CUBO, A Mathematical Journal
Vol. 26, no. 3, pp. 541–558, December 2024
DOI: 10.56754/0719-0646.2603.541

# Investigating the existence and multiplicity of solutions to $\varphi(x)$-Kirchhoff problem

Abolfazl Sadeghi[1]
Ghasem Alizadeh Afrouzi[1,✉]
Maryam Mirzapour[2]

[1] *Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.*
sadeghi31587@mail.com
afrouzi@umz.ac.ir✉

[2] *Department of Mathematics Education, Farhangian University, P.O. Box 14665-889, Tehran, Iran.*
m.mirzapour@cfu.ac.ir

**ABSTRACT**

In this article, we want to discuss variational methods such as the Mountain pass theorem and the Symmetric Mountain pass theorem, without the Ambrosetti-Rabinowitz condition. We prove the existence and multiplicity of nontrivial weak solutions for the problem of the following form

$$\begin{cases} -\left(\alpha - \beta \displaystyle\int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)} dx\right) \Delta_{\varphi(x)} \upsilon + |\upsilon|^{\psi(x)-2}\upsilon \\ \qquad = \lambda \eta(x,\upsilon), \\ \qquad x \in \Omega, \\ \left(\alpha - \beta \displaystyle\int_{\partial\Omega} \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)} \, dx\right) |\nabla \upsilon|^{\varphi(x)-2} \frac{\partial \upsilon}{\partial \nu} = 0 \\ \qquad x \in \partial\Omega, \end{cases}$$

where $\alpha \geq \beta > 0$, $\Delta_{\varphi(x)}\upsilon$ is the $\varphi(x)$-Laplacian operator, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $\nu$ is the outer unit normal to $\partial\Omega$, $\varphi(x), \psi(x) \in C(\bar{\Omega})$ with $1 < \varphi(x) < N$, $\varphi(x) < \psi(x) < \varphi^*(x) := \dfrac{N\varphi(x)}{N - \varphi(x)}$, $\lambda > 0$ is a real parameter and $\eta(x,t) \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$.

Published: 13 December, 2024
Accepted: 29 November, 2024
Received: 14 April, 2024

**RESUMEN**

En este artículo discutimos métodos variacionales, como el teorema del paso de la montaña y el teorema simétrico del paso de la montaña, sin la condición de Ambrosetti-Rabinowitz. Demostramos la existencia y multiplicidad de soluciones débiles no triviales para el problema de la siguiente forma

$$\begin{cases} -\left(\alpha - \beta \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)} dx\right) \Delta_{\varphi(x)} \upsilon + |\upsilon|^{\psi(x)-2}\upsilon \\ \qquad = \lambda \eta(x,\upsilon), \\ \qquad x \in \Omega, \\ \left(\alpha - \beta \int_{\partial\Omega} \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx\right) |\nabla \upsilon|^{\varphi(x)-2} \frac{\partial \upsilon}{\partial \nu} = 0 \\ \qquad x \in \partial\Omega, \end{cases}$$

donde $\alpha \geq \beta > 0$, $\Delta_{\varphi(x)}\upsilon$ es el $\varphi(x)$ operador Laplaciano, $\Omega$ es un dominio acotado y suave en $\mathbb{R}^N$ con borde suave $\partial\Omega$ y $\nu$ es la normal unitaria exterior a $\partial\Omega$, $\varphi(x), \psi(x) \in C(\bar{\Omega})$ con $1 < \varphi(x) < N,\ \varphi(x) < \psi(x) < \varphi^*(x) := \dfrac{N\varphi(x)}{N - \varphi(x)}$, $\lambda > 0$ es un parámetro real y $\eta(x,t) \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$.

**Keywords and Phrases:** Generalized Lebesgue-Sobolev spaces, weak solutions, mountain pass theorem, symmetric mountain pass theorem.

**2020 AMS Mathematics Subject Classification:** 35J60, 35J20.

# 1 Introduction

In this article, we consider the following problem

$$\begin{cases} -\left(\alpha-\beta\int_{\Omega}\dfrac{1}{\varphi(x)}|\nabla\upsilon|^{\varphi(x)}\,dx\right)\Delta_{\varphi(x)}\upsilon+|\upsilon|^{\psi(x)-2}\upsilon=\lambda\,\eta(x,\upsilon), & x\in\Omega,\\ \left(\alpha-\beta\int_{\partial\Omega}\dfrac{1}{\varphi(x)}|\nabla\upsilon|^{\varphi(x)}\,dx\right)|\nabla\upsilon|^{\varphi(x)-2}\dfrac{\partial\upsilon}{\partial\nu}=0, & x\in\partial\Omega,\end{cases} \tag{1.1}$$

where $\alpha\geq\beta>0$, $\Delta_{\varphi(x)}\upsilon$ is the $\varphi(x)$-Laplacian operator, defined as $\Delta_{\varphi(x)}\upsilon:=\operatorname{div}(|\nabla\upsilon|^{\varphi(x)-2}\nabla\upsilon)=\sum_{i=1}^{N}\left(|\nabla\upsilon|^{\varphi(x)-2}\dfrac{\partial\upsilon}{\partial x_i}\right)$, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $\nu$ is the outer unit normal to $\partial\Omega$ and $\varphi(x),\psi(x)\in C(\bar{\Omega})$ with $1<\varphi(x)<N$, $\varphi(x)<\psi(x)<\varphi^*(x):=\dfrac{N\varphi(x)}{N-\varphi(x)}$, $\lambda>0$ is a real parameter. We define $\varphi_l$ and $\varphi_s$ for convenience as follows: $\varphi_l:=\inf_{\Omega}\varphi(x)$ and $\varphi_s:=\sup_{\Omega}\varphi(x)$, for all $\varphi(x)\in C(\bar{\Omega})$. The function $\eta(x,t)\in C(\bar{\Omega}\times\mathbb{R},\mathbb{R})$ satisfies:

($\eta_1$) $|\eta(x,t)|\leq c(1+|t|^{r(x)-1})$, $\forall(x,t)\in\Omega\times\mathbb{R}$, where $c>0$ and $\varphi(x)<r(x)<\varphi^*(x)$,

($\eta_2$) $\displaystyle\lim_{t\to0}\frac{\eta(x,t)}{|t|^{\varphi(x)-2}t}=0$, uniformly a.e. $x\in\Omega$,

($\eta_3$) $\displaystyle\lim_{|t|\to\infty}\frac{\eta(x,t)}{|t|^{\varphi_s}}=+\infty$, uniformly a.e. $x\in\Omega$,

($\eta_4$) there exists a constant $c_0>0$ such that $\hat{H}(x,t)\leq\hat{H}(x,s)+c_0$ for each $x\in\Omega$, $0<|t|<s$, where $\hat{H}(x,t):=t\,\eta(x,t)-\varphi_sH(x,t)$ and $H(x,t):=\int_0^t\eta(x,s)ds$,

($\eta_5$) $\eta(x,-t)=-\eta(x,t)$ for all $(x,t)\in\Omega\times\mathbb{R}$.

In addition to the conditions given for $\eta$, the functions $\varphi(x),\psi(x),r(x)$ must satisfy the following condition, which we call the $(\varphi\psi r)$-condition:

$$1<\varphi_l<\varphi(x)<\varphi_s<\psi_l<\psi(x)<\psi_s<2\varphi_l<r_l<r(x)<r_s<\varphi^*(x).$$

Sobolev spaces are essential in contemporary analysis, especially in the study of partial differential equations (PDEs) and functional analysis. These spaces generalize the classical concepts of differentiability and integrability, offering a more adaptable structure for analyzing functions whose derivatives might not be classically well-defined. By incorporating weak derivatives, Sobolev spaces allow for the examination of broader issues in areas such as mathematical physics, fluid dynamics, and engineering applications, see [1,4,5,7–9,12,20,21,26,27,32,34,38].

The necessity of Sobolev spaces arises from their ability to handle irregularities and discontinuities in functions that appear naturally in real-world problems. For instance, solutions to PDEs often lack classical differentiability but possess weak derivatives that allow their analysis within Sobolev spaces. This makes them indispensable in addressing variational problems and boundary value problems.

Kirchhoff's problems, named after the German physicist Gustav Kirchhoff [28], are fundamental in the study of mechanics and mathematical physics, particularly in understanding wave propagation and elasticity theory. Kirchhoff's equations describe the motion of elastic surfaces and play a key role in modeling vibrating systems, such as strings, membranes, and plates. Recent research in this field has focused on nonlinear versions of Kirchhoff's equations, particularly in higher dimensions, where the complexity of the problem increases, see [2, 6, 10, 11, 14, 17–19, 24, 25, 31, 34, 37].

Variational methods have a relatively long history. Many scientists have studied in this field and have achieved many successes. Due to the applicability of this method in experimental sciences, it has always been of interest [**?**, 3, 8, 13, 15, 16, 22, 23, 26, 29, 33, 35, 36]. In these methods, especially those used to solve boundary value problems, the Palais-Smale condition ($(PS)$-condition in short) plays a crucial role in ensuring the existence of critical points, which correspond to solutions of the problem. This condition provides a framework for the analysis of functionals in infinite-dimensional spaces, such as Sobolev spaces. On the other hand, the Cerami condition ($(C)$-condition in short) is a variation of the $(PS)$-condition that is particularly useful in dealing with problems where the $(PS)$-condition might not hold. This modified condition is often more applicable in certain classes of problems, particularly those involving non-compact domains or complex geometries.

Now we state our main results.

**Theorem 1.1.** *Suppose* $(\eta_1) - (\eta_4)$ *and the* $(\varphi\psi r)$*-condition hold. Then problem* (1.1) *has at least a nontrivial weak solution for all* $\lambda < \lambda_0$ *(*$\lambda_0$ *which has been given in Section 3).*

**Theorem 1.2.** *Suppose* $(\eta_1), (\eta_2), (\eta_4), (\eta_5)$ *and the* $(\varphi\psi r)$*-condition hold. Then problem* (1.1) *has infinitely many weak solutions for all* $\lambda < \lambda_0$ *(*$\lambda_0$ *which has been given in Section 3).*

To prove our results, we will use inequalities and applied theorems such as Hölder and Poincaré inequalities and the embedding, Mountain pass and Symmetric Mountain pass theorems.

## 2 Preliminary results

In this section, we recall some important definitions and essential characteristics of the generalized Lebesgue-Sobolev spaces $L^{\varphi(x)}(\Omega)$ and $W^{1,\varphi(x)}(\Omega)$ where $\Omega \subset \mathbb{R}^N$ is an open set. In this regard, we refer readers to the book of Musielak [32] and the papers [20, 21]. Set

$$C_+(\bar{\Omega}) := \{h : h \in C(\bar{\Omega}), h(x) > 1 \quad \text{for all} \quad x \in \bar{\Omega}\},$$

and for each $\varphi(x) \in C_+(\bar{\Omega})$

$$L^{\varphi(x)}(\Omega) = \left\{ \upsilon : \text{a measurable real-valued function such that } \int_\Omega |\upsilon(x)|^{\varphi(x)}\, dx < \infty \right\},$$

is the definition of variable exponent Lebesgue space, that by mentioned the following norm (so-called Luxemburg norm) is reflexive and separable Banach space

$$\|\upsilon\|_{\varphi(x)} := \inf \left\{ \mu > 0; \int_\Omega \left| \frac{\upsilon(x)}{\mu} \right|^{\varphi(x)} dx \leq 1 \right\}.$$

These spaces are similar to classical Lebesgue spaces in many aspects [35]:

a) If $0 < |\Omega| < \infty$ and $\varphi_1(x), \varphi_2(x)$ are variable exponents so that $\varphi_1(x) \leq \varphi_2(x)$ a.e. $x \in \Omega$, then there is a continuous embedding

$$L^{\varphi_2(x)}(\Omega) \hookrightarrow L^{\varphi_1(x)}(\Omega).$$

b) The Hölder inequality holds, *i.e.*, if $L^{\varphi'(x)}(\Omega)$ is a conjugate of $L^{\varphi(x)}(\Omega)$, where $\dfrac{1}{\varphi(x)} + \dfrac{1}{\varphi'(x)} = 1$, we have

$$\left| \int_\Omega u\upsilon\, dx \right| \leq \left( \frac{1}{\varphi_l} + \frac{1}{\varphi_l'} \right) \|u\|_{\varphi(x)} \|\upsilon\|_{\varphi'(x)}, \quad \forall u \in L^{\varphi(x)}(\Omega), \quad \forall \upsilon \in L^{\varphi'(x)}(\Omega).$$

The modular plays an essential role in manipulating the $L^{\varphi(x)}$ spaces and is defined by the following relation, $\rho_{\varphi(x)} : L^{\varphi(x)} \to \mathbb{R}$;

$$\rho_{\varphi(x)}(\upsilon) = \int_\Omega |\upsilon|^{\varphi(x)} dx.$$

**Proposition 2.1** ([20])**.** *If $\upsilon, \upsilon_n \in L^{\varphi(x)}(\Omega)$ and $\varphi_s < +\infty$, then the following relations hold*

(1) $\|\upsilon\|_{\varphi(x)} > 1 \implies \|\upsilon\|_{\varphi(x)}^{\varphi_l} \leq \rho_{\varphi(x)}(\upsilon) \leq \|\upsilon\|_{\varphi(x)}^{\varphi_s}$;

(2) $\|\upsilon\|_{\varphi(x)} < 1 \implies \|\upsilon\|_{\varphi(x)}^{\varphi_s} \leq \rho_{\varphi(x)}(\upsilon) \leq \|\upsilon\|_{\varphi(x)}^{\varphi_l}$;

(3) $\|\upsilon\|_{\varphi(x)} < 1$ *(respectively,* $= 1; > 1$*)* $\iff \rho_{\varphi(x)}(\upsilon) < 1$ *(respectively,* $= 1; > 1$*);*

(4) $\|\upsilon_n\|_{\varphi(x)} \to 0$ *(respectively,* $\to +\infty$*)* $\iff \rho_{\varphi(x)}(\upsilon) = 0$ *(respectively,* $\to +\infty$*);*

(5) $\lim_{n\to\infty} \|\upsilon_n - \upsilon\|_{\varphi(x)} = 0 \iff \lim_{n\to\infty} \rho_{\varphi(x)}(\upsilon_n - \upsilon) = 0$;

(6) *For* $\upsilon \neq 0$, $\|\upsilon\|_{\varphi(x)} = \lambda \iff \rho\left(\frac{\upsilon}{\lambda}\right) = 1$.

**Definition 2.2** ([21])**.** *If $\Omega \subset \mathbb{R}^N$, the Sobolev space with variable exponent $W^{1,\varphi(x)}(\Omega)$ is defined as*

$$W^{1,\varphi(x)}(\Omega) := \{\upsilon : \Omega \to \mathbb{R} : \upsilon \in L^{\varphi(x)}(\Omega),\ |\nabla\upsilon| \in L^{\varphi(x)}(\Omega)\},$$

*endowed with the following norm*

$$\|\upsilon\|_{W^{1,\varphi(x)}} := |||\upsilon||| = \|\upsilon\|_{\varphi(x)} + \|\nabla\upsilon\|_{\varphi(x)},$$

*or equivalently*

$$|||\upsilon||| = \inf\left\{\mu > 0, \int_\Omega \frac{\|\nabla\upsilon(x)\|_{\varphi(x)}^{\varphi(x)} + \|\upsilon\|_{\varphi(x)}^{\varphi(x)}}{\mu^{\varphi(x)}} dx \leq 1\right\}.$$

**Proposition 2.3** ([20])**.** *The Poincaré inequality in $W^{1,\varphi(x)}(\Omega)$ holds, that is, there exists a positive constant $c$ so that*

$$\|\upsilon\|_{\varphi(x)} \leq c\|\nabla\upsilon\|_{\varphi(x)}, \quad \forall \upsilon \in W^{1,\varphi(x)}(\Omega). \tag{2.1}$$

**Proposition 2.4** (Sobolev embedding [21])**.** *If $\varphi(x), \psi(x) \in C_+(\bar{\Omega})$ and $1 \leq \psi(x) \leq \varphi^*(x)$ for each $x \in \bar{\Omega}$, then there exists a continuous embedding*

$$W^{1,\varphi(x)}(\Omega) \hookrightarrow L^{\psi(x)}(\Omega). \tag{2.2}$$

*If $1 < \psi(x) < \varphi^*(x)$, the continuous embedding is compact.*

In the sequel, the constant $c_{emb}$ represents the Sobolev embedding quantity, and we denote by $X := W^{1,\varphi(x)}(\Omega); X^* = (W^{1,\varphi(x)}(\Omega))^*$, the dual space and $\langle\cdot,\cdot\rangle$, the dual pair.

**Lemma 2.5** ([21])**.** *Suppose*

$$J(\upsilon) = \int_\Omega \frac{1}{\varphi(x)}|\nabla\upsilon|^{\varphi(x)}\, dx, \quad \forall \upsilon \in X,$$

*then $J(\upsilon) \in C^1(X, \mathbb{R})$ and the derivative operator $J'$ of $J$ is*

$$\langle J'(\upsilon), \vartheta\rangle = \int_\Omega |\nabla\upsilon|^{\varphi(x)-2}\nabla\upsilon\nabla\vartheta\, dx, \quad \forall \upsilon, \vartheta \in X$$

*and the following relations hold:*

(1) *$J$ is a convex functional,*

(2) *$J' : X \to X^*$ is a strictly monotone operator and bounded homeomorphism,*

(3) *$J'$ is a mapping of type $(S_+)$, it means, $\upsilon_n \rightharpoonup \upsilon$ (weakly) and $\lim_{n\to+\infty} \sup\langle J'(\upsilon), \upsilon_n - \upsilon\rangle \leq 0$, imply $\upsilon_n \to \upsilon$ (strongly) in $W_0^{1,\varphi(x)}(\Omega)$.*

**Definition 2.6.** *$\upsilon \in X$ is a weak solution of problem* (1.1)*, if*

$$\left(\alpha - \beta \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx\right) \int_\Omega |\nabla \upsilon|^{\varphi(x)-2} \nabla \upsilon \nabla \nu \, dx + \int_\Omega |\upsilon|^{\psi(x)-2} \upsilon \nu \, dx = \lambda \int_\Omega \eta(x,\upsilon) \nu \, dx, \quad \forall \nu \in X.$$

*The energy functional related to our problem, $J_\lambda : X \to \mathbb{R}$ such that*

$$J_\lambda(\upsilon) = \alpha \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx - \frac{\beta}{2} \left( \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx \right)^2 + \int_\Omega \frac{1}{\psi(x)} |\upsilon|^{\psi(x)}\, dx - \lambda \int_\Omega H(x,\upsilon)\, dx, \quad \forall \upsilon \in X, \tag{2.3}$$

*which is also well defined and of class $C^1$ in $(X, \mathbb{R})$.*

Now we define $J'_\lambda$ as the derivative operator of $J_\lambda$ in the weak sense, by the following formula,

$$\langle J'_\lambda(\upsilon), \nu \rangle = \left(\alpha - \beta \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx\right) \int_\Omega |\nabla \upsilon|^{\varphi(x)-2} \nabla \upsilon \nabla \nu \, dx + \int_\Omega |\upsilon|^{\psi(x)-2} \upsilon \nu \, dx - \lambda \int_\Omega \eta(x,\upsilon) \nu \, dx, \quad \forall \upsilon, \nu \in X. \tag{2.4}$$

A critical point of $J_\lambda$ is clearly a weak solution of problem (1.1).

**Definition 2.7.** *If $(X, \|\cdot\|)$ is a real Banach space and $J \in C^1(X, \mathbb{R})$, then we can say that $J$ ensures Cerami-condition in level $c$ ($(C)_c$-condition in short), if for all sequence $\{\upsilon_n\} \subset X$ satisfying*

$$J(\upsilon_n) \to c \quad \text{and} \quad \|J'(\upsilon_n)\|_{X^*}(1 + \|\upsilon_n\|_X) \to 0, \tag{2.5}$$

*then, $\{\upsilon_n\}$ contains a convergent subsequence.*

If this condition holds for each $c \in \mathbb{R}$, it can be called $(C)$-condition.

## 3 Proof of Theorem 1.1

To prove Theorem 1.1, we will use the following Mountain pass theorem.

**Theorem 3.1** (Mountain pass theorem [8])**.** *Let $X$ be a real Banach space, let $J_\lambda : X \to \mathbb{R}$ as $J_\lambda \in C^1(X, \mathbb{R})$ that ensures the $(C)_c$-condition and $J_\lambda(0) = 0$, such that*

*(a) there exists $R > 0$ and $\alpha > 0$, so that $J_\lambda(\upsilon) \geq \alpha$ for each $\upsilon \in X$ with $|||\upsilon||| = R$,*

*(b) there is a function $e \in X$ such that $|||e||| > R$ and $J_\lambda(e) \leq 0$.*

*So, $J_\lambda$ has a critical value $c \geq \alpha$, that is $\upsilon \in X$, such that $J_\lambda(\upsilon) = c$ and $J'_\lambda(\upsilon) = 0$ in $X^*$.*

First, we prove that $J_\lambda$ has the geometry of the above Mountain pass theorem.

**Lemma 3.2.** *(a) Under the condition $(\eta_3)$ the functional $J_\lambda$ is unbounded from below.*

*(b) Under the conditions $(\eta_1)$ and $(\eta_2)$, $\upsilon = 0$ is a strict local minimum for $J_\lambda$.*

*Proof.* (a) By $(\eta_3)$, we have

$$\forall M > 0,\ \exists c_M > 0; \quad \eta(x,t) \geq M|t|^{\varphi_s} - c_M, \quad \forall x \in \Omega, \quad t \in \mathbb{R}. \tag{3.1}$$

If $\upsilon \in X$ for $\upsilon > 0$, and (3.1), we have

$$\begin{aligned} J_\lambda(t\upsilon) &= \alpha \int_\Omega \frac{t^{\varphi(x)}}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx - \frac{\beta}{2}\left(\int_\Omega \frac{t^{\varphi(x)}}{\varphi(x)}\ \nabla \upsilon|^{\varphi(x)}\, dx\right)^2 + \int_\Omega \frac{t^{\psi(x)}}{\psi(x)} |\upsilon|^{\psi(x)}\, dx \\ &- \lambda \int_\Omega H(x, t\upsilon)\, dx \\ &\leq \alpha t^{\varphi_s} \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx - \frac{\beta}{2} t^{2\varphi_l} \left(\int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx\right)^2 + t^{\psi_s} \int_\Omega \frac{1}{\psi(x)} |\upsilon|^{\psi(x)}\, dx \\ &- M\lambda t^{\varphi_s} \int_\Omega |\upsilon|^{\varphi(x)}\, dx + \lambda c_M |\Omega| \to -\infty, \quad \text{as} \quad t \to +\infty, \end{aligned}$$

since $\varphi_s < \psi_s < 2\varphi_l$, thus, $J_\lambda$ is unbounded from below.

(b) According to the conditions $(\eta_1)$ and $(\eta_2)$, we have

$$\forall \varepsilon > 0,\ \exists c_\varepsilon > 0; \quad H(x,t) \leq \varepsilon |t|^{\varphi(x)} + c_\varepsilon |t|^{r(x)}, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Therefore, if $\upsilon \in X$ with $|||\upsilon||| \leq 1$, by Poincaré inequality and Sobolev embedding (2.2), we have

$$\begin{aligned} J_\lambda(\upsilon) &= \alpha \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx - \frac{\beta}{2}\left(\int_\Omega |\nabla \upsilon|^{\varphi(x)}\, dx\right)^2 + \int_\Omega \frac{1}{\psi(x)} |\upsilon|^{\psi(x)}\, dx - \lambda \int_\Omega H(x,\upsilon)\, dx, \\ &\geq \frac{\alpha}{\varphi_s} \int_\Omega |\nabla \upsilon|^{\varphi(x)} dx - \frac{\beta}{2\varphi_l^2}\left(\int_\Omega |\nabla \upsilon|^{\varphi(x)}\, dx\right)^2 - \varepsilon\lambda \int_\Omega |\upsilon|^{\varphi(x)}\, dx - c_\varepsilon \lambda \int_\Omega |\upsilon|^{r(x)}\, dx \\ &\geq \left(\frac{\alpha}{\varphi_s} - c_2 \lambda \varepsilon\right) \int_\Omega |\nabla \upsilon|^{\varphi(x)}\, dx - \frac{\beta}{2\varphi_l}\left(\int_\Omega |\nabla \upsilon|^{\varphi(x)}\, dx\right)^2 - c_\varepsilon \lambda \left(\|\upsilon\|_{r(x)}^{r_l} + \|\upsilon\|_{r(x)}^{r_s}\right) \\ &\geq \left(\frac{\alpha}{\varphi_s} - c_2 \varepsilon \lambda\right) |||\upsilon|||^{\varphi_s} - \frac{\beta}{2\varphi_l^2} |||\upsilon|||^{2\varphi_l} - c_\varepsilon \lambda \left(c_{emb}^{r_l} |||\upsilon|||^{r_l} + c_{emb}^{r_s} |||\upsilon|||^{r_s}\right) \\ &\geq \left(\frac{\alpha}{\varphi_s} - c_2 \varepsilon \lambda\right) |||\upsilon|||^{\varphi_s} - \frac{\beta}{2\varphi_l^2} |||\upsilon|||^{2\varphi_l} - c_\varepsilon \lambda \left(c_{emb}^{r_l} + c_{emb}^{r_s}\right) |||\upsilon|||^{r_l}, \end{aligned}$$

where embedding constant $c_{emb} > 0$. By selecting $\varepsilon \leq \dfrac{\alpha}{2c_2\varphi_s\lambda}$, we have

$$J_\lambda(\upsilon) \geq \frac{\alpha}{2\varphi_s} |||\upsilon|||^{\varphi_s} - \frac{\beta}{2\varphi_l^2} |||\upsilon|||^{2\varphi_l} - c_\varepsilon \lambda \left(c_{emb}^{r_l} + c_{emb}^{r_s}\right) |||\upsilon|||^{r_l}.$$

By dividing the previous inequality sides on the positive value $|||v|||^{\varphi_s}$ and since, we know that $\varphi_s < 2\varphi_l < r_l$, we have

$$J_\lambda(v) \geq |||v|||^{\varphi_s}\left[\frac{\alpha}{2\varphi_s} - \frac{\beta}{2\varphi_l^2}|||v|||^{2\varphi_l-\varphi_s} - c_\varepsilon\lambda\left(c_{emb}^{r_l} + c_{emb}^{r_s}\right)|||v|||^{r_l-\varphi_s}\right],$$

now, we can choose $|||v||| = R > 0$, such that

$$\frac{\alpha}{2\varphi_s} - \frac{\beta}{2\varphi_l^2}R^{2\varphi_l-\varphi_s} - c_\varepsilon\lambda\left(c_{emb}^{r_l} + c_{emb}^{r_s}\right)R^{r_l-\varphi_s} > 0. \tag{3.2}$$

We can infer that

$$c_\varepsilon\lambda\left(c_{emb}^{r_l} + c_{emb}^{r_s}\right)R^{r_l-\varphi_s} < \frac{\alpha}{2\varphi_s} - \frac{\beta}{2\varphi_l^2}R^{2\varphi_l-\varphi_s} = \frac{\alpha\varphi_l^2 - \beta\varphi_s R^{2\varphi_l-\varphi_s}}{2\varphi_s\varphi_l^2},$$

since $c_\varepsilon$ and $c_{emb} > 0$, we can infer that

$$\lambda < \frac{\alpha\varphi_l^2 - \beta\varphi_s R^{2\varphi_l-\varphi_s}}{2c_\varepsilon\left(c_{emb}^{r_l} + c_{emb}^{r_s}\right)\varphi_s\varphi_l^2 R^{r_l-\varphi_s}} := \lambda_0, \tag{3.3}$$

therefore, by (3.2) and (3.3) we have

$$\frac{\alpha}{2\varphi_s} - \frac{\beta}{2\varphi_l^2}R^{2\varphi_l-\varphi_s} - c_\varepsilon\lambda\left(c_{emb}^{r_l} + c_{emb}^{r_s}\right)R^{r_l-\varphi_s} > 0, \quad \forall\lambda \in (0, \lambda_0).$$

So, there exists $\delta > 0$ so that $J_\lambda(v) \geq \delta > 0$ for all $v \in X$ with $|||v||| = R$. Thus, the proof is complete. □

Now, we prove that $J_\lambda$ ensures the $(C)_c$-condition.

**Lemma 3.3.** *If $(\eta_1) - (\eta_4)$ hold, then for all $\lambda \geq 0$, $J_\lambda$ ensures the $(C)_c$-condition at any level $c \in \left(-\infty, \dfrac{\alpha^2}{2\beta}\right)$.*

*Proof.* At the beginning, we consider the boundary condition for $\{v_n\}$, let $\{v_n\} \subset X$ be a $(C)_c$ sequence related to the $J_\lambda$, such that

$$J_\lambda(v_n) \to c \quad \text{and} \quad \|J'_\lambda(v_n)\|_{X^*}(1 + |||v_n|||) \to 0. \tag{3.4}$$

Using $(\eta_3)$ and (3.4), we can write

$$\begin{aligned}
\varphi_s c + O_n(1) &\geq \varphi_s J_\lambda(v_n) - \langle J'_\lambda(v_n), v_n\rangle \\
&= \alpha\int_\Omega\left(\frac{\varphi_s}{\varphi(x)} - 1\right)|\nabla v_n|^{\varphi(x)}\,dx + \int_\Omega\left(\frac{\psi_s}{\psi(x)} - 1\right)|v_n|^{\psi(x)}\,dx \\
&+ \lambda\int_\Omega \hat{H}(x, v_n)\,dx - \beta\left(\int_\Omega\frac{1}{\varphi(x)}|\nabla v_n|^{\varphi(x)}\,dx\right)\left(\int_\Omega\left[\frac{\varphi_s}{2\varphi(x)} - 1\right]|\nabla v_n|^{\varphi(x)}\,dx\right).
\end{aligned}$$

Since $\alpha \geq \beta$ and $2\varphi_l > \varphi_s$ we have

$$\begin{aligned}\varphi_s c + O_n(1) &\geq \beta\left(\frac{1}{\varphi_s} - \frac{1}{2\varphi_l}\right)\left(\int_\Omega |\nabla \upsilon_n|^{\varphi(x)}\,dx\right)^2 + \int_\Omega \left(\frac{\psi_s}{\psi(x)} - 1\right)|\upsilon_n|^{\psi(x)}\,dx \\ &\quad + \lambda\int_\Omega (\hat{H}(x,0) - c_0)\,dx \\ &\geq \beta\left(\frac{1}{\varphi_s} - \frac{1}{2\varphi_l}\right)|||\upsilon_n|||^{2\varphi_l} + \int_\Omega \left(\frac{\psi_s}{\psi(x)} - 1\right)|\upsilon_n|^{\psi(x)}\,dxa + \lambda\int_\Omega (\hat{H}(x,0) - c_0)\,dx,\end{aligned}$$

therefore

$$\varphi_s c + O_n(1) \geq \beta\left(\frac{1}{\varphi_s} - \frac{1}{2\varphi_l}\right)|||\upsilon_n|||^{2\varphi_l} + \int_\Omega \left(\frac{\psi_s}{\psi(x)} - 1\right)|\upsilon_n|^{\psi(x)}\,dx + \lambda\int_\Omega (\hat{H}(x,0) - c_0)\,dx.$$

Since $\lambda \geq 0$, we have

$$\varphi_s c + O_n(1) \geq \beta\left(\frac{1}{\varphi_s} - \frac{1}{2\varphi_l}\right)|||\upsilon_n|||^{2\varphi_l} - \lambda c_0|\Omega|,$$

thus

$$\beta\left(\frac{1}{\varphi_s} - \frac{1}{2\varphi_l}\right)|||\upsilon_n|||^{2\varphi_l} \leq \varphi_s c + O_n(1) + \lambda c_0|\Omega|.$$

Since $\varphi_s < 2\varphi_l$, $\beta > 0$ and $\lambda \geq 0$, it is clear that $\{\upsilon_n\}$ is bounded in $X$. Then

$$\upsilon_n \rightharpoonup \upsilon \quad \text{weakly in} \quad X. \tag{3.5}$$

By Sobolev embedding (2.2), we have the following compact embedding

$$X \hookrightarrow L^{s(x)}(\Omega) \quad \text{for} \quad 1 \leq s(x) < \varphi^*(x). \tag{3.6}$$

From (3.5) and (3.6), we can infer that

$$\upsilon_n \rightharpoonup \upsilon \quad \text{in} \quad X, \quad \upsilon_n \to \upsilon \quad \text{in} \quad L^{s(x)}(\Omega), \quad \upsilon_n(x) \to \upsilon(x), \quad \text{a.e. in} \quad \Omega. \tag{3.7}$$

Using Hölder inequality and (3.7), we have

$$\begin{aligned}\left|\int_\Omega |\upsilon_n|^{\psi(x)-2}\upsilon_n(\upsilon_n - \upsilon)\,dx\right| &\leq \int_\Omega |\upsilon_n|^{\psi(x)-1}|\upsilon_n - \upsilon|\,dx \\ &\leq \||\upsilon_n|^{\psi(x)-1}\|_{\frac{\psi(x)}{\psi(x)-1}}\|\upsilon_n - \upsilon\|_{\psi(x)} \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

thus

$$\int_\Omega |\upsilon_n|^{\psi(x)-2}\upsilon_n(\upsilon_n - \upsilon)\,dx \to 0, \quad \text{as} \quad n \to \infty. \tag{3.8}$$

By $(\eta_1)$ and $(\eta_2)$, we have that for each $\varepsilon \in (0,1)$, there is $c_\varepsilon > 0$ so that

$$|\eta(x, \upsilon_n)| \leq \varepsilon |\upsilon_n|^{\varphi(x)-1} + c_\varepsilon |\upsilon_n|^{r(x)-1}. \tag{3.9}$$

By Sobolev embedding (2.2), Hölder inequality and (3.9), we have

$$\begin{aligned}\left|\int_\Omega \eta(x, \upsilon_n)(\upsilon_n - \upsilon)\,dx\right| &\leq \int_\Omega \left(\varepsilon |\upsilon_n|^{\varphi(x)-1}|\upsilon_n - \upsilon| + c_\varepsilon |\upsilon_n|^{r(x)-1}|\upsilon_n - \upsilon|\right) dx \\ &\leq \varepsilon \| |\upsilon_n|^{\varphi(x)-1}\|_{\frac{\varphi(x)}{\varphi(x)-1}} \|\upsilon_n - \upsilon\|_{\varphi(x)} + c_\varepsilon \varepsilon \| |\upsilon_n|^{r(x)-1}\|_{\frac{r(x)}{r(x)-1}} \|\upsilon_n - \upsilon\|_{r(x)} \to 0,\end{aligned}$$

as $n \to \infty$. Therefore

$$\int_\Omega \eta(x, \upsilon_n)(\upsilon_n - \upsilon)\,dx \to 0, \quad \text{as} \quad n \to \infty. \tag{3.10}$$

From (3.4), we have $\langle J'_\lambda(\upsilon_n), \upsilon_n - \upsilon\rangle \to 0$, as $n \to \infty$, so, we can infer that

$$\begin{aligned}\left(\alpha - \beta \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon_n|^{\varphi(x)}\,dx\right) \int_\Omega |\nabla \upsilon_n|^{\varphi(x)-2} \nabla \upsilon_n (\nabla \upsilon_n - \nabla \upsilon)\,dx \\ + \int_\Omega |\upsilon_n|^{\psi(x)-2}\upsilon_n(\upsilon_n - \upsilon)\,dx - \lambda \int_\Omega \eta(x, \upsilon_n)(\upsilon_n - \upsilon)\,dx \to 0.\end{aligned} \tag{3.11}$$

From (3.8), (3.10), (3.11), we can write

$$\left(\alpha - \beta \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon_n|^{\varphi(x)}\,dx\right) \int_\Omega |\nabla \upsilon_n|^{\varphi(x)-2} \nabla \upsilon_n (\nabla \upsilon_n - \nabla \upsilon)\,dx \to 0, \quad \text{as} \quad n \to \infty. \tag{3.12}$$

Since $\{\upsilon_n\}$ is bounded in $X$, therefore, it is necessary for the following positive sequence to converge to a non-negative value such as $\upsilon_p$, which means,

$$\int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon_n|^{\varphi(x)}\,dx \to \upsilon_p \geq 0, \quad \text{as} \quad n \to \infty.$$

Similar to the proof of Lemma 3.1 in [23], we have the sequence $\left\{\alpha - \beta \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon_n|^{\varphi(x)}\,dx\right\}$ is bounded, when $n$ is large enough. So, it follows from (3.12) that

$$\int_\Omega |\nabla \upsilon_n|^{\varphi(x)-2} \nabla \upsilon_n (\nabla \upsilon_n - \nabla \upsilon)\,dx \to 0,$$

as $n \to \infty$. So, by the $(S_+)$ property (see Lemma 2.5), we get $|||\upsilon_n||| \to |||\upsilon|||$ (strongly) in $X$, that means $J_\lambda$ ensures the $(C)_c$-condition. Moreover, considering the proof of Lemma 3.1, Lemma 3.2 and Remark 3.1 in [23], we deduce that the $(C)_c$-condition is satisfied for $c < \dfrac{\alpha^2}{2\beta}$. □

### 3.1 Proof of Theorem 1.1

*Proof.* It is clear that $J_\lambda(0) = 0$, by Lemma 3.3, $J_\lambda$ ensures the $(C)_c$-condition where $c \in \left(-\infty, \dfrac{\alpha^2}{2\beta}\right)$. Considering Lemma 3.2, we prove that $J_\lambda$ has the geometry of the Mountain pass theorem, thus, all the assumptions of Mountain pass theorem are satisfied, therefore, for each $\lambda < \lambda_0$, our problem has at least a nontrivial weak solution in $X$. □

## 4 Proof of Theorem 1.2

In this section, we will prove that problem (1.1) has many pairs of solutions by using the following Symmetric Mountain pass theorem.

**Theorem 4.1** ([8])**.** *Let $X$ be a real Banach space, and $J_\lambda \in C^1(X, \mathbb{R})$ that ensures the $(C)_c$-condition and $J_\lambda(0) = 0$ and $J_\lambda$ be an even functional, such as*

(A) *there exist two constants $a, R > 0$, so that $J_\lambda(\upsilon) \geq a$ for each $u \in X$ with $|||\upsilon||| = R$,*

(B) *for each finite dimensional subspace $E \subset X$, there exists $R_E > 0$ so that $J_\lambda(\upsilon) \leq 0$ on $E \backslash B_R$.*

*Then $J_\lambda$ has a sequence of critical points $\{\upsilon_n\}$ such that $J_\lambda(\upsilon_n) \to +\infty$.*

It is clear that for the even functional $J_\lambda$, we have $J_\lambda(0) = 0$ and by Lemma 3.3, $J_\lambda$ ensures the $(C)_c$-condition where $c \in \left(-\infty, \dfrac{\alpha^2}{2\beta}\right)$. Therefore, it suffices to prove that the two conditions (A) and (B) of the Theorem 4.1 are true for the functional $J_\lambda$. On the other hand by the proof of Lemma 3.2 (a), where

$$a_0 = \frac{\alpha\varphi_l^2 - \beta\varphi_s R^{2\varphi_l - \varphi_s}}{2c_\varepsilon\left(c_{emb}^{r_l} + c_{emb}^{r_s}\right)\varphi_s\varphi_l^2 R^{r_l - \varphi_s}}$$

and $a = a_0 R^{\varphi_s}$ for each $\lambda \in (0, a_0)$, there is $a > 0$ so that for each $\upsilon \in X$ with $|||\upsilon||| = R$, we have $J_\lambda(\upsilon) \geq a > 0$. Thus, it suffices to consider only the condition (B).

We use the indirect proof method, thus assume that $\{\upsilon_n\} \subset E$ such that if $|||\upsilon_n||| \to +\infty$ as $n \to +\infty$, then there is $M \in \mathbb{R}$ so that it is a fixed constant, then

$$J_\lambda(\upsilon_n) \geq M, \quad \forall n \in N. \tag{4.1}$$

Now, for any $\upsilon_n \in E \subseteq X$, put $V_n := \dfrac{\upsilon_n}{|||\upsilon_n|||}$. It is clear that $|||V_n||| = 1$. On the other hand, since $\dim E < +\infty$, we have

$$\exists V \in E\backslash\{0\}; \quad |||V_n - V||| \to 0.$$

We can infer that

$$V_n(x) \to V(x) \quad \text{a.e.} \quad x \in \Omega, \quad \text{as} \quad n \to \infty,$$

since $V(x) \neq 0 \rightarrow |\upsilon_n(x)| \rightarrow +\infty$, as $n \rightarrow +\infty$, (by (4.1)).

By $(\eta_3)$, we can infer that

$$\lim_{n\to+\infty} \frac{H(x,\upsilon_n(x))}{|||\upsilon_n|||^{\varphi_s}} = \lim_{n\to+\infty} \frac{H(x,\upsilon_n(x))}{|\upsilon_n(x)|^{\varphi_s}} |V_n(x)|^{\varphi_s} = +\infty,$$

for all $x \in \Omega_0 := \{x \in \Omega : V(x) \neq 0\}$ and by $(\eta_4)$, there is $s_0$, such that

$$\frac{H(x,s)}{|s|^{\varphi_s}} > 1, \quad \forall x \in \Omega \quad \text{and} \quad |s| > s_0. \tag{4.2}$$

Now by $(\eta_1)$, we can write

$$\exists C_2 > 0; \quad |H(x,s)| \leq C_2, \quad \forall (x,s) \in \Omega \times [-s_0, s_0]. \tag{4.3}$$

Using (4.2) and (4.3), we conclude that

$$\exists C_4 \in \mathbb{R}, \quad H(x,s) \geq C_4, \quad \forall (x,s) \in \Omega \times \mathbb{R}. \tag{4.4}$$

Thus

$$\frac{H(x,\upsilon_n) - C_4}{|||\upsilon_n|||^{\varphi_s}} \geq 0, \quad \forall x \in \Omega, \quad \forall n \in \mathbb{N}.$$

Then, we have

$$\frac{H(x,\upsilon_n)}{|\upsilon_n(x)|^{\varphi_s}} |V_n(x)|^{\varphi_s} - \frac{C_4}{|||\upsilon_n|||^{\varphi_s}} \geq 0, \quad \forall x \in \Omega, \quad \forall n \in \mathbb{N}. \tag{4.5}$$

Thus, by Poincaré inequality, (4.1) and (4.5), we can infer that

$$\begin{aligned} 0 &\leq \lim_{n\to+\infty} \frac{J_\lambda(\upsilon_n)}{|||\upsilon_n|||^{\varphi_s}} \\ &\leq \lim_{n\to+\infty} \left[ \frac{\alpha \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon_n|^{\varphi(x)}\,dx + \int_\Omega \frac{1}{\psi(x)} |\upsilon_n|^{\psi(x)}\,dx}{|||\upsilon_n|||^{\varphi_s}} - \lambda \int_\Omega \frac{H(x,\upsilon_n)}{|||\upsilon_n|||^{\varphi_s}}\,dx \right]. \end{aligned}$$

Since $\psi_s > \varphi_s$, and $\lambda > 0$, we have

$$\begin{aligned} 0 &\leq \lim_{n\to+\infty} \left[ \frac{\alpha \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon_n|^{\varphi(x)}\,dx}{|||\upsilon_n|||^{\varphi_s}} + \frac{\int_\Omega \frac{1}{\psi(x)} |\upsilon_n|^{\psi(x)}\,dx}{|||\upsilon_n|||^{\psi_s}} - \lambda \int_\Omega \frac{H(x,\upsilon_n)}{|||\upsilon_n|||^{\varphi_s}}\,dx \right] \\ &\leq \frac{\alpha}{\varphi_l} + \frac{C_5}{\psi_s} - \lambda \lim_{n\to+\infty} \int_\Omega \frac{H(x,\upsilon_n) - C_4}{|||\upsilon_n|||^{\varphi_s}}\,dx \\ &\leq \frac{\alpha}{\varphi_l} + \frac{C_5}{\psi_s} - \lambda \liminf_{n\to+\infty} \int_{\Omega_0} \frac{H(x,\upsilon_n) - C_4}{|||\upsilon_n|||^{\varphi_s}}\,dx \\ &\leq \frac{\alpha}{\varphi_l} + \frac{C_5}{\psi_s} - \lambda \liminf_{n\to+\infty} \int_{\Omega_0} \frac{H(x,\upsilon_n)}{|\upsilon_n(x)|^{\varphi_s}} |V_n(x)|^{\varphi_s}\,dx \rightarrow -\infty, \end{aligned}$$

which is a contradiction. Then, the proof of (B) in the Theorem 4.1 is complete.

## 4.1 Proof of Theorem 1.2

*Proof.* Now, by Theorem 4.1, we can deduce that $J_\lambda$ has a sequence of critical points $\{\upsilon_n\}$ such that $J_\lambda(\upsilon_n) \to +\infty$, thus, we prove that our problem has infinitely many weak solutions and the Theorem 1.2 is proven. □

## Acknowledgement

The authors gratefully ackowledge the anonymous referees for their useful comments that lead to an improved version of the paper.

# References

[1] E. Acerbi and G. Mingione, "Gradient estimates for the $p(x)$-Laplacean system," *J. Reine Angew. Math.*, vol. 584, pp. 117–148, 2005, doi: 10.1515/crll.2005.2005.584.117.

[2] G. A. Afrouzi and M. Mirzapour, "Existence and multiplicity of solutions for nonlocal $\overrightarrow{p}(x)$-Laplacian problem," *Taiwanese J. Math.*, vol. 18, no. 1, pp. 219–236, 2014, doi: 10.11650/tjm.18.2014.2596.

[3] G. A. Afrouzi, M. Mirzapour, and N. T. Chung, "Existence and multiplicity of solutions for a $p(x)$-Kirchhoff type equation," *Rend. Semin. Mat. Univ. Padova*, vol. 136, pp. 95–109, 2016, doi: 10.4171/RSMUP/136-8.

[4] G. A. Afrouzi, M. Mirzapour, and V. D. Rădulescu, "Qualitative analysis of solutions for a class of anisotropic elliptic equations with variable exponent," *Proc. Edinb. Math. Soc. (2)*, vol. 59, no. 3, pp. 541–557, 2016, doi: 10.1017/S0013091515000346.

[5] G. A. Afrouzi, M. Mirzapour, and V. D. Rădulescu, "Variational analysis of anisotropic Schrödinger equations without Ambrosetti-Rabinowitz-type condition," *Z. Angew. Math. Phys.*, vol. 69, no. 1, 2018, Art. ID 9, doi: 10.1007/s00033-017-0900-y.

[6] G. A. Afrouzi, N. T. Chung, and Z. Naghizadeh, "Multiple solutions for $p(x)$-Kirchhoff type problems with Robin boundary conditions," *Electron. J. Differential Equations*, 2022, Art. ID 24, doi: 10.58997/ejde.2022.24.

[7] M. Alimohammady and F. Fattahi, "Existence of solutions to hemivariational inequalities involving the $p(x)$-biharmonic operator," *Electron. J. Differential Equations*, 2015, Art. ID 79.

[8] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *J. Functional Analysis*, vol. 14, pp. 349–381, 1973, doi: 10.1016/0022-1236(73)90051-7.

[9] M. Avci, R. A. Ayazoglu, and B. Cekic, "Solutions of an anisotropic nonlocal problem involving variable exponent," *Adv. Nonlinear Anal.*, vol. 2, no. 3, pp. 325–338, 2013, doi: 10.1515/anona-2013-0010.

[10] R. Ayazoglu, S. Akbulut, and E. Akkoyunlu, "Existence of multiple solutions of Schrödinger-Kirchhoff-type equations involving the $p(.)$-Laplacian in $\mathbb{R}^N$," *Math. Methods Appl. Sci.*, vol. 43, no. 17, pp. 9598–9614, 2020, doi: 10.1002/mma.6626.

[11] R. Ayazoğlu, S. Akbulut, and E. Akkoyunlu, "Existence and multiplicity of solutions for $p(.)$-Kirchhoff-type equations," *Turkish J. Math.*, vol. 46, no. 4, pp. 1342–1359, 2022, doi: 10.55730/1300-0098.3164.

[12] A. Bensedik, "On existence results for an anisotropic elliptic equation of Kirchhoff-type by a monotonicity method," *Funkcial. Ekvac.*, vol. 57, no. 3, pp. 489–502, 2014, doi: 10.1619/fesi.57.489.

[13] J. Chabrowski and Y. Fu, "Existence of solutions for $p(x)$-Laplacian problems on a bounded domain," *J. Math. Anal. Appl.*, vol. 306, no. 2, pp. 604–618, 2005, doi: 10.1016/j.jmaa.2004.10.028.

[14] B. Cheng, "A new result on multiplicity of nontrivial solutions for the nonhomogenous Schrödinger-Kirchhoff type problem in $R^N$," *Mediterr. J. Math.*, vol. 13, no. 3, pp. 1099–1116, 2016, doi: 10.1007/s00009-015-0527-1.

[15] N. T. Chung, "Multiplicity results for a class of $p(x)$-Kirchhoff type equations with combined nonlinearities," *Electron. J. Qual. Theory Differ. Equ.*, 2012, Art. ID 42.

[16] D. G. Costa and O. H. Miyagaki, "Nontrivial solutions for perturbations of the $p$-Laplacian on unbounded domains," *J. Math. Anal. Appl.*, vol. 193, no. 3, pp. 737–755, 1995, doi: 10.1006/jmaa.1995.1264.

[17] G. Dai and R. Hao, "Existence of solutions for a $p(x)$-Kirchhoff-type equation," *J. Math. Anal. Appl.*, vol. 359, no. 1, pp. 275–284, 2009, doi: 10.1016/j.jmaa.2009.05.031.

[18] G. Dai and R. Ma, "Solutions for a $p(x)$-Kirchhoff type equation with Neumann boundary data," *Nonlinear Anal. Real World Appl.*, vol. 12, no. 5, pp. 2666–2680, 2011, doi: 10.1016/j.nonrwa.2011.03.013.

[19] M. Dreher, "The Kirchhoff equation for the $p$-Laplacian," *Rend. Semin. Mat. Univ. Politec. Torino*, vol. 64, no. 2, pp. 217–238, 2006.

[20] X.-L. Fan and Q.-H. Zhang, "Existence of solutions for $p(x)$-Laplacian Dirichlet problem," *Nonlinear Anal.*, vol. 52, no. 8, pp. 1843–1852, 2003, doi: 10.1016/S0362-546X(02)00150-5.

[21] X. Fan, "On nonlocal $p(x)$-Laplacian Dirichlet problems," *Nonlinear Anal.*, vol. 72, no. 7-8, pp. 3314–3323, 2010, doi: 10.1016/j.na.2009.12.012.

[22] W. Guo, J. Yang, and J. Zhang, "Existence results of nontrivial solutions for a new $p(x)$-biharmonic problem with weight function," *AIMS Math.*, vol. 7, no. 5, pp. 8491–8509, 2022, doi: 10.3934/math.2022473.

[23] M. K. Hamdani, A. Harrabi, F. Mtiri, and D. D. Repovš, "Existence and multiplicity results for a new $p(x)$-Kirchhoff problem," *Nonlinear Anal.*, vol. 190, 2020, Art. ID 1111598, doi: 10.1016/j.na.2019.111598.

[24] S. Heidarkhani, "Infinitely many solutions for systems of $n$ two-point Kirchhoff-type boundary value problems," *Ann. Polon. Math.*, vol. 107, no. 2, pp. 133–152, 2013, doi: 10.4064/ap107-2-3.

[25] S. Heidarkhani and J. Henderson, "Infinitely many solutions for nonlocal elliptic systems of $(p_1, \dots, p_n)$-Kirchhoff type," *Electron. J. Differential Equations*, 2012, Art. ID 69.

[26] S. Heidarkhani, S. Khademloo, and A. Solimaninia, "Multiple solutions for a perturbed fourth-order Kirchhoff type elliptic problem," *Port. Math.*, vol. 71, no. 1, pp. 39–61, 2014, doi: 10.4171/PM/1940.

[27] N. Kikuchi and J. T. Oden, *Contact problems in elasticity: a study of variational inequalities and finite element methods*, ser. SIAM Studies in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1988, vol. 8, doi: 10.1137/1.9781611970845.

[28] G. Kirchhoff, *Vorlesungen über Mechanik*. B.G. Teubner, Leipzig, Germany, 1883.

[29] S. Liang, H. Pu, and V. D. Rădulescu, "High perturbations of critical fractional Kirchhoff equations with logarithmic nonlinearity," *Appl. Math. Lett.*, vol. 116, 2021, Art. ID 107027, doi: 10.1016/j.aml.2021.107027.

[30] J.-L. Lions, "On some questions in boundary value problems of mathematical physics," in *Contemporary developments in continuum mechanics and partial differential equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977)*, ser. North-Holland Math. Stud. North-Holland, Amsterdam-New York, 1978, vol. 30, pp. 284–346.

[31] M. Mirzapour, "Existence and multiplicity of solutions for Neumann boundary value problems involving nonlocal $p(x)$-Laplacian equations," *International Journal of Nonlinear Analysis and Applications*, vol. 14, no. 8, pp. 237–247, 2023, doi: 10.22075/ijnaa.2022.7212.

[32] J. Musielak, *Orlicz spaces and modular spaces*, ser. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1983, vol. 1034, doi: 10.1007/BFb0072210.

[33] P. Pucci, M. Xiang, and B. Zhang, "Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional $p$-Laplacian in $\mathbb{R}^N$," *Calc. Var. Partial Differential Equations*, vol. 54, no. 3, pp. 2785–2806, 2015, doi: 10.1007/s00526-015-0883-5.

[34] X. H. Tang, "Infinitely many solutions for semilinear Schrödinger equations with sign-changing potential and nonlinearity," *J. Math. Anal. Appl.*, vol. 401, no. 1, pp. 407–415, 2013, doi: 10.1016/j.jmaa.2012.12.035.

[35] N. Thanh Chung, "Multiple solutions for a $p(x)$-Kirchhoff-type equation with sign-changing nonlinearities," *Complex Var. Elliptic Equ.*, vol. 58, no. 12, pp. 1637–1646, 2013, doi: 10.1080/17476933.2012.701289.

[36] M. Xiang, B. Zhang, and M. Ferrara, "Existence of solutions for Kirchhoff type problem involving the non-local fractional $p$-Laplacian," *J. Math. Anal. Appl.*, vol. 424, no. 2, pp. 1021–1041, 2015, doi: 10.1016/j.jmaa.2014.11.055.

[37] Q.-L. Xie, X.-P. Wu, and C.-L. Tang, "Existence of solutions for Kirchhoff type equations," *Electron. J. Differential Equations*, 2015, Art. ID 47.

[38] A. Zang, "$p(x)$-Laplacian equations satisfying Cerami condition," *J. Math. Anal. Appl.*, vol. 337, no. 1, pp. 547–555, 2008, doi: 10.1016/j.jmaa.2007.04.007.