-1}) - \mathbb{M}_{\psi_{n-1},1,0}[v_{n-1}, w_{n-1}; G]\mathbb{M}_{\psi_{n-1},0,1}\big)\Big)N_n, \end{aligned}$$

where ϕ is a family of minimizing geodesics such that $\phi(0) = u_{n-1}$, $\phi(1) = u_n$.

Proof. We know that

$$[\phi(s+h), \phi(s); G]\phi^{'}(s) = \frac{1}{h}\Big(\mathbb{M}_{\phi,s+h,s}G(\phi(s+h)) - G(\phi(s))\Big),$$

put $s = 0$ and $h = 1$ in above equality, we get

$$[u_n, u_{n-1}; G]\phi^{'}(0) = \mathbb{M}_{\phi,1,0}G(u_n) - G(u_{n-1}).$$

Since $\phi(t) = \exp_{u_{n-1}}(tN_n)$, we have $\phi^{'}(0) = N_n$.

We obtain that

$$[u_n, u_{n-1}; G]N_n = \mathbb{M}_{\phi,1,0}G(u_n) - G(u_{n-1}). \tag{3.4}$$

By (3.1), we have

$$N_n = -\mathbb{M}_{\psi_{n-1},1,0}[v_{n-1}, w_{n-1}; G]^{-1}\mathbb{M}_{\psi_{n-1},0,1}G(u_{n-1})$$

or

$$G(u_{n-1}) = -\mathbb{M}_{\psi_{n-1},1,0}[v_{n-1}, w_{n-1}; G]\mathbb{M}_{\psi_{n-1},0,1}N_n. \tag{3.5}$$

By (3.4) and (3.5), we obtain

$$\begin{aligned}\mathbb{M}_{\phi,1,0}G(u_n) &= \Big([u_n, u_{n-1}; G] - \mathbb{M}_{\psi_{n-1},1,0}[v_{n-1}, w_{n-1}; G]\mathbb{M}_{\psi_{n-1},0,1}\Big)N_n \\ &= \left(\int_0^1 \big(\mathbb{M}_{\phi,1,0}DG(\phi(t))\mathbb{M}_{\phi,0,1} - DG(u_{n-1})\big)dt \right.\\ &\quad \left. + \big(DG(u_{n-1}) - \mathbb{M}_{\psi_{n-1},1,0}[v_{n-1}, w_{n-1}; G]\mathbb{M}_{\psi_{n-1},0,1}\big)\right)N_n.\end{aligned}$$

□

Lemma 3.2. *Suppose the sequence $\{f_n\}$ is generated by* (3.3)*. If $L\xi c^2 \leq \frac{1}{2}$ and $f \in [0, f^*]$, then the sequence $\{f_n\}$ is increasing and bounded above. Hence converges to f^*.*

Proof. We define the function h by

$$h(f) = f - \frac{z(f)}{z'(f)}.$$

Then differentiating both sides, we get

$$h'(f) = \frac{z(f)z''(f)}{\big(z'(f)\big)^2},$$

as $z(f) \geq 0,\ z''(f) > 0,\ z'(f) < 0$ in $[0, f^*]$. We have

$$h'(f) = \frac{z(f)z''(f)}{\big(z'(f)\big)^2} \geq 0, \quad \forall f \in [0, f^*].$$

It shows that the function h is increasing on $[0, f^*]$. So, if $f_k \in [0, f^*]$ for some $k \in \mathbb{N}$, then

$$f_k \leq f_k - \frac{z(f_k)}{z'(f_k)} = f_{k+1}$$

and

$$f_{k+1} = f_k - \frac{z(f_k)}{z'(f_k)} \leq f^* - \frac{z(f^*)}{z'(f^*)} = f^*.$$

Thus, it completes the proof of Lemma 3.2. □

Now we can demonstrate the convergence of our method.

Theorem 3.3. *Let Q be a complete Riemannian manifold, $\mho \subseteq Q$ be an open convex set and $G \in \mathfrak{X}(Q)$ satisfies the conditions* $(1) - (3)$ *with:*

$$(3a+b)\xi K\zeta_0 < 2, \quad L\xi c^2 \leq \frac{1}{2}, \quad \zeta_0 K f^* < 1, \quad K\zeta_0(3f^* + \xi + f^{**}) < 2, \quad V(u_0, f^*) \subseteq \mho.$$

Then, the method given by (3.1) *converges to a singular point u^* of the vector field G in $V[u_0, f^*]$ and the solution u^* is unique in $V[u_0, f^{**} + \xi]$.*

Proof. To prove the theorem, at first we shall prove some conditions for all $i = 0, 1, 2, \ldots$

$(C1)$ $u_i \in V[u_0, f^*]$,

$(C2)$ $v_i \in V[u_0, f^*]$,

$(C3)$ $w_i \in V[u_0, f^*]$,

$(C4)$ $\|DG(u_i)^{-1}\| \leq \zeta_i$.

For $i = 0$, $(C1)$ and $(C4)$ are trivial and since

$$d(v_0, u_0) = a\xi \leq f^*, \quad d(w_0, u_0) = b\xi \leq f^*,$$

therefore $(C1) - (C4)$ are true for $i = 0$. Now we will prove for $i \in \mathbb{N}$. We have

$$d(u_1, u_0) \leq \|[v_0, w_0; G]^{-1}\| \|G(u_0)\| \leq c\xi = f_1 - f_0 \leq f^*,$$

therefore $u_1 \in V[u_0, f^*]$. By Lemma 3.1, we have

$$\mathbb{M}_{\phi,1,0} G(u_n) = \Bigg(\int_0^1 \big(\mathbb{M}_{\phi,1,0} DG(\phi(t)) \mathbb{M}_{\phi,0,1} - DG(u_{n-1}) \big) dt \\ + \big(DG(u_{n-1}) - \mathbb{M}_{\psi_{n-1},1,0} [v_{n-1}, w_{n-1}; G] \mathbb{M}_{\psi_{n-1},0,1} \big) \Bigg) N_n.$$

For $n = 1$, we obtain that

$$\begin{aligned}
\|G(u_1)\| = \|\mathbb{M}_{\phi,1,0} G(u_1)\| &= \Bigg\| \Bigg(\int_0^1 \big(\mathbb{M}_{\phi,1,0} DG(\phi(t)) \mathbb{M}_{\phi,0,1} - DG(u_0) \big) dt \\
&\quad + \big(DG(u_0) - \mathbb{M}_{\psi_0,1,0} [v_0, w_0; G] \mathbb{M}_{\psi_0,0,1} \big) \Bigg) N_1 \Bigg\| \\
&\leq \int_0^1 \|\mathbb{M}_{\phi,1,0} DG(\phi(t)) \mathbb{M}_{\phi,0,1} - DG(u_0)\| \|N_1\| dt \\
&\quad + \|DG(u_0) - \mathbb{M}_{\psi_0,1,0} [v_0, w_0; G] \mathbb{M}_{\psi_0,0,1} \\
&\quad + \mathbb{M}_{\psi_0,1,0} DG(v_0) \mathbb{M}_{\psi_0,0,1} - \mathbb{M}_{\psi_0,1,0} DG(v_0) \mathbb{M}_{\psi_0,0,1}\| \|N_1\| \\
&\leq \frac{K}{2} d(u_1, u_0)^2 + \|\mathbb{M}_{\psi_0,1,0} DG(v_0) \mathbb{M}_{\psi_0,0,1} - DG(u_0)\| \|N_1\| \\
&\quad + \|[v_0, w_0; G] - DG(v_0)\| \|N_1\| \\
&= \frac{K}{2} d(u_1, u_0)^2 + \|\mathbb{M}_{\psi_0,1,0} DG(v_0) \mathbb{M}_{\psi_0,0,1} - DG(u_0)\| \|N_1\| \\
&\quad + \int_0^1 \|\big(\mathbb{M}_{\alpha_0,1,0} DG(\alpha_0(t)) \mathbb{M}_{\alpha_0,0,1} - DG(v_0) \big)\| \|N_1\| dt \\
&\leq \frac{K}{2} d(u_1, u_0)^2 + \frac{K(3a + b)}{2} \|G(u_0)\| d(u_1, u_0) \\
&\leq \frac{K}{2} (f_1 - f_0)^2 + \frac{K(3a + b)}{2} z(f_0)(f_1 - f_0) \leq \frac{L}{2} (f_1 - f_0)^2 = z(f_1).
\end{aligned}$$

As the sequence (3.3) is increasing and the polynomial (3.2) is decreasing in $[0, f^*]$, we have

$$d(v_1, u_0) \leq d(u_1, u_0) + d(v_1, u_1) = d(u_1, u_0) + \|L_1\| = d(u_1, u_0) + a\|G(u_1)\| \leq f^*,$$
$$d(w_1, u_0) \leq d(u_1, u_0) + d(w_1, u_1) = d(u_1, u_0) + \|M_1\| = d(u_1, u_0) + b\|G(u_1)\| \leq f^*,$$

so that $v_1,\ w_1 \in V[u_0, f^*]$. We suppose that $u_i, v_{i-1}, w_{i-1} \in V[u_0, f^*]$, for $i = 2, 3, 4, \ldots, n$. Then we will prove for $i = n+1$. Since

$$\begin{aligned} z(f_n) &= \int_0^1 \Big(z'\big(f_{n-1} + x(f_n - f_{n-1})\big) - z'(f_{n-1})\Big) dx (f_n - f_{n-1}) \\ &= L\int_0^1 x(f_n - f_{n-1})^2 dx = \frac{L}{2}(f_n - f_{n-1})^2, \end{aligned}$$

we have $\|G(u_n)\| \leq z(f_n)$, for all $n \in \mathbb{N}$, as

$$\begin{aligned} \|G(u_n)\| = \|\mathbb{M}_{\phi,1,0} G(u_n)\| &= \bigg\| \bigg(\int_0^1 \big(\mathbb{M}_{\phi,1,0} DG(\phi(t)) \mathbb{M}_{\phi,0,1} - DG(u_{n-1})\big) dt \\ &\quad + \big(DG(u_{n-1}) - \mathbb{M}_{\psi_{n-1},1,0}[v_{n-1}, w_{n-1}; G] \mathbb{M}_{\psi_{n-1},0,1}\big) \bigg) N_n \bigg\| \\ &\leq \int_0^1 \|\mathbb{M}_{\phi,1,0} DG(\phi(t)) \mathbb{M}_{\phi,0,1} - DG(u_{n-1})\| \|N_n\| dt \\ &\quad + \|DG(u_{n-1}) - \mathbb{M}_{\psi_{n-1},1,0}[v_{n-1}, w_{n-1}; G] \mathbb{M}_{\psi_{n-1},0,1} \\ &\quad + \mathbb{M}_{\psi_{n-1},1,0} DG(v_{n-1}) \mathbb{M}_{\psi_{n-1},0,1} - \mathbb{M}_{\psi_{n-1},1,0} DG(v_{n-1}) \mathbb{M}_{\psi_{n-1},0,1}\| \|N_n\| \\ &\leq \frac{K}{2} d(u_n, u_{n-1})^2 + \|\mathbb{M}_{\psi_{n-1},1,0} DG(v_{n-1}) \mathbb{M}_{\psi_{n-1},0,1} - DG(u_{n-1})\| \|N_n\| \\ &\quad + \|[v_{n-1}, w_{n-1}; G] - DG(v_{n-1})\| \|N_n\| \\ &= \frac{K}{2} d(u_n, u_{n-1})^2 + \|\mathbb{M}_{\psi_{n-1},1,0} DG(v_{n-1}) \mathbb{M}_{\psi_{n-1},0,1} - DG(u_{n-1})\| \|N_n\| \\ &\quad + \int_0^1 \|\big(\mathbb{M}_{\alpha_{n-1},1,0} DG(\alpha_{n-1}(t)) \mathbb{M}_{\alpha_{n-1},0,1} - DG(v_{n-1})\big)\| \|N_n\| dt \\ &\leq \frac{K}{2} d(u_n, u_{n-1})^2 + \frac{K(3a+b)}{2} \|G(u_{n-1})\| d(u_n, u_{n-1}) \\ &\leq \frac{K}{2} (f_n - f_{n-1})^2 + \frac{K(3a+b)}{2} z(f_{n-1})(f_n - f_{n-1}) \leq \frac{L}{2}(f_n - f_{n-1})^2 = z(f_n). \end{aligned}$$

We have

$$d(v_n, u_0) \leq d(u_n, u_0) + d(v_n, u_n) = d(u_n, u_0) + a\|L_n\| = d(u_n, u_0) + a\|G(u_n)\| \leq f^*,$$
$$d(w_n, u_0) \leq d(u_n, u_0) + d(w_n, u_n) = d(u_n, u_0) + b\|M_n\| = d(u_n, u_0) + b\|G(u_n)\| \leq f^*,$$

so that $v_n, w_n \in V[u_0, f^*]$. Now we will show that the operator $[v_n, w_n; G]^{-1}$ is bounded. Let π

be a minimizing geodesic such that $\pi(0) = u_0$, $\pi(1) = v_n$. We have

$$\begin{aligned}\|DG(u_0)^{-1}\mathbb{M}_{\pi,1,0}[v_n, w_n; G]\mathbb{M}_{\pi,0,1} - I_{u_0}\| &\leq \|DG(u_0)^{-1}\mathbb{M}_{\pi,1,0}([v_n, w_n; G] - DG(v_n))\mathbb{M}_{\pi,0,1}\| \\ &+ \|DG(u_0)^{-1}(\mathbb{M}_{\pi,1,0}DG(v_n)\mathbb{M}_{\pi,0,1} - DG(u_0))\| \\ &\leq \|DG(u_0)^{-1}\| \int_0^1 \|\mathbb{M}_{\alpha_n,1,0}DG(\alpha_n(t))\mathbb{M}_{\alpha_n,0,1} - DG(v_n)\|dt \\ &+ \|DG(u_0)^{-1}\|\|\mathbb{M}_{\pi,1,0}DG(v_n)\mathbb{M}_{\pi,0,1} - DG(u_0)\| \\ &\leq \frac{K\zeta_0}{2}\big(2(f_n - f_0) + (3a+b)z(f_n)\big) < 1,\end{aligned}$$

therefore $\mathbb{M}_{\pi,1,0}[v_n, w_n; G]\mathbb{M}_{\pi,0,1}$ is invertible and

$$\begin{aligned}\|[v_n, w_n; G]^{-1}\| &= \|\mathbb{M}_{\pi,1,0}[v_n, w_n; G]^{-1}\mathbb{M}_{\pi,0,1}\| \\ &\leq \frac{\|DG(u_0)^{-1}\|}{1 - \|DG(u_0)^{-1}\|\|\mathbb{M}_{\pi,1,0}[v_n, w_n; G]^{-1}\mathbb{M}_{\pi,0,1} - DG(u_0)\|} \leq \frac{-1}{z'(f_n)}.\end{aligned}$$

We have

$$d(u_{n+1}, u_n) \leq \|[v_n, w_n; G]^{-1}\|\|G(u_n)\| \leq \frac{-z(f_n)}{z'(f_n)} = f_{n+1} - f_n \tag{3.6}$$

and

$$d(u_{n+1}, u_0) \leq d(u_{n+1}, u_n) + d(u_n, u_0) \leq f_{n+1} - f_n + f_n - f_0 = f_{n+1} - f_0 \leq f^*.$$

So that $u_{n+1} \in V[u_0, f^*]$. Suppose $(C4)$ holds for $i = 1, 2, \ldots, n$ and then we will prove for $i = n+1$. Let δ be a minimizing geodesic δ from $[0,1]$ to Q such that $\delta(0) = u_0$, $\delta(1) = u_{n+1}$, and $\|\delta'(0)\| = d(u_{n+1}, u_0)$.

We obtain that

$$\|\mathbb{M}_{\delta,1,0}DG(u_{n+1})\mathbb{M}_{\delta,0,1} - DG(u_0)\| \leq K\int_0^1 \|\delta'(0))\|ds = Kd(u_{n+1}, u_0) \leq Kf^*$$

and

$$\|DG(u_0)^{-1}\|\|\mathbb{M}_{\delta,1,0}DG(u_{n+1})\mathbb{M}_{\delta,0,1} - DG(u_0)\| \leq \zeta_0 Kf^* < 1,$$

as $\zeta_0 Kf^* < 1$. Therefore $\mathbb{M}_{\delta,1,0}DG(u_{n+1})\mathbb{M}_{\delta,0,1}$ is invertible by Banach's lemma and

$$\begin{aligned}\|DG(u_{n+1})^{-1}\| = \|\mathbb{M}_{\delta,1,0}DG(u_{n+1})^{-1}\mathbb{M}_{\delta,0,1}\| &\leq \frac{\|DG(u_0)^{-1}\|}{1 - \|DG(u_0)^{-1}\|\|\mathbb{M}_{\delta,1,0}DG(u_{n+1})\mathbb{M}_{\delta,0,1} - DG(u_0)\|} \\ &\leq \frac{\zeta_0}{1 - \zeta_0 Kd(u_{n+1}, u_0)} = \zeta_{n+1},\end{aligned}$$

therefore it holds for $i = n+1$. Thus $(C1) - (C4)$ hold for all $i \in \mathbb{N}$.

Now we will prove the Theorem. Since $\{f_n\}$ is a convergent sequence and hence it is a Cauchy sequence therefore from (3.6) the sequence $\{u_n\}$ is also a convergent sequence and let the sequence

$\{u_n\}$ converges to $u^* \in V[u_0, f^*]$. Now we will show that u^* is a singularity of G. As for all $n \in \mathbb{N}$,

$$\|G(u_n)\| \leq z(f_n),$$

taking $n \to \infty$ both sides, we get

$$\|G(u^*)\| \leq z(f^*) = 0.$$

Then, we have $G(u^*) = 0$. Finally, we will show that the singularity is unique in $V[u_0, f^{**} + \xi]$. Let v^* be another singularity of G in $V[u_0, f^{**} + \xi]$. Let ρ be a minimizing geodesic from $[0,1]$ to Q such that $\rho(0) = u^*,\ \rho(1) = v^*$, and $\|\rho'(0)\| = d(u^*, v^*)$.

We obtain

$$\|\mathbb{M}_{\rho,t,0} DG(\rho(t))\mathbb{M}_{\rho,0,t} - DG(u^*)\| \leq K \int_0^t \|\rho'(0)\| ds = Ktd(u^*, v^*) \leq Kt\left(d(u_0, u^*) + d(u_0, v^*)\right)$$

and

$$\begin{aligned}
\|DG(u^*)^{-1}\| \int_0^1 \|\mathbb{M}_{\rho,t,0} DG(\rho(t))\mathbb{M}_{\rho,0,t} - DG(u^*)\| dt &\leq \left(\frac{1}{\zeta_0} - Kf^*\right)^{-1} \int_0^1 Kt\left(d(u_0, u^*) + d(u_0, v^*)\right) dt \\
&\leq \left(\frac{1}{\zeta_0} - Kf^*\right)^{-1} \frac{K}{2}(f^* + f^{**} + \xi) < 1.
\end{aligned}$$

It shows that the operator

$$T = \int_0^1 \mathbb{M}_{\rho,t,0} DG(\rho(t))\mathbb{M}_{\rho,0,t} dt$$

is invertible by Banach's lemma and we have

$$0 = \mathbb{M}_{\rho,1,0} G(v^*) - G(u^*) = \int_0^1 \mathbb{M}_{\rho,t,0} DG(\rho(t))\mathbb{M}_{\rho,0,t}(\rho'(0)) dt.$$

So that $\rho'(0) = 0$. We have $0 = \|\rho'(0)\| = d(u^*, v^*)$, implies that $u^* = v^*$. Thus it completes the proof. □

Theorem 3.4. *Suppose that u^* is a singular point of G in $V[u_0, f^*]$, if $V(u_0, f^{**}) \subseteq \mho$, then the only singular point of G in $V[u_0, r]$ is u^*, where $f^* < r \leq f^{**}$.*

Proof. Let v^* be a singular point of G in $V[u_0, r]$. Let Λ be a minimizing geodesic such that $\Lambda(0) = u_0, \quad \Lambda(1) = v^*$. Then by (2.3), we have

$$\begin{aligned}
\mathbb{M}_{\Lambda,1,0} G(v^*) &= \mathbb{M}_{\Lambda,1,0} G(v^*) - G(u_0) + G(u_0) + DG(u_0)\Lambda'(0) - DG(u_0)\Lambda'(0) \\
&= \int_0^1 \mathbb{M}_{\Lambda,t,0} DG(\Lambda(t))\mathbb{M}_{\Lambda,0,t}\Lambda'(0) dt - DG(u_0)\Lambda'(0) + G(u_0) + DG(u_0)\Lambda'(0) \\
&= \int_0^1 \big(\mathbb{M}_{\Lambda,t,0} DG(\Lambda(t))\mathbb{M}_{\Lambda,0,t} - DG(u_0)\big)\Lambda'(0) dt + G(u_0) + DG(u_0)\Lambda'(0).
\end{aligned}$$

Thus, we have

$$\frac{Ld(u_0,v^*)^2}{2} \geq \frac{Kd(u_0,v^*)^2}{2} \geq \|G(u_0)+DG(u_0)\Lambda'(0)\| \geq \frac{1}{\|DG(u_0)^{-1}\|}\|DG(u_0)^{-1}G(u_0)+\Lambda'(0)\|$$
$$\geq \frac{1}{\zeta_0}\Big(\|\Lambda'(0)\| - \|DG(u_0)^{-1}G(u_0)\|\Big) \geq \left(\frac{d(u_0,v^*)}{\zeta_0} - \xi\right) \geq \left(\frac{d(u_0,v^*)}{c} - \xi\right).$$

Therefore

$$z(d(u_0,v^*)) = \frac{Ld(u_0,v^*)^2}{2} - \frac{d(u_0,v^*)}{c} + \xi \geq 0.$$

Since $d(u_0,v^*) \leq r \leq f^{**}$, we have $d(u_0,v^*) \leq f^*$, hence by Theorem 3.3, $u^* = v^*$. □

4 Numerical examples

In this section, two examples are given to show the application of our theorem.

Example 4.1. *Let us consider the vector field G from $\mho = (-1,1)^3 \subseteq Q = \mathbb{R}^3$ to $\mho = (-1,1)^3$ given by*

$$G\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = \begin{pmatrix}e^{u_1}-1\\u_2^2+u_2\\u_3\end{pmatrix}$$

with the max norm $\|\cdot\|_\infty$. For the point $\boldsymbol{u} = (u_1,u_2,u_3)^T$, the first and second Fréchet derivatives of G are:

$$DG(\boldsymbol{u}) = \begin{bmatrix}e^{u_1} & 0 & 0\\ 0 & 2u_2+1 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad D^2G(\boldsymbol{u}) = \left[\begin{array}{ccc|ccc|ccc}e^{u_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{array}\right].$$

Initially for $\boldsymbol{u}_0 = (-0.005,-0.005,-0.005)^T$, we obtain

$$\|G(\boldsymbol{u}_0)\| = \max(|-0.005|,|-0.005|,|-0.005|) = 0.005 = \xi,$$
$$\|DG(\boldsymbol{u}_0)^{-1}\| = 1.0101 = \zeta_0, \quad \|D^2G(\boldsymbol{u})\| = \max(0.995,2,0) = 2 = K.$$

Now, for $a = 1$, $b = 1$, all the assumptions of the convergence theorem are satisfied and the Steffensen-like method can be applied to get the desired singular point.

Example 4.2. *Let us consider the vector field G from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by*

$$G\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}\frac{\cos u_1+4u_1}{4}\\u_2\end{pmatrix}$$

with the max norm $\|\cdot\|_\infty$*. For the point* $u = (u_1, u_2)^T$*, the first and second Fréchet derivatives of* G *are:*

$$DG(u) = \begin{bmatrix} \frac{-\sin u_1 + 4}{4} & 0 \\ 0 & 1 \end{bmatrix}, \quad D^2G(u) = \left[\begin{array}{cc|cc} \frac{-\cos u_1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Initially for $u_0 = (0,0)^T$*, we obtain*

$$\|G(u_0)\| = \frac{1}{4} = \xi, \quad \|DG(u_0)^{-1}\| = 1 = \zeta_0, \quad \|D^2G(u)\| \le \frac{1}{4} = K.$$

Now, for $a = 1,\ b = 1,$ *all the assumptions for convergence are satisfied and the Steffensen-like method can be applied to get the desired singular point.*

5 Conclusion

In this paper, we have studied the semilocal convergence of Steffensen-like method for approximating the zeros of a vector field in Riemannian manifolds and established convergence theorem under Lipschitz continuity condition on the first order covariant derivative of a vector field. Finally, two examples are given to show the application of our theorem.

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