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Minkowski type inequalities for a generalized fractional integral

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ABSTRACT

In this paper we introduce a new generalized fractional integral unifying most of previous existing fractional integrals. Then, we prove some essential properties of this new operator under some classical assumptions. As application, we use this novel fractional integral to establish a several inequalities of Minkowski type. Our results recover a large number of a well known inequalities in the literature.

RESUMEN

En este artículo introducimos una nueva integral fraccionaria generalizada, que unifica la mayoría de las integrales fraccionarias existentes. Luego demostramos algunas propiedades esenciales de este nuevo operador bajo algunas suposiciones clásicas. Como aplicación, usamos esta nueva integral fraccionaria para establecer varias desigualdades de tipo Minkowski. Nuestros resultados recuperan un amplio número de desigualdades bien conocidas en la literatura.

Keywords and Phrases: Fractional calculus, fractional integral, Riemann-Liouville integral, reverse Minkowski inequality.

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1 Introduction

Fractional calculus has been the subject of a lot of works during the last years. Fractional models has been used to diverse problems in various domains of science, see [43]. In fact, it is mainly used in modeling different phenomena, as mechanics [6], economy [45], human body modeling [14], visco-elasticity [18, 30], biology [28], circuits [31], material sciences [41], porous-medium equations [36] and many other domains. In order to modeling such problems, different integral operators or differential operators were defined. Nevertheless, some of fractional operators defined with a special "kernel" are used only in some cases. In [27], the authors defined a fractional integral according to another function ψ as a general integral. Choosing a particular function ψ, we obtain a pre-existing non-integer integral. This allows us to select the most adapted integral for proving the result under examination.

In [47], Sousa-Oliveira defined a new fractional-derivative according to another function; the "ψ-Hilfer fractional derivative". They proved many interesting properties and they presented also a large number of integrals and derivatives as a special cases of the ψ-Hilfer derivative and the integral according to another function.

In [25], Katugampola defined the following new fractional integral

$$({}^{\rho}I^{\alpha,\beta}_{a+;\eta,k}f)(x) = \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_a^x \frac{u^{\rho(1+\eta)-1}}{(x^\rho - u^\rho)^{1-\alpha}}f(u)du.$$

He proved that the above integral unifies six pre-existing fractional integrals.

With the numerous propositions of fractional derivatives and integrals, it was very important to propose a new definition of fractional integral that unifies most of the pre-existing definitions. The new generalized ψ-fractional integral proposed in this paper, will be the first step in order to obtain a single general model, which can be used to different problems and to prove different results only for this general model, rather than proving similar results each time in each different model. In the first part of this paper, our purpose is to define this new fractional integral. Then, we prove some important properties to justify the originality of this new generalization. Among other, we show that the new operator is well defined, bounded and satisfies the semigroup property.

As application of the numerous fractional integrals proposed in the last years, a large number of works are interested to several important inequalities for different definitions of fractional integrals. See for example [2, 8, 19, 48, 49] for the Ostrowski type inequalities, [7, 10, 16, 20, 40] for the Grüss type inequalities, [4, 11, 12, 17, 26, 35] for the Hermite-Hadamard type inequalities, [21, 23, 34, 42, 46] for the Čebyšev type inequalities, [5, 13, 15, 17, 32, 33, 37–39, 44] for the Minkowski type inequalities and many others, (see [3, 29]). Such types of inequalities are very important in different areas of science, (see [32, 38]).

Motivated by the above large literature and as application of the new generalized ψ-fractional integral defined in the first part, our second aim in this work is to generalize the Minkowski type inequalities using the new generalized ψ- fractional integral. Our results recover the Minkowski type inequalities proved in [5, 13, 15, 17, 33, 37–39] and [44]. Then, we prove different other inequalities related to the Minkowski's inequalities.

The remainder of the paper is organized as follows. In the next section we present the definition of the new generalized ψ-fractional integral and some examples. In section three we give some principal properties of this new operators. In section four, we prove the main results related to Minkowski inequality and in the last section, we prove other inequalities related to the new fractional integral.

2 Definition and examples

Definition 2.1 ([24]). *Let $f \in L_1(a,b)$ and ψ be a positive function such that its derivative is continuous and satisfying $\psi'(x) > 0, \forall x \in (a,b)$. For $1 \leq p < \infty$, we denote*

$$X^p_\psi(a,b) := \{f : (a,b) \to \mathbb{R},\ \textit{Lebesgue-measurable s.t. } \|f\|_{X^p_\psi} < \infty\},$$

where

$$\|f\|^p_{X^p_\psi} = \int_a^b |f(s)|^p \psi'(s) ds.$$

For $p = \infty$,

$$\|f\|_{X^\infty_\psi} = \operatorname*{ess\,sup}_{s\in(a,b)} |\psi'(s) f(s)|.$$

When $\psi(s) = s$, the space $X^p_\psi(a,b)$, $(1 \leq p < \infty)$, is identical to $L_p(a,b)$.

Definition 2.2. *For $1 \leq p \leq \infty$, let $f \in X^p_\psi(a,b)$ and ψ as defined in the previous Definition 2.1. For $\alpha > 0$, $\beta, \gamma, \eta, k, \rho, \in \mathbb{R}$, we define the following new generalized ψ-fractional integral, (left side and right side), by*

$$I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x) = \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta} \int_a^x [\psi(u)]^\eta \psi'(u) \exp\big(-\gamma\psi(u)\big)\big(\psi(x) - \psi(u)\big)^{\alpha-1} f(u) du \tag{2.1}$$

$$I^{\alpha,\beta;\psi}_{b-;\eta,k,\gamma,\rho} f(x) = \frac{[\psi(x)]^\eta \exp(-\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta} \int_x^b [\psi(u)]^k \psi'(u) \exp\big(\gamma\psi(u)\big)\big(\psi(u) - \psi(x)\big)^{\alpha-1} f(u) du \tag{2.2}$$

Remark 2.3. *Most of the pre-existing fractional integrals are a particular cases of integrals* (2.1) *and* (2.2). *For example, if $\psi(x) = x$, $\alpha > 0$, $\gamma = 0$, $k = 0$, $\eta = 0$, $\rho > 0$, $\beta = 0$, then we obtain the integral of Riemann Liouville (left sided). For $a = -\infty$, we obtain the integral of Liouville ${}^L I^\alpha_+ f(x)$. If $a = 0$ then* (2.1) *is the analogue of the integral of Riemann ${}^R I^\alpha_+ f(x)$. For a general case of function ψ,* (2.1) *is reduced to the integral of Riemann Liouville according to a function ψ,*

$$ {}^{RL}I_{a+}^{\alpha;\psi}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(s)(\psi(x)-\psi(s))^{\alpha-1} f(s)ds. $$

If $\psi(x) = \ln x$, $\alpha > 0$, $\gamma = 0$, $k = 0$, $\eta = 0$, $\rho > 0$, $\beta = 0$ *then* (2.1) *is reduced to the integral of Hadamard* ${}^{H}I_{a+}^{\alpha}f(x)$ *and for* $\gamma \in \mathbb{R}$ *and* $a = 0$, *we get the integral of Hadamard type (called also Butzer et al. integral),*

$$ {}^{H}I_{a+;\gamma}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(\frac{u}{x}\right)^{-\gamma} \left(\ln\left(\frac{x}{u}\right)\right)^{\alpha-1} f(u)\frac{du}{u}. $$

If $\psi(x) = x^\rho$, $\alpha > 0$, $\gamma = 0$, $k = -\alpha - \eta$, $\rho > 0$, $\beta = 0$, *then we get the fractional-integral of "Erdélyi-Kober",*

$$ {}^{EK}I_{a+;\eta,\rho}^{\alpha}f(x) = \frac{\rho x^{-\rho(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x f(s)s^{\rho(1+\eta)-1}\big(x^\rho - s^\rho\big)^{\alpha-1}ds. $$

For $a = 0$, *we get the fractional-integral of "Erdélyi"* ${}^{E}I_{a+;\eta,\rho}^{\alpha}f(x)$ *and for* $\rho = 1$, $a = 0$, *we get the fractional-integral of "Kober",* ${}^{K}I_{a+;\eta,\rho}^{\alpha}f(x)$.

If $\psi(x) = x^\rho$, $\alpha > 0, \rho \in \mathbb{R}$, $\gamma = 0$, $\beta = \alpha$, $\eta = k = 0$, *then* (2.1) *is reduced to "Katugampola" integral, and for* $\eta \in \mathbb{R}$, $\beta \in \mathbb{R}$, $k = s/\rho$, *we get the "generalized Katugampola" fractional integral*

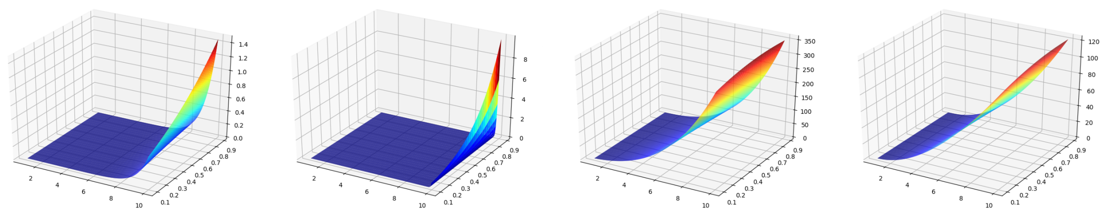
$$ {}^{\rho}I_{a+;\eta,s}^{\alpha,\beta}f(x) = \frac{x^s}{\Gamma(\alpha)\rho^{\beta-1}} \int_a^x u^{\rho(1+\eta)-1}\big(x^\rho - u^\rho\big)^{\alpha-1}f(u)du. $$

If $\psi(x) = x$, $\gamma = \frac{\rho-1}{\rho}$, $\alpha > 0$, $k = \eta = 0$, $\rho \in (0,1]$, $\beta = \alpha$, *we obtain the fractional (left sided) generalized proportional integral* $I_{a+;\rho}^{\alpha}f(x)$ *(Jarad-Abdeljawad-Alzabut integral) and for a general case of* ψ, *we obtain the fractional (left sided) proportional integral in the general form according to a function* ψ,

$$ I_{a+;\rho}^{\alpha;\psi}f(x) = \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^x \psi'(u)\exp\left[\frac{\rho-1}{\rho}\big(\psi(x)-\psi(s)\big)\right]\big(\psi(x)-\psi(s)\big)^{\alpha-1}f(s)ds. $$

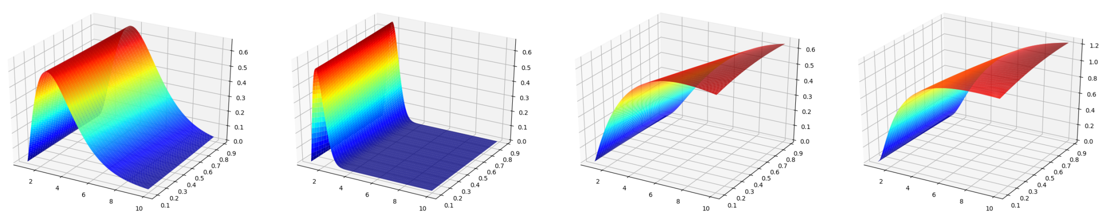
If $\psi(x) = \frac{x^{\rho+r}}{\rho+r}$, $\gamma = 0$, $\alpha > 0$, $k = \eta = \beta = 0$, $\rho \in (0,1]$, *and* $r \in \mathbb{R}$, *we obtain the generalized conformable fractional (left sided) integrals,* ${}^{r}K_{a+;\rho}^{\alpha}f(x)$. *If* $\psi(x) = \ln x$, $\gamma = \frac{\rho-1}{\rho}$, $\alpha > 0$, $k = \eta = 0$, $\rho \in (0,1]$, $\beta = \alpha$, *we obtain the generalized proportional integral of "Hadamard" (left sided),* $I_{a+;\rho}^{\alpha;\psi}f(x)$.

In the following, we plot some examples of the new ψ-fractional integral of the function $f(x)$ from Theorem 3.2, in the case $\rho = 0.5$, $\beta = 1$, $k = 1$ for a different example of ψ and different values of γ. The first two figures plot the expression of $I_{a+;\eta,k,\gamma,\rho}^{\alpha,\beta;\psi}f(x)$ against the variables x and α. The third and fourth figures plot the expression of $I_{a+;\eta,k,\gamma,\rho}^{\alpha,\beta;\psi}f(x)$ against the variables x and λ. Since the fractional integral of the above mentioned function $f(x) = [\psi(x)]^{-\eta}\exp\big(\gamma\psi(x)\big)\big(\psi(x)-\psi(a)\big)^{\lambda-1}$ is the solution of many well known fractional differential equations (see [43]), each figure is the solution of a specific differential equation. This fact will be the subject of a forthcoming work.



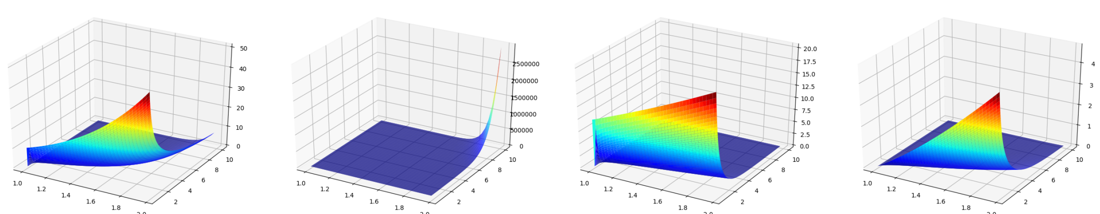
$\psi(x) = x$ $\psi(x) = x^2$ $\psi(x) = \sqrt{x}$ $\psi(x) = \log(x)$

Figure 1: ψ-fractional integral $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)$ where $f(x) = [\psi(x)]^{-\eta} \exp\big(\gamma\psi(x)\big)\big(\psi(x) - \psi(a)\big)^{\lambda-1}$ with $\gamma = 1$, $\lambda = 2$, $1 \le x \le 10$ and $0.1 \le \alpha \le 0.9$.



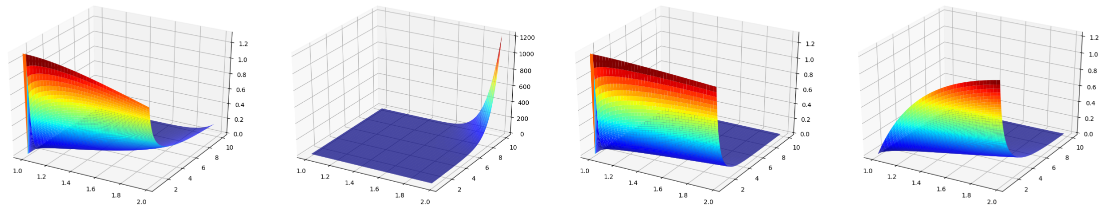
$\psi(x) = x$ $\psi(x) = x^2$ $\psi(x) = \sqrt{x}$ $\psi(x) = \log(x)$

Figure 2: ψ-fractional integral $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)$ where $f(x) = [\psi(x)]^{-\eta} \exp\big(\gamma\psi(x)\big)\big(\psi(x) - \psi(a)\big)^{\lambda-1}$ with $\gamma = -1$, $\lambda = 2$, $0.1 \le \alpha \le 0.9$ and $1 \le x \le 10$.



$\psi(x) = x$ $\psi(x) = x^2$ $\psi(x) = \sqrt{x}$ $\psi(x) = \log(x)$

Figure 3: ψ-fractional integral $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)$ where $f(x) = [\psi(x)]^{-\eta} \exp\big(\gamma\psi(x)\big)\big(\psi(x) - \psi(a)\big)^{\lambda-1}$ with $\gamma = 1$, $\alpha = 0.5$, $1 \le x \le 2$ and $0.5 \le \lambda \le 10$.



$\psi(x) = x$ $\psi(x) = x^2$ $\psi(x) = \sqrt{x}$ $\psi(x) = \log(x)$

Figure 4: ψ-fractional integral $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f(x)$ where $f(x) = [\psi(x)]^{-\eta}\exp\big(\gamma\psi(x)\big)\big(\psi(x)-\psi(a)\big)^{\lambda-1}$ with $\gamma = -1$, $\alpha = 0.5$, $1 \le x \le 2$ and $0.5 \le \lambda \le 10$.

3 Main properties of the new generalized ψ-fractional integral

We present in this section some essential properties of the new generalized ψ-fractional integral. First, we give some elementary properties having obvious proofs.

Theorem 3.1. *Let $\alpha > 0$, $\beta, \gamma, k, \eta, \rho, \in \mathbb{R}$ and ψ as defined in Definition 2.1. For $1 \le p \le \infty$ and $f \in X^p_\psi(a,b)$, we have the following properties:*

- $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}\psi(x)^\lambda f(x) = I^{\alpha,\beta;\psi}_{a+;\eta+\lambda,k,\gamma,\rho}f(x)$,
- $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}\exp(\lambda\psi(x))\ f(x) = \exp(\lambda\psi(x))\ I^{\alpha,\beta;\psi}_{a+;\eta+\lambda,k,\gamma-\lambda,\rho}f(x)$,
- $I^{\alpha,\beta;\psi}_{b-;\eta,k,\gamma,\rho}\psi(x)^\lambda f(x) = I^{\alpha,\beta;\psi}_{b-;\eta,k+\lambda,\gamma,\rho}f(x)$,
- $I^{\alpha,\beta;\psi}_{b-;\eta,k,\gamma,\rho}\exp(\lambda\psi(x))\ f(x) = \exp(\lambda\psi(x))\ I^{\alpha,\beta;\psi}_{b-;\eta+\lambda,k,\gamma+\lambda,\rho}f(x)$.

Theorem 3.2. *Let $\alpha > 0$, $\beta, \gamma, k, \eta, \rho, \in \mathbb{R}$ and ψ as defined in Definition 2.1. We have*

- $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}[\psi(t)]^{-\eta}\exp\big(\gamma\psi(t)\big)\Big(\psi(t)-\psi(a)\Big)^{\lambda-1} = \dfrac{\Gamma(\lambda)[\psi(t)]^k\exp\big(\gamma\psi(t)\big)}{\Gamma(\lambda+\alpha)\rho^\beta}\Big(\psi(t)-\psi(a)\Big)^{\lambda+\alpha-1}$,
- $I^{\alpha,\beta;\psi}_{b-;\eta,k,\gamma,\rho}[\psi(t)]^{-k}\exp\big(-\gamma\psi(t)\big)\Big(\psi(b)-\psi(t)\Big)^{\lambda-1} = \dfrac{\Gamma(\lambda)[\psi(t)]^\eta\exp\big(-\gamma\psi(t)\big)}{\Gamma(\lambda+\alpha)\rho^\beta}\Big(\psi(b)-\psi(t)\Big)^{\lambda+\alpha-1}$.

Next we prove that for a positive increasing function ψ on (a,b), the new ψ-fractional operator $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}$ is well-defined and bounded on the space $X^p_\psi(a,b)$.

Theorem 3.3. *Let $\gamma > 0$, $\alpha > 0$, $\beta, k, \eta, \rho \in \mathbb{R}$ and ψ as defined in Definition 2.1. For $1 \le p \le \infty$ and $f \in X^p_\psi(a,b)$, we have*

$$\left\| I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f \right\|_{X^p_\psi} \le K\|f\|_{X^p_\psi},$$

where

$$K = \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta}[\psi(b)]^{\alpha+\eta+k}\int_{\psi^{-1}(1)}^{\psi^{-1}\left(\frac{\psi(b)}{\psi(a)}\right)}[\psi(u)]^{-\alpha-\eta}\big[\psi(u)-1\big]^{\alpha-1}\psi'(u)du.$$

Proof. Let $1 \leq p < \infty$. Using (2.1) and Definition 2.1, we have

$$\begin{aligned}\left\| I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f \right\|_{X^p_\psi} &= \left(\int_a^b \left| \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta} \int_a^x \psi'(u)[\psi(u)]^\eta \exp(-\gamma\psi(u)) \right.\right. \\ &\qquad \left.\left. \times \big(\psi(x)-\psi(u)\big)^{\alpha-1} f(u)du \right|^p \psi'(x)dx \right)^{\frac{1}{p}} \\ &= \frac{1}{\Gamma(\alpha)\rho^\beta} \left(\int_a^b \left| \int_a^x [\psi(x)]^k \exp\big(\gamma(\psi(x)-\psi(u))\big)[\psi(u)]^\eta \psi'(u)\big(\psi(x)-\psi(u)\big)^{\alpha-1} f(u)du \right|^p \psi'(x)dx \right)^{\frac{1}{p}} \\ &\leq \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta} \left(\int_a^b \left| \int_a^x [\psi(x)]^k \psi'(u)[\psi(u)]^{\eta+\alpha-1} \left(\frac{\psi(x)}{\psi(u)} - 1 \right)^{\alpha-1} f(u)du \right|^p \psi'(x)dx \right)^{\frac{1}{p}}.\end{aligned}$$

If we suppose $\frac{\psi(x)}{\psi(u)} = \psi(s)$, we get

$$\begin{aligned}\left\| I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f \right\|_{X^p_\psi} \leq \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta} &\left(\int_a^b \left| \int_{\psi^{-1}(1)}^{\psi^{-1}\left(\frac{\psi(x)}{\psi(a)}\right)} [\psi(x)]^{k+\eta+\alpha} \psi'(s)[\psi(s)]^{-1-\eta-\alpha} \big[\psi(s)-1\big]^{\alpha-1} \right.\right. \\ &\left.\left. \times f\left(\psi^{-1}\left(\frac{\psi(x)}{\psi(s)} \right) \right) ds \right|^p \psi'(x)dx \right)^{\frac{1}{p}}.\end{aligned}$$

Using the generalized Minkowski-inequality ([1]), we obtain

$$\begin{aligned}\left\| I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f \right\|_{X^p_\psi} &\leq \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta} \int_{\psi^{-1}(1)}^{\psi^{-1}\left(\frac{\psi(b)}{\psi(a)}\right)} (\psi(s))^{-\eta-\alpha} \big(\psi(s)-1\big)^{\alpha-1} \psi'(s)[\psi(b)]^{\eta+k+\alpha} \\ &\qquad \times \left(\int_{\psi^{-1}(\psi(a)\psi(t))}^b \frac{\psi'(x)}{\psi(s)} \left| f\left(\psi^{-1}\left(\frac{\psi(x)}{\psi(s)} \right) \right) \right|^p dx \right)^{\frac{1}{p}} ds \\ &\leq \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta} \int_{\psi^{-1}(1)}^{\psi^{-1}\left(\frac{\psi(b)}{\psi(a)}\right)} (\psi(s))^{-\eta-\alpha} \big(\psi(s)-1\big)^{\alpha-1} \psi'(s)[\psi(b)]^{k+\eta+\alpha} \left(\int_a^{\psi^{-1}\left(\frac{\psi(b)}{\psi(s)}\right)} \big|f(t)\big|^p \psi'(t)dt \right)^{\frac{1}{p}} ds \\ &\leq K \|f\|_{X^p_\psi},\end{aligned}$$

where

$$K = \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta} [\psi(b)]^{k+\eta+\alpha} \int_{\psi^{-1}(1)}^{\psi^{-1}\left(\frac{\psi(b)}{\psi(a)}\right)} (\psi(u))^{-\eta-\alpha} \big(\psi(u)-1\big)^{\alpha-1} \psi'(u)du.$$

Thus, the result is proved for $1 \leq p < \infty$. For $p = \infty$, we have

$$\begin{aligned}\left\| I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f \right\|_{X^\infty_\psi} &= \operatorname*{ess\,sup}_{t\in(a,b)} \left| \psi'(t)\ I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(t) \right| \\ &\leq \frac{[\psi(b)]^k \exp(\gamma\psi(b))}{\Gamma(\alpha)\rho^\beta} \int_a^x [\psi(u)]^\eta \psi'(x)\big(\psi(x)-\psi(u)\big)^{\alpha-1} \big|\psi'(u)f(u)\big| du \\ &\leq \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta} [\psi(b)]^{k+\eta+\alpha} \int_{\psi^{-1}(1)}^{\psi^{-1}\left(\frac{\psi(b)}{\psi(a)}\right)} [\psi(s)]^{-\eta-\alpha} \big(\psi(s)-1\big)^{\alpha-1} \psi'(s)ds\ \|f\|_{X^\infty_\psi}.\end{aligned}$$

Theorem 3.3 is thereby proved. □

In the next result, we prove that the new generalized ψ-fractional integral satisfies the property of semigroup.

Theorem 3.4. *Let $\alpha > 0$, $\beta, k, \eta, \rho, \gamma \in \mathbb{R}$ and ψ as defined in Definition 2.1. For $1 \le p \le \infty$ and $f \in X^p_\psi(a,b)$, we have:*

$$I^{\alpha_1,\beta_1;\psi}_{a+;\eta_1,k_1,\gamma,\rho}\, I^{\alpha_2,\beta_2;\psi}_{a+;\eta_2,-\eta_1,\gamma,\rho} f(x) = I^{\alpha_1+\alpha_2,\beta_1+\beta_2;\psi}_{a+;\eta_2,k_1,\gamma,\rho} f(x),$$
$$I^{\alpha_1,\beta_1;\psi}_{b-;\eta_1,-\eta_2,\gamma,\rho}\, I^{\alpha_2,\beta_2;\psi}_{b-;\eta_2,k_2,\gamma,\rho} f(x) = I^{\alpha_1+\alpha_2,\beta_1+\beta_2;\psi}_{b-;\eta_1,k_2,\gamma,\rho} f(x).$$

Proof. Using Definition 2.1, we have

$$\begin{aligned} I^{\alpha_1,\beta_1;\psi}_{a+;\eta_1,k_1,\gamma,\rho} I^{\alpha_2,\beta_2;\psi}_{a+;\eta_2,k_2,\gamma,\rho} f(x) &= \frac{[\psi(x)]^{k_1}\exp(\gamma\psi(x))}{\Gamma(\alpha_1)\rho^{\beta_1}} \int_a^x [\psi(t)]^{\eta_1} \exp(-\gamma\psi(t)) \big(\psi(x)-\psi(t)\big)^{\alpha_1-1} \\ &\quad \times \frac{[\psi(t)]^{k_2}\exp(\gamma\psi(t))}{\Gamma(\alpha_2)\rho^{\beta_2}} \psi'(t) \int_a^t \psi'(s)[\psi(s)]^{\eta_2}\exp(-\gamma\psi(s))\big[\psi(t)-\psi(s)\big]^{\alpha_2-1} f(s)ds\, dt \\ &= \frac{[\psi(x)]^{k_1}\exp(\gamma\psi(x))}{\Gamma(\alpha_1)\Gamma(\alpha_2)\rho^{\beta_1+\beta_2}} \int_a^x \psi'(s)[\psi(s)]^{\eta_2}\exp(-\gamma\psi(s)) f(s) \int_s^x \psi'(t)[\psi(t)]^{\eta_1+k_2} \\ &\quad \times \big(\psi(x)-\psi(t)\big)^{\alpha_1-1}\big(\psi(t)-\psi(s)\big)^{\alpha_2-1} dt\, ds. \end{aligned}$$

For $k_2 = -\eta_1$ and supposing that $u := \dfrac{\psi(t)-\psi(s)}{\psi(x)-\psi(s)}$, we derive that

$$\begin{aligned} &\int_s^x (\psi(t))^{\eta_1+k_2}\psi'(t)\big(\psi(x)-\psi(t)\big)^{\alpha_1-1}\big(\psi(t)-\psi(s)\big)^{\alpha_2-1}dt \\ &\qquad = \big(\psi(x)-\psi(s)\big)^{\alpha_1+\alpha_2-1}\int_0^1 (1-u)^{\alpha_1-1}u^{\alpha_2-1}du \\ &\qquad = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\big(\psi(x)-\psi(s)\big)^{\alpha_1+\alpha_2-1}. \end{aligned}$$

Thus,

$$\begin{aligned} &I^{\alpha_1,\beta_1;\psi}_{a+;\eta_1,k_1,\gamma,\rho}\, I^{\alpha_2,\beta_2;\psi}_{a+;\eta_2,-\eta_1,\gamma,\rho} f(x) \\ &\qquad = \frac{[\psi(x)]^{k_1}\exp(\gamma\psi(x))}{\Gamma(\alpha_1+\alpha_2)\rho^{\beta_1+\beta_2}} \int_a^x \psi'(s)[\psi(s)]^{\eta_2}\exp(-\gamma\psi(s))\big(\psi(x)-\psi(s)\big)^{\alpha_1+\alpha_2-1} f(s)ds \\ &\qquad = I^{\alpha_1+\alpha_2,\beta_1+\beta_2;\psi}_{a+;\eta_2,k_1,\gamma,\rho} f(x). \end{aligned}$$

The first identity in Theorem 3.4 is thereby proved. The second one follows using the same arguments. □

Theorem 3.5. *Let $\alpha > 0$, $\beta, k, \eta, \rho, \gamma \in \mathbb{R}$ and ψ as defined in Definition 2.1. For $1 \le p \le \infty$, we have*

$$\int_a^b f(u)\Big[I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g\Big](u)\psi'(u)du = \int_a^b g(u)\Big[I^{\alpha,\beta;\psi}_{b-;\eta,k,\gamma,\rho} f\Big](u)\psi'(u)du.$$

Proof. We have

$$\begin{aligned}\int_a^b f(u)\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}g\Big)(u)\psi'(u)du &= \int_a^b f(u)\psi'(u)\frac{[\psi(u)]^k\exp(\gamma\psi(u))}{\Gamma(\alpha)\rho^\beta}\\ &\quad\times\int_a^u[\psi(t)]^\eta\exp\big(-\gamma\psi(t)\big)\big(\psi(u)-\psi(t)\big)^{\alpha-1}g(t)\psi'(t)dt\,du\\ &= \int_a^b g(t)\frac{[\psi(t)]^\eta\exp(-\gamma\psi(t))}{\Gamma(\alpha)\rho^\beta}\psi'(t)\int_t^b[\psi(u)]^k\psi'(u)\exp\big(\gamma\psi(u)\big)\big(\psi(u)-\psi(t)\big)^{\alpha-1}f(u)du\,dt\\ &= \int_a^b\psi'(u)g(u)\Big(I^{\alpha,\beta;\psi}_{b-;\eta,k,\gamma,\rho}f\Big)(u)du.\end{aligned}$$

□

Theorem 3.6. *Let $\alpha>0$, $\beta,k,\eta,\rho,\gamma\in\mathbb{R}$ and ψ as defined in Definition 2.1. For $f\in X^\infty_\psi(a,b)$ and $x,y\in(a,b)$, we have*

$$\Big\|I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f(x)-I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f(y)\Big\|\le\frac{2\Big\|[\psi(u)]^\eta\exp(-\gamma\psi(u))f(u)\Big\|_{X^\infty_\psi}}{\Gamma(\alpha+1)\rho^\beta}[\psi(y)]^k\exp(\gamma\psi(y))\Big[\psi(y)-\psi(x)\Big]^\alpha.$$

Proof. We have

$$\begin{aligned}\Big\|I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f(x)-I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f(y)\Big\| &= \Bigg\|\frac{\exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x[\psi(u)]^\eta\psi'(u)\exp\big(-\gamma\psi(u)\big).\big[\psi(x)-\psi(u)\big]^{\alpha-1}f(u)du\\ &\quad-\frac{\exp(\gamma\psi(y))}{\Gamma(\alpha)\rho^\beta}\int_a^y[\psi(u)]^\eta\psi'(u)\exp\big(-\gamma\psi(u)\big)\big(\psi(y)-\psi(u)\big)^{\alpha-1}f(u)du\Bigg\|\\ &= \Bigg\|\frac{1}{\Gamma(\alpha)\rho^\beta}\int_a^x f(u)(\psi(u))^\eta\psi'(u)\exp\big(-\gamma(\psi(u))\big)\\ &\quad\times\Big(\exp(\gamma(\psi(x)))[\psi(x)]^k\big(\psi(x)-\psi(u)\big)^{\alpha-1}-\exp(\gamma(\psi(y)))[\psi(y)]^k\big[\psi(y)-\psi(u)\big]^{\alpha-1}\Big)dt\\ &\quad-\frac{\exp(\gamma\psi(y))}{\Gamma(\alpha)\rho^\beta}\int_x^y[\psi(u)]^\eta\psi'(u)\exp\big(-\gamma\psi(u)\big)\big(\psi(y)-\psi(u)\big)^{\alpha-1}f(u)du\Bigg\|\\ &\le\frac{\|[\psi(u)]^\eta\exp(-\gamma\psi(u))f(u)\|_{X^\infty_\psi}}{\Gamma(\alpha)\rho^\beta}\int_a^x\psi'(u)\Big([\psi(x)]^k\exp(\gamma(\psi(x)))\big(\psi(x)-\psi(u)\big)^{\alpha-1}\\ &\quad-[\psi(y)]^k\exp(\gamma(\psi(y)))\big(\psi(y)-\psi(u)\big)^{\alpha-1}\Big)du\\ &\quad+\frac{\|[\psi(u)]^\eta\exp\big(-\gamma\psi(u)\big)f(u)\|_{X^\infty_\psi}}{\Gamma(\alpha)\rho^\beta}\int_x^y\psi'(u)[\psi(y)]^k\exp(\gamma\psi(y))\big(\psi(y)-\psi(u)\big)^{\alpha-1}du\\ &\le\frac{\|[\psi(u)]^\eta\exp(-\gamma\psi(u))f(u)\|_{X^\infty_\psi}}{\Gamma(\alpha+1)\rho^\beta}\Big([\psi(x)]^k\exp(\gamma(\psi(x)))\Big(\psi(x)-\psi(a)\Big)^\alpha\\ &\quad-\exp(\gamma(\psi(y)))[\psi(y)]^k\Big(\big[\psi(y)-\psi(a)\big]^\alpha-\Big[\psi(y)-\psi(x)\Big]^\alpha\Big)\Big)\\ &\quad+\frac{\|[\psi(u)]^\eta\exp(-\gamma\psi(u))f(u)\|_{X^\infty_\psi}}{\Gamma(\alpha+1)\rho^\beta}[\psi(y)]^k\exp(\gamma\psi(y))\Big(\psi(y)-\psi(x)\Big)^\alpha\\ &\le\frac{2\|[\psi(u)]^\eta\exp(-\gamma\psi(u))f(u)\|_{X^\infty_\psi}}{\Gamma(\alpha+1)\rho^\beta}[\psi(y)]^k\exp(\gamma\psi(y))\Big[\psi(y)-\psi(x)\Big]^\alpha.\end{aligned}$$

□

Theorem 3.7. *Let $n-1<\alpha<n$, $\beta,\eta,k,\rho,\gamma\in\mathbb{R}$ and ψ as defined in Definition 2.1. For $(f_n)_{n\geq 1}$ a sequence uniformly convergent in $X_\psi^\infty(a,b)$, we have*

$$I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f_n(x).$$

Proof. Let $f(x)=\lim\limits_{n\to\infty} f_n(x)$. We have

$$\begin{aligned}&\Big|I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f_n(x)-I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f(x)\Big|\\ &\leq \frac{([\psi(x)]^k\exp\big(\gamma(\psi(x))\big)}{\Gamma(\alpha)\rho^\beta}\int_a^x[\psi(u)]^\eta\psi'(u)\exp\big(-\gamma(\psi(u))\big)\big(\psi(x)-\psi(u)\big)^{\alpha-1}\big|f_n(u)-f(u)\big|du\\ &\leq \Big\|\exp(-\gamma(\psi(u)))[\psi(u)]^\eta\big(f_n(u)-f(u)\big)\Big\|_{X_\psi^\infty}\frac{([\psi(x)]^k\exp(\gamma(\psi(x)))}{\Gamma(\alpha+1)\rho^\beta}\big(\psi(x)-\psi(a)\big)^\alpha.\end{aligned}$$

Since the sequence $(f_n)_{n\geq1}$ is uniformly convergence, the result follows. ☐

Theorem 3.8. *Let f be a uniformly continuous function on $[0,b]$. For $\beta,\eta,k,\rho,\gamma\in\mathbb{R}$ and ψ as defined in Definition 2.1, if there exists $\alpha\in(0,1]$ satisfying*

$$\lim_{x\to\infty} I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}|f(x)|=0,$$

then

$$\lim_{x\to\infty}|f(x)|=0.$$

Proof. Arguing by contradiction, we assume that there exists an unbounded sequence $(x_i)_{i\in\mathbb{N}}$ and $\varepsilon>0$ such that

$$|f(x_i)|\geq\varepsilon,\quad \forall x_i\in[0,b].$$

Using the fact that f is uniformly continuous, we deduce that for each x_i, $\exists\,\mu>0$ such that

$$\big|f(x_i)-f(x)\big|<\frac{\varepsilon}{2},\quad \forall x\in[x_i-\mu,x_i+\mu]$$

Thus, for all $x\in[x_i-\mu,x_i+\mu]$ we have:

$$|f(x)|\geq\Big||f(x_i)|-|f(x_i)-f(x)|\Big|\geq\frac{\varepsilon}{2}. \tag{3.1}$$

From another part, we have

$$\begin{aligned}I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}|f(x_i)|=\frac{([\psi(x_i)]^k\exp(\gamma(\psi(x_i)))}{\Gamma(\alpha)\rho^\beta}\bigg(&\int_{x_0}^{x_i-1}[\psi(u)]^\eta\psi'(u)\exp\big(-\gamma(\psi(u))\big)\big(\psi(x_i)-\psi(u)\big)^{\alpha-1}|f(u)|du\\ &+\int_{x_i-1}^{x_i-\mu}[\psi(u)]^\eta\psi'(u)\exp\big(-\gamma(\psi(u))\big)\big(\psi(x_i)-\psi(u)\big)^{\alpha-1}|f(u)|du\end{aligned}$$

$$+ \int_{x_i-\mu}^{x_i} [\psi(u)]^\eta \psi'(u) \exp\big(-\gamma(\psi(u))\big)\big(\psi(x_i)-\psi(u)\big)^{\alpha-1} |f(u)| du\Big).$$

If we suppose that $[\psi(u)]^\eta \psi'(u) \exp\big(-\gamma(\psi(u))\big)\big(\psi(x_i)-\psi(u)\big)^{\alpha-1} \geq 1$, $\forall t \in [x_i-1, x_i]$, then

$$I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}|f(x_i)| \geq \frac{([\psi(x_i)]^k \exp(\gamma(\psi(x_i)))}{\Gamma(\alpha)\rho^\beta} \Big(\int_{x_0}^{x_i-1} [\psi(u)]^\eta \psi'(u) \exp\big(-\gamma(\psi(t))\big) \times \big(\psi(x_i)-\psi(u)\big)^{\alpha-1} \big|f(u)\big| du + \int_{x_i-1}^{x_i-\mu} \big|f(u)\big| du + \int_{x_i-\mu}^{x_i} \big|f(u)\big| du\Big). \tag{3.2}$$

Using (3.1) and (3.2) and denoting $c = \big([\psi(0)]^k \exp(\gamma(\psi(0)))\big)$, we obtain

$$I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}|f(x_i)| \geq \frac{c\varepsilon}{2\Gamma(\alpha)\rho^\beta},$$

which contradicts the hypothesis of the Theorem. □

4 On a Minkowski type inequality

First, we recall the celebrated Minkowski inequality as follows, (see [1, 22]).

Theorem 4.1. *If $p \geq 1$ and f, g two positives functions in $L^p([a,b])$, then*

$$\left(\int_a^b |f(t)+g(t)|^p dt\right)^{1/p} \leq \left(\int_a^b |f(t)|^p dt\right)^{1/p} + \left(\int_a^b |g(t)|^p dt\right)^{1/p}.$$

As a reverse of Minkowski's inequality, Bougoffa [9] proved the following result.

Theorem 4.2. *If $p \geq 1$, f and g two positives functions satisfying $0 < m \leq \frac{f(t)}{g(t)} \leq M$, $\forall t \in [a,b]$, then*

$$\left(\int_a^b |f(t)|^p dt\right)^{1/p} + \left(\int_a^b |g(t)|^p dt\right)^{1/p} \leq c\left(\int_a^b |f(t)+g(t)|^p dt\right)^{1/p},$$

where $c = \frac{M}{M+1} + \frac{1}{m+1}$.

The above result was generalized by Dahmani [17] using Riemann-Liouville fractional integral, by Chinchane-Pachpatte [13] and Taf-Brahim [44] using the Hadamard fractional integral, by Sousa-Oliveira [15] using Katugampola generalized fractional integral, by Aljaaidi-Pachpatte [5] using the ψ Riemann Liouville integral, by Rahman *et al.* [37] using generalized proportional fractional integral, by Rachid-Jarad-Chu [39] using generalized proportional integral according to another function, by Rachid *et al.* [38] using generalized conformable integral, by Nale-Panchal-Chinchane [33] using generalized proportional Hadamard fractional integral.

In the following, we prove the reverse of the Minkowski inequality using the new generalized ψ-Hilfer integral, recovering the results of the above cited papers.

Theorem 4.3. *Let $\beta, k, \eta, \rho, \gamma \in \mathbb{R}, \alpha > 0$, $p \geq 1$ and ψ as defined in Definition 2.1. Let also f, g be two positive functions in $X_\psi^p(a,b)$. If $0 < m \leq \dfrac{f(t)}{g(t)} \leq M$, $\forall t \in [a,b]$ for m and M two strictly positive constants, then:*

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(t)\right)^{\frac{1}{p}} + \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(t)\right)^{\frac{1}{p}} \leq c \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} (f+g)^p(t)\right)^{\frac{1}{p}},$$

where $c = \dfrac{1}{m+1} + \dfrac{M}{M+1}$.

Proof. Since $\dfrac{f(s)}{g(s)} \leq M$, $\forall s \in [a,b]$, then

$$f(s) + Mf(s) \leq M\big(g(s) + f(s)\big), \quad \forall s \in [a,b],$$

thus

$$(M+1)^p f^p(s) \leq M^p\big(f(s) + g(s)\big)^p, \quad \forall s \in [a,b].$$

Multiplying both sides by $\dfrac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1}$ and integrating with respect to s, we obtain

$$\begin{aligned}(M+1)^p \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta} \int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1} f^p(s)ds \\ \leq M^p \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta} \int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1} (f+g)^p(s)ds.\end{aligned}$$

Which implies that

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}} \leq \frac{M}{M+1} \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} (g+f)^p(x)\right)^{\frac{1}{p}}. \tag{4.1}$$

From another part, since $0 < m \leq \dfrac{f(s)}{g(s)}$, for all $s \in [a,b]$, then

$$g(s) \leq \frac{f(s)}{m}, \quad \forall s \in [a,b].$$

Thus

$$g(s)\left(1 + \frac{1}{m}\right) \leq \frac{f(s)}{m} + \frac{g(s)}{m}, \quad \forall s \in [a,b],$$

and consequently

$$g^p(s)\left(1 + \frac{1}{m}\right)^p \leq \left(\frac{1}{m}\right)^p \Big[g(s) + f(s)\Big]^p.$$

Multiplying both sides by $\frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1}$ and integrating with respect to s, we derive that

$$\left(1+\frac{1}{m}\right)^p \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1} g^p(s)ds$$
$$\leq \left(\frac{1}{m}\right)^p \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1}\Big[g(s)+f(s)\Big]^p ds.$$

Thus,

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq \frac{1}{m+1}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}(g+f)^p(x)\right)^{\frac{1}{p}}. \tag{4.2}$$

Using (4.1) and (4.2), the result follows. □

Theorem 4.4. *Let $\beta,\rho,\gamma,k,\eta\in\mathbb{R}$, $\alpha>0$, $p\geq 1$ and ψ as defined in Definition 2.1. Let also f,g be two positive functions in $X^p_\psi(a,b)$. If $0<m\leq\frac{f(t)}{g(t)}\leq M$, $\forall t\in[a,b]$ for $m>0, M>0$, then:*

$$\left[I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right]^{\frac{2}{p}} + \left[I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right]^{\frac{2}{p}} \geq \hat{c}\left[I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right]^{\frac{1}{p}}\left[I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right]^{\frac{1}{p}},$$

where $\hat{c}=\frac{(M+1)(m+1)}{M}-2$.

Proof. From (4.1) and (4.2), we have:

$$\frac{(M+1)(m+1)}{M}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}(g+f)^p(x)\right)^{\frac{2}{p}}. \tag{4.3}$$

Using Minkowski's inequality, we obtain

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}(g+f)^p(x)\right)^{\frac{1}{p}} \leq \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}} + \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \tag{4.4}$$

Using (4.3) and (4.4), we deduce that

$$\frac{(M+1)(m+1)}{M}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq \left(\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}} + \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}}\right)^2.$$

Thus,

$$\left(\frac{(M+1)(m+1)}{M}-2\right)\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{2}{p}} + \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{2}{p}}.$$ □

Remark 4.5. *Using Remark 2.3, it is easy to see that Theorems 4.3 and 4.4 recover Theorems 2.1 and 2.3 of [17], Theorems 3.1 and 3.2 of [13], Theorems 2.9 and 2.10 of [44], Theorems 7 and 8 of [15], Theorems 3.1 and 3.2 of [5], Theorems 3.1 and 3.2 of [37], Theorems 5 and 6 of [39], Theorems 3.1 and 3.2 of [38] and Theorems 3.1 and 3.2 of [33].*

5 Other inequalities related to the Minkowski type inequality

In this section, we state other inequalities related to the Minkowski type inequality, using generalized ψ-fractional integral.

Theorem 5.1. *Let $\beta, \eta, \rho, \gamma, k \in \mathbb{R}$, $\alpha > 0$, $p \geq 1$ and ψ as defined in Definition 2.1. For f, g two positive functions in $X_\psi^p(a,b)$, if $0 < m \leq f(s) \leq M$ and $0 < n \leq g(s) \leq N$ for all $s \in [a,b]$, then*

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}} + \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq \tilde{c}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} (g+f)^p(x)\right)^{\frac{1}{p}}.$$

Here $\tilde{c} = \dfrac{M}{M+n} + \dfrac{N}{N+m}$.

Proof. Since $0 < n \leq g(s) \leq N$ for all $s \in [a,b]$, then

$$\frac{1}{N} \leq \frac{1}{g(s)} \leq \frac{1}{n}, \quad \forall s \in [a,b].$$

Thus,

$$\frac{m}{N} \leq \frac{f(s)}{g(s)} \leq \frac{M}{n}. \tag{5.1}$$

From (5.1), we deduce that

$$g(s)\left(\frac{m}{N}+1\right) \leq g(s) + f(s), \tag{5.2}$$

$$\left(\frac{n}{M}+1\right) f(s) \leq g(s) + f(s). \tag{5.3}$$

Thus,

$$g^p(s) \leq \left(\frac{N}{m+N}\right)^p (g(s)+f(s))^p, \tag{5.4}$$

$$f^p(s) \leq \left(\frac{M}{n+M}\right)^p (g(s)+f(s))^p. \tag{5.5}$$

Multiplying both sides of (5.4) by $\dfrac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1}$ and integrating with respect to s, we obtain

$$\begin{aligned}
&\frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1} g^p(s)ds \\
&\leq \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1}\left(\frac{N}{m+N}\right)^p (f+g)^p(s)ds,
\end{aligned}$$

which implies that

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq \frac{N}{m+N} \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} (g+f)^p(x)\right)^{\frac{1}{p}}. \tag{5.6}$$

From another part, using the same argument to equation (5.5), we derive that

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}} \leq \frac{M}{n+M} \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} (g+f)^p(x)\right)^{\frac{1}{p}}. \tag{5.7}$$

Adding (5.6) and (5.7), the result follows. □

Theorem 5.2. *Let $\beta, \rho, k, \eta, \gamma, \in \mathbb{R}$, $\alpha > 0$, $p \geq 1$ and ψ as defined in Definition 2.1. Let f, g two positive functions in $X^p_\psi(a,b)$. If $0 < m \leq \dfrac{f(s)}{g(s)} \leq M, \forall s \in [a,b]$ for $m, M \in \mathbb{R}^*_+$, then*

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}} + \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq 2 \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} h^p(g(x)+f(x))\right)^{\frac{1}{p}},$$

where $h(g(x)+f(x)) = \max\left\{ \left(\dfrac{M}{m}+1\right) f(x) - Mg(x),\ \dfrac{(M+m)g(x)-f(x)}{m} \right\}.$

Proof. Since $0 < m \leq \dfrac{f(s)}{g(s)} \leq M$, $\forall s \in [a,b]$, then

$$0 < m \leq M - \frac{f(s)}{g(s)} + m.$$

Thus

$$g(s) \leq \frac{(M+m)g(s) - f(s)}{m},$$

which implies that

$$g(s) \leq h(f(s), g(s)). \tag{5.8}$$

From another part, since $0 < \dfrac{1}{M} \leq \dfrac{g(s)}{f(s)} \leq \dfrac{1}{m}$, then

$$\frac{1}{M} \leq \frac{1}{M} + \frac{1}{m} - \frac{g(s)}{f(s)}.$$

Thus,

$$\frac{1}{M} \leq \frac{\left(\frac{1}{M}+\frac{1}{m}\right) f(s) - g(s)}{f(s)},$$

which implies that

$$f(s) \leq M\left(\frac{1}{M}+\frac{1}{m}\right) f(s) - Mg(s) \leq \left(\frac{M}{m}+1\right) f(s) - Mg(s) \leq h(f(s), g(s)). \tag{5.9}$$

From (5.8) and (5.9), we get

$$f^p(s) \leq h^p(f(s), g(s)), \tag{5.10}$$

$$g^p(s) \leq h^p(f(s), g(s)). \tag{5.11}$$

Multiplying both sides of (5.10) by $\dfrac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1}$ and integrating with respect to s, we derive that

$$\begin{aligned}&\frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1} f^p(s)ds\\ &\qquad \leq \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1} h^p(f(s), g(s))ds.\end{aligned}$$

Which implies that

$$\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\Big)^{\frac{1}{p}} \leq \Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} h^p(g(x), f(x))\Big)^{\frac{1}{p}}. \tag{5.12}$$

Using the same argument to equation (5.11), we obtain

$$\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\Big)^{\frac{1}{p}} \leq \Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} h^p(g(x), f(x))\Big)^{\frac{1}{p}} \tag{5.13}$$

and the result follows. □

Theorem 5.3. *Under the hypothesis of Theorem 5.2, we have*

$$\frac{1}{M}\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)g(x)\Big) \leq \frac{1}{(M+1)(m+1)}\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}(f+g)^2(x)\Big) \leq \frac{1}{m}\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)(g)(x)\Big).$$

Proof. Since $0 < m \leq \dfrac{f(s)}{g(s)} \leq M$ for all $s \in [a,b]$, then

$$g(s)(1+m) \leq f(s) + g(s) \leq g(s)(1+M). \tag{5.14}$$

Additionally, using the fact that $0 < \dfrac{1}{M} \leq \dfrac{g(s)}{f(s)} \leq \dfrac{1}{m}, \forall s \in [a,b]$, we obtain

$$f(s)\left(\frac{1}{M}+1\right) \leq f(s) + g(s) \leq f(s)\left(1+\frac{1}{m}\right). \tag{5.15}$$

From (5.14) and (5.15), we deduce that

$$\frac{g(s)f(s)}{M} \leq \frac{\big(g(s)+f(s)\big)^2}{(1+m)(1+M)} \leq \frac{f(s)g(s)}{m}. \tag{5.16}$$

Multiplying both sides of equation (5.16) by $\dfrac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x) - \psi(s)\big]^{\alpha-1}$ and integrating with respect to s, we obtain

$$\begin{aligned}
&\frac{[\psi(x)]^k \exp(\gamma\psi(x))}{M\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1} f(s)g(s)ds \\
&\quad\le \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{(m+1)(M+1)\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1}(f+g)^2(s)ds \\
&\quad\le \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{m\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1} f(s)g(s)ds.
\end{aligned}$$

Thus,

$$\frac{1}{M}\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)g(x)\Big) \le \frac{1}{(1+m)(1+M)}\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}(f+g)^2(x)\Big) \le \frac{1}{m}\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)g(x)\Big). \quad \square$$

6 Conclusion

Minkowski type inequalities play a crucial role in various fields of science. In recent years, these inequalities have been proved by numerous researchers using different fractional integrals. The aim of this work was to prove a generalized Minkowski type inequality which recovers most of the previous results. For this purpose, we defined a new generalized ψ fractional integral, which generalizes most of the pre-existing fractional integrals. Then, we gave some essential properties of this new operator and we presented some examples. As an application, we used this generalized ψ fractional integral to prove a Minkowski type inequality and several related ones. These inequalities recover a large number of a well known results. Many other interesting inequalities as Grüss-type, Hermite-Hadamard type or Čebyšev type inequalities can be proved using the newly defined integral operator. These questions will be discussed in a forthcoming paper.

Conflict of interest statement:

The author declares that he has no conflict of interest.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, ser. National Bureau of Standards Applied Mathematics Series. U. S. Government Printing Office, Washington, DC, 1964, vol. 55.

[2] M. Adil Khan, S. Begum, Y. Khurshid, and Y.-M. Chu, "Ostrowski type inequalities involving conformable fractional integrals," *J. Inequal. Appl.*, 2018, Art. ID 70, doi: 10.1186/s13660-018-1664-4.

[3] R. Agarwal, N. Kumar, R. K. Parmar, and S. D. Purohit, "Some families of the general Mathieu-type series with associated properties and functional inequalities," *Math. Methods Appl. Sci.*, vol. 45, no. 4, pp. 2132–2150, 2022, doi: 10.1002/mma.7913.

[4] S. Ali, S. Mubeen, R. S. Ali, G. Rahman, A. Morsy, K. S. Nisar, S. D. Purohit, and M. Zakarya, "Dynamical significance of generalized fractional integral inequalities via convexity," *AIMS Math.*, vol. 6, no. 9, pp. 9705–9730, 2021, doi: 10.3934/math.2021565.

[5] T. A. Aljaaidi and D. B. Pachpatte, "The Minkowski's inequalities via Ψ-Riemann-Liouville fractional integral operators," *Rend. Circ. Mat. Palermo (2)*, vol. 70, no. 2, pp. 893–906, 2021, doi: 10.1007/s12215-020-00539-w.

[6] T. M. Atanacković, S. Pilipović, B. Stanković, and D. Zorica, *Fractional calculus with applications in mechanics: Wave propagation, impact and variational principles*, ser. Mechanical Engineering and Solid Mechanics Series. ISTE, London; John Wiley & Sons, Inc., Hoboken, NJ, 2014.

[7] D. Baleanu, S. D. Purohit, and F. Uçar, "On Grüss type integral inequality involving the Saigo's fractional integral operators," *J. Comput. Anal. Appl.*, vol. 19, no. 3, pp. 480–489, 2015.

[8] Y. Basci and D. Baleanu, "Ostrowski type inequalities involving ψ-Hilfer fractional integrals," *Mathematics*, vol. 7, no. 9, pp. 9705–9730, 2019, Art. ID 770, doi: 10.3390/math7090770.

[9] L. Bougoffa, "On Minkowski and Hardy integral inequalities," *JIPAM. J. Inequal. Pure Appl. Math.*, vol. 7, no. 2, 2006, Art. ID 60.

[10] S. I. Butt, A. O. Akdemir, M. Nadeem, and M. A. Raza, "Grüss type inequalities via generalized fractional operators," *Math. Methods Appl. Sci.*, vol. 44, no. 17, pp. 12 559–12 574, 2021, doi: 10.1002/mma.7563.

[11] F. Chen, "Extensions of the Hermite-Hadamard inequality for convex functions via fractional integrals," *J. Math. Inequal.*, vol. 10, no. 1, pp. 75–81, 2016, doi: 10.7153/jmi-10-07.

[12] H. Chen and U. N. Katugampola, "Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals," *J. Math. Anal. Appl.*, vol. 446, no. 2, pp. 1274–1291, 2017, doi: 10.1016/j.jmaa.2016.09.018.

[13] V. L. Chinchane and D. B. Pachpatte, "New fractional inequalities via Hadamard fractional integral," *Int. J. Funct. Anal. Oper. Theory Appl.*, vol. 5, no. 3, pp. 165–176, 2013.

[14] Y. Cho, I. Kim, and D. Sheen, "A fractional-order model for MINMOD Millennium," *Math. Biosci.*, vol. 262, pp. 36–45, 2015, doi: 10.1016/j.mbs.2014.11.008.

[15] J. V. da C. Sousa and E. C. de Oliveira, "The Minkowski's inequality by means of a generalized fractional integral," *AIMS Mathematics*, vol. 3, no. 1, pp. 131–147, 2018, doi: 10.3934/Math.2018.1.131.

[16] J. V. da C. Sousa, D. S. Oliveira, and E. C. de Oliveira, "Grüss-type inequalities by means of generalized fractional integrals," *Bull. Braz. Math. Soc. (N.S.)*, vol. 50, no. 4, pp. 1029–1047, 2019.

[17] Z. Dahmani, "On Minkowski and Hermite-Hadamard integral inequalities via fractional integration," *Ann. Funct. Anal.*, vol. 1, no. 1, pp. 51–58, 2010.

[18] J. Espíndola, J. Neto, and E. Lopes, "A generalised fractional derivative approach to viscoelastic material properties measurement," *Appl. Math. and Comp.*, vol. 164, pp. 493–506, 2005, doi: 10.1016/j.amc.2004.06.099.

[19] G. Farid, "Some new Ostrowski type inequalities via fractional integrals," *Int. J. Anal. Appl.*, vol. 14, pp. 64–68, 2017.

[20] P. Guzmán and J. Nápoles Valdés, "Generalized fractional Grüss-type inequalities," *Int. J. Anal. Appl.*, vol. 2, pp. 16–21, 2020, doi: 10.47443/cm.2020.0029.

[21] S. Habib, S. Mubeen, and M. N. Naeem, "Chebyshev type integral inequalities for generalized k-fractional conformable integrals," *J. Inequal. Spec. Funct.*, vol. 9, no. 4, pp. 53–65, 2018.

[22] G. H. Hardy, J. Littlewood, and G. Pólya, *Inequalities.* Cambridge University Press, Cambridge, England, 1934.

[23] K. Jangid, S. D. Purohit, K. S. Nisar, and S. Araci, "Chebyshev type inequality containing a fractional integral operator with a multi-index Mittag-Leffler function as a kernel," *Analysis (Berlin)*, vol. 41, no. 1, pp. 61–67, 2021, doi: 10.1515/anly-2020-0051.

[24] E. Kacar, Z. Kacar, and H. Yildirim, "Integral inequalities for Riemann-Liouville fractional integrals of a function with respect to another function," *Iran. J. Math. Sci. Inform.*, vol. 13, no. 1, pp. 1–13, 2018.

[25] U. Katugampola, "New fractional integral unifying six existing fractional integrals," 2016, *arXiv:1612.08596*.

[26] M. A. Khan, Y. Khurshid, T.-S. Du, and Y.-M. Chu, "Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals," *J. Funct. Spaces*, 2018, Art. ID 5357463, doi: 10.1155/2018/5357463.

[27] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, ser. North-Holland Mathematics Studies. Elsevier Science B.V., Amsterdam, 2006, vol. 204.

[28] R. L. Magin, "Fractional calculus models of complex dynamics in biological tissues," *Comput. Math. Appl.*, vol. 59, no. 5, pp. 1586–1593, 2010, doi: 10.1016/j.camwa.2009.08.039.

[29] B. Meftah and D. Foukrach, "Some new Gronwall-Bellman-Bihari type integral inequality associated with ψ-Hilfer fractional derivative," *Analysis (Berlin)*, vol. 43, no. 2, pp. 117–127, 2023.

[30] F. C. Meral, T. J. Royston, and R. Magin, "Fractional calculus in viscoelasticity: an experimental study," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 15, no. 4, pp. 939–945, 2010, doi: 10.1016/j.cnsns.2009.05.004.

[31] M. A. Moreles and R. Lainez, "Mathematical modelling of fractional order circuit elements and bioimpedance applications," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 46, pp. 81–88, 2017, doi: 10.1016/j.cnsns.2016.10.020.

[32] S. Mubeen, S. Habib, and M. N. Naeem, "The Minkowski inequality involving generalized k-fractional conformable integral," *J. Inequal. Appl.*, 2019, Art. ID 81, doi: 10.1186/s13660-019-2040-8.

[33] A. B. Nale, S. K. Panchal, and V. L. Chinchane, "Minkowski-type inequalities using generalized proportional Hadamard fractional integral operators," *Filomat*, vol. 35, no. 9, pp. 2973–2984, 2021.

[34] K. S. Nisar, G. Rahman, and K. Mehrez, "Chebyshev type inequalities via generalized fractional conformable integrals," *J. Inequal. Appl.*, 2019, Art. ID 245.

[35] S. Panwar, R. M. Pandey, P. Rai, S. D. Purohit, and D. L. Suthar, "Hermite-Hadamard-Mercer's type inequalities for ABK-fractional integrals via strong convexity and its applications," *Discontinuity, Nonlinearity, and Complexity*, vol. 13, no. 4, pp. 663–674, 2024, doi: 10.5890/DNC.2024.12.007.

[36] Ł. Płociniczak, "Analytical studies of a time-fractional porous medium equation. Derivation, approximation and applications," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 24, no. 1-3, pp. 169–183, 2015, doi: 10.1016/j.cnsns.2015.01.005.

[37] G. Rahman, A. Khan, T. Abdeljawad, and K. S. Nisar, "The Minkowski inequalities via generalized proportional fractional integral operators," *Adv. Difference Equ.*, 2019, Art. ID 287, doi: 10.1186/s13662-019-2229-7.

[38] S. Rashid, A. O. Akdemir, K. S. Nisar, T. Abdeljawad, and G. Rahman, "New generalized reverse Minkowski and related integral inequalities involving generalized fractional conformable integrals," *J. Inequal. Appl.*, 2020, Art. ID 177, doi: 10.1186/s13660-020-02445-2.

[39] S. Rashid, F. Jarad, and Y.-M. Chu, "A note on reverse Minkowski inequality via generalized proportional fractional integral operator with respect to another function," *Math. Probl. Eng.*, 2020, Art. ID 7630260, doi: 10.1155/2020/7630260.

[40] S. Rashid, F. Jarad, M. A. Noor, K. I. Noor, D. Baleanu, and J.-B. Liu, "On Grüss inequalities within generalized $\mathcal{K}$-fractional integrals," *Adv. Difference Equ.*, p. Art. ID 203, 2020.

[41] E. Reyes-Melo, J. Martinez-Vega, C. Guerrero-Salazar, and U. Ortiz-Mendez, "Application of fractional calculus to the modeling of dielectric relaxation phenomena in polymeric materials," *J. Appl. Polymer Sci.*, vol. 98, no. 2, pp. 923–935, 2005, doi: 10.1002/app.22057.

[42] E. Set, M. E. Özdemir, and S. Demirbaş, "Chebyshev type inequalities involving extended generalized fractional integral operators," *AIMS Math.*, vol. 5, no. 4, pp. 3573–3583, 2020, doi: 10.3934/math.2020232.

[43] H. Sun, Y. Zhang, D. Baleanu, W. Chen, and Y. Chen, "A new collection of real world applications of fractional calculus in science and engineering," *Communications in Nonlinear Science and Numerical Simulation*, vol. 64, pp. 213–231, 2018.

[44] S. Taf and K. Brahim, "Some new results using Hadamard fractional integral," *Int. J. Nonlinear Anal. Appl.*, pp. 103–109, 2016, doi: 10.22075/ijnaa.2015.299.

[45] V. E. Tarasov and V. V. Tarasova, "Macroeconomic models with long dynamic memory: fractional calculus approach," *Appl. Math. Comput.*, vol. 338, pp. 466–486, 2018, doi: 10.1016/j.amc.2018.06.018.

[46] F. Usta, H. Budak, and M. Z. Sarıkaya, "Some new Chebyshev type inequalities utilizing generalized fractional integral operators," *AIMS Math.*, vol. 5, no. 2, pp. 1147–1161, 2020.

[47] J. Vanterler da C. Sousa and E. Capelas de Oliveira, "On the ψ-Hilfer fractional derivative," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 60, pp. 72–91, 2018.

[48] H. Yaldiz and E. Set, "Some new Ostrowski type inequalities for generalized fractional integrals," *AIP Conference Proceedings*, pp. 103–109, 1991, doi: 10.1063/1.5047891.

[49] Ç. Yildiz, M. Emin Özdemir, and M. Zeki Sarikaya, "New generalizations of Ostrowski-like type inequalities for fractional integrals," *Kyungpook Math. J.*, vol. 56, no. 1, pp. 161–172, 2016, doi: 10.5666/KMJ.2016.56.1.161.