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A note on Buell's theorem on length four Büchi sequences

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ABSTRACT

Büchi sequences are sequences whose second difference of squares is the sequence $(2,\dots,2)$, like for instance $(6,23,32,39)$ — so they can be seen as a generalization of arithmetic progressions. No (non-trivial) length 5 Büchi sequence is known to exist. Length four Büchi sequences were parameterized by D. A. Buell in 1987. We revisit his theorem, fixing the statement (about 26% of the Büchi sequences from R. G. E. Pinch's 1993 table were missed), and giving a much simpler proof.

RESUMEN

Las secuencias de Büchi son secuencias para las cuales la segunda diferencia de sus cuadrados es la sucesión $(2,\dots,2)$, como por ejemplo $(6,23,32,39)$ — luego pueden ser vistas como una generalización de las progresiones aritméticas. No se sabe de la existencia de ninguna secuencia de Büchi (no-trivial) de largo 5. Las secuencias de Büchi de largo 4 fueron parametrizadas por D. A. Buell en 1987. Revisitamos este teorema, corrigiendo el enunciado (faltan alrededor del 26% de las secuencias de Büchi de la tabla de R. G. E. Pinch de 1993), y dando una demostración bastante más simple.

Keywords and Phrases: Representation of systems of quadratic forms, Büchi's n-squares problem, second difference of squares.

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1 Introduction and result

Recall that the first (forward) difference of a sequence $(y_n)_n$ is the sequence $(y_{n+1} - y_n)_n$, so the second difference is $((y_{n+2} - y_{n+1}) - (y_{n+1} - y_n))_n = (y_{n+2} - 2y_{n+1} + y_n)_n$. A Büchi sequence is a sequence $(x_1, \dots, x_M)$ whose second difference of its sequence of squares is the constant sequence $(\dots, 2, \dots)$, namely, it is a sequence which satisfy the system of Büchi equations $x_{n+2}^2 - 2x_{n+1}^2 + x_n^2 = 2$, for $n = 1, \dots, M-2$. We call *trivial Büchi sequence* any such sequence such that $x_{n+1}^2 = (x_n \pm 1)^2$ for every $n = 1, \dots, M-1$. Büchi's problem asks whether there exists an M such that every Büchi sequence of integers of length M is trivial. It is not known whether any such M exists, and actually no non-trivial length 5 Büchi sequence of integers is known to exist. However, Büchi's problem has a positive answer, namely, an M can be proved to exist, under some classical conjectures in Number Theory — see [11] and [6]. For a general survey on Büchi's problem and variations, see [5] and the references therein.

Length 3 Büchi sequences of integers were characterized by D. Hensley [2,3] through a parametrization in two variables coming from the line and circle method, and later by P. Sáez and the second author [8] using matrices. In [1], D. A. Buell builds on Hensley's parametrization to find a parametrized family, say by a pair (k, ℓ) of integers, of quadratic equations whose solutions correspond to length 4 Büchi sequences of integers (BS4 in the sequel) — see Equation (1.1) below. As J. Lipman pointed out in [4, page 4], it is not clear how to characterize the pairs (k, ℓ) for which the equation is solvable.

See [7], [10] and [9] for other approaches to the problem of understanding the BS4.

In this short note, we fix two mistakes in the statement of the original theorem — see the comments before the proof — and give a much simpler and more transparent proof.

Theorem 1.1 (D. A. Buell, 1987, *revisited*)**.** *A sequence $\sigma = (x_1, \dots, x_4)$ is a Büchi sequence of integers if and only if there exist coprime integers k and ℓ of opposite parity, an integer x, and a rational number y such that $3y \in \mathbb{Z}$, which satisfy*

$$\begin{cases} x_1 &= x(-2\ell + 3k) + y(-3\ell + 6k) \\ x_2 &= x(-\ell + 2k) + y(-2\ell + 3k) \\ x_3 &= xk + y\ell \\ x_4 &= x\ell + 3yk \end{cases}$$

and

$$(\ell - k)^2 x^2 + (2\ell^2 - 6k\ell + 6k^2)xy + (\ell - 3k)^2 y^2 = 1. \tag{1.1}$$

The proof below allows to find easily some of the possible parameters k and ℓ from a given BS4

— this was our original motivation, as this is not clear how to do it from [1]. This is also how we realized that the possibility of having a 3 in the denominator of the y cannot be dropped, as can be seen with the Büchi sequence $(16, 87, 122, 149)$, for which $yk = -\frac{40}{3}$. Indeed, about 26% of the sequences with some entry at most 1000 need a 3 in the denominator (see [7] for the list). This phenomenon was overlooked in Buell's statement, though one could detect it while going through his intricate proof: his quotient $\frac{a+t}{\ell-3k}$, line 4 before the Theorem, can actually have a 3 in the denominator. The other issue in Buell's original statement has to do with trivial sequences, which cannot be put aside in the statement, as our proof shows.

Proof. If direction. When computing the second difference of squares of the x_i, one obtains the left hand-side of Equation (1.1) multiplied by two. So σ is a Büchi sequence. If y is an integer, there is nothing else to prove. Otherwise, replacing y by $\frac{y'}{3}$ in Equation (1.1), then multiplying by 9 and taking modulo 3, we see that 3 divides ℓ, so the x_i are indeed integers.

Only if direction. Assume that $(x_1, \ldots, x_4)$ is a Büchi sequence of integers. The idea is to pretend that $\omega_1 := xk$ is a variable, as well as $\omega_2 := x\ell$, $\omega_3 := yk$ and $\omega_4 := y\ell$, so that the system of the statement can be seen as a linear system:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 6 & -3 \\ 2 & -1 & 3 & -2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} xk \\ x\ell \\ yk \\ y\ell \end{pmatrix}. \tag{1.2}$$

By inverting the system we get:

$$\begin{cases} 2\omega_1 &= -x_1 + 2x_2 + x_3 \\ 2\omega_2 &= -2x_1 + 3x_2 + x_4 \\ 6\omega_3 &= 2x_1 - 3x_2 + x_4 \\ 2\omega_4 &= x_1 - 2x_2 + x_3. \end{cases} \tag{1.3}$$

Observe that, since x_i and x_{i+1} have opposite parity for each i (which can be easily seen from the Büchi equations), ω_1, ω_2, $3\omega_3$ and ω_4 are integers.

If $\omega_1 = \omega_2 = 0$, then one can choose $x = 0$, and $y = 1$, $\ell = x_3$, $k = \frac{x_4}{3}$ if 3 divides x_4, and $y = \frac{1}{3}$, $\ell = 3x_3$ and $k = x_4$ if not. From (1.3), we get $x_2 + 2x_3 - x_4 = 0$, which, together with the Büchi equation $x_4^2 = 2x_3^2 - x_2^2 + 2$ gives $(x_2 + x_3)^2 = 1$, hence the sequence is trivial. Similarly, if $\omega_3 = \omega_4 = 0$, then one can choose $y = 0$, $x = 1$, $k = x_3$ and $\ell = x_4$, and again the sequence is trivial. Since in both cases the sequence is trivial, we have $x_4 = \pm x_3 \pm 1$, so in particular, k and ℓ are coprime and of opposite parity. One readily checks that (1.2) and (1.1) are satisfied in both cases.

We assume now that $(\omega_1, \omega_2) \neq (0,0)$ and $(\omega_3, \omega_4) \neq (0,0)$. A direct computation gives

$$12(\omega_1\omega_4 - \omega_2\omega_3) = x_1^2 - 3x_2^2 + 3x_3^2 - x_4^2 = (x_1^2 - 2x_2^2 + x_3^2) - (x_2^2 - 2x_3^2 + x_4^2) = 0,$$

so we have

$$\omega_1\omega_4 = \omega_2\omega_3. \tag{1.4}$$

Hence $\omega_1 = 0$ if and only if $\omega_3 = 0$, in which case we choose $k = 0$, $\ell = 1$, $x = \omega_2$ and $y = \omega_4$, so that $xk = \omega_1$, $x\ell = \omega_2$, $yk = \omega_3$ and $y\ell = \omega_4$. Similarly, $\omega_2 = 0$ if and only if $\omega_4 = 0$, in which case we choose $\ell = 0$, $k = 1$, $x = \omega_1$ and $y = \omega_3$, so that $xk = \omega_1$, $x\ell = \omega_2$, $yk = \omega_3$ and $y\ell = \omega_4$.

Assume that $\omega_1\omega_2\omega_3\omega_4 \neq 0$. Let ε be the sign of $\omega_1\omega_3$. Choose $x = \varepsilon \gcd(\omega_1, \omega_2)$, $k = \frac{\omega_1}{x}$, $\ell = \frac{\omega_2}{x}$ (so k and ℓ are coprime integers), and $y = \frac{y'}{3}$, where $y' = \gcd(3\omega_3, 3\omega_4)$. Note that if both ω_1 and ω_3 are positive, then we obtain

$$3\omega_3 \gcd(\omega_1, \omega_2) = \gcd(3\omega_1\omega_3, 3\omega_2\omega_3) = \gcd(3\omega_1\omega_3, 3\omega_1\omega_4) = \omega_1 \gcd(3\omega_3, 3\omega_4).$$

In general, we have $3\omega_3 \gcd(\omega_1, \omega_2) = \varepsilon\omega_1 \gcd(3\omega_3, 3\omega_4)$, hence

$$3\omega_3 = \frac{\varepsilon\omega_1}{\gcd(\omega_1, \omega_2)} \times \gcd(3\omega_3, 3\omega_4) = ky'$$

hence $\omega_3 = yk$. Since $\omega_1 \neq 0$, we have $\omega_4 = \frac{\omega_2\omega_3}{\omega_1} = \frac{x\ell \cdot yk}{xk} = y\ell$. By inverting the system (1.3), we see that the system (1.2) is satisfied.

Equation (1.1) comes from replacing the x_i in $x_4^2 - 2x_3^2 + x_2^2 = 2$ (for instance) by their expression in terms of x, y, k and ℓ. Equation (1.1) implies immediately that k and ℓ cannot have the same parity. □

While working on this note, we realized that the solutions of (1.1) with $k = \ell + 1$, described in Section 5 of [1], are precisely the BS4 that were found by the second author in [10] with a different method.

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