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Almost automorphic solutions for some nonautonomous evolution equations under the light of integrable dichotomy

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ABSTRACT

In this work, we prove the existence and uniqueness of μ-pseudo almost automorphic solutions for a class of semilinear nonautonomous evolution equations of the form: $u'(t) = A(t)u(t) + f(t, u(t))$, $t \in \mathbb{R}$ where $(A(t))_{t\in\mathbb{R}}$ is a family of closed linear operators acting in a Banach space X that generates an evolution family having an integrable dichotomy on $\mathbb{R}$ and $f : \mathbb{R} \times X \longrightarrow X$ is μ-pseudo almost automorphic with respect to t and Lipshitzian in the second variable. Moreover we provide an application illustrating our results.

RESUMEN

En este trabajo, demostramos la existencia y unicidad de soluciones μ-pseudo casi automorfas para una clase de ecuaciones de evolución semilineales no autónomas de la forma: $u'(t) = A(t)u(t)+f(t, u(t))$, $t \in \mathbb{R}$ donde $(A(t))_{t\in\mathbb{R}}$ es una familia de operadores lineales cerrados actuando en un espacio de Banach X que genera una familia de evolución que posee una dicotomía integrable en $\mathbb{R}$ y $f : \mathbb{R} \times X \longrightarrow X$ es μ-pseudo casi automorfa con respecto a t y Lipschitziana en la segunda variable. Más aún presentamos una aplicación ilustrando nuestros resultados.

Keywords and Phrases: Evolution family, delay evolution equations, exponential dichotomy, integrable dichotomy, μ-pseudo almost automorphic functions.

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1 Introduction

The current paper deals with the existence and uniqueness of μ-pseudo almost automorphic solutions for the following evolution equations:

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R} \tag{1.1}$$

and

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

and the perturbed delay system

$$u'(t) = A(t)u(t) + f(t, u(t), u(t-\tau)), \quad t \in \mathbb{R}, \tag{1.3}$$

where $(A(t), D(A(t)))$, $t \in \mathbb{R}$ is a family of closed linear operators that generates a strongly continuous evolution family $(U(t,s))_{t\geq s}$ on a Banach space X which has an integrable dichotomy on $\mathbb{R}$. The function f is μ-pseudo almost automorphic in t for each $x \in X$ and Lipschitzian with respect to the second and third arguments, $\tau > 0$ is a fixed constant . This work is a continuation of the works done in [21, 22].

In the theory of differential equations, exponential dichotomy is a classical concept and it plays a central role for getting important results. So, there exist many researchs on this topics see [15,20]. It is well-known that the concept of integrable dichotomy is a generalization of exponential dichotomy [1, 21, 22]. This concept was introduced by Pinto *et al.* [21], they proved the existence and uniqueness of bounded periodic solutions of nonlinear integro-differential equations with infinite delay. In [22], the authors proved the existence and uniqueness of almost periodic and pseudo-almost periodic mild solutions of equations (4.1) and (4.2) under the light of integrable bi-almost periodic Green's functions. In fact, the authors established some examples of purely integrale dichotomy (*i.e.*, which is not necessarily of exponential type). Recently, in [1], Abadias *et al.* investigate the semi-linear differential equation $x'(t) = A(t)x(t) + f(t, x(t), \varphi[\alpha(t, x(t))])$, $t \in \mathbb{R}$, where $(A(t), D(A(t)))$, $t \in \mathbb{R}$, generate an evolution family which has an integrable dichotomy. They obtained several results of existence and uniqueness of (ω, c)-periodic mild solutions under some assumptions on the nonlinear term. To our knowledge in the literature, there are few papers which deal with integrable dichotomy.

The concept of almost periodic functions is introduced by H. Bohr [12]. This notion has been much invested before being generalized by the concept of almost automorphic functions introduced by S. Bochner [8–11]. In [24], the authors introduced the notion of pseudo almost automorphic functions which is more general than the notion of almost automorphic functions. Moreover, they proved that the space $(PAA(\mathbb{R}, X), \|\cdot\|_0)$ is complete and they obtained an existence and uniqueness result of pseudo almost automorphic mild solutions to equation (4.1) in Banach spaces. In [4], Blot *et al.* introduced the notion of weighted pseudo almost automorphic functions which generalizes the concept of pseudo almost automorphic functions. For more details on these topics, one can see [19, 26]. More recently, the concept of μ-pseudo almost automorphy due to Ezzinbi *et al.* [5, 16] generalizes both notions of pseudo almost automorphy and weighted pseudo almost automorphy. For more details, one can see [4, 14, 17, 24].

In this work, our main results are Theorems 3.1 and 4.3. We show that equations (4.1) and (4.2) have respectively, unique bounded almost automorphic and μ-pseudo almost automorphic solutions. It should be noted that we obtained these results under light of integrable dichotomy, dominated convergence Theorem, Banach fixed point, standard and locally Lipschitz conditions. The nonlinear term f is in $PAA(\mathbb{R}, X, \mu)$.

The rest of this paper is organized as follows. Section 2 is devoted to some preliminaries. In sections 3 and 4, we present some criteria ensuring the existence of μ-pseudo almost automorphic mild solutions to equations (4.1) and (4.2). An example is given to illustrate our theoretical result in section 5.

2 Almost automorphic functions and integrable dichotomy

This section is concerned with some notations and preliminary facts that are used in the sequel of this work.

Definition 2.1 ([12]). *A continuous function $f : \mathbb{R} \to X$ is to be almost periodic if for every $\varepsilon > 0$, there exists $l_\varepsilon > 0$, such that for every $a \in \mathbb{R}$, there exists $\tau \in [a, a + l_\varepsilon]$ satisfying:*

$$\|f(t+\tau) - f(t)\| < \varepsilon \quad \text{for all } t \in \mathbb{R}$$

The space of all such functions is denoted by $\mathrm{AP}(\mathbb{R}, X)$.

Definition 2.2 ([9]). *A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for every sequence $(s'_n)_{n\geq 0}$ of real numbers, there exist a subsequence $(s_n)_{n\geq 0} \subset (s'_n)_{n\geq 0}$ and a measurable function $g : \mathbb{R} \to X$, such that*

$$g(t) = \lim_{n\to\infty} f(t+s_n) \quad \text{and} \quad f(t) = \lim_{n\to\infty} g(t-s_n) \quad \text{for all } t \in \mathbb{R}.$$

The space of all such functions is denoted by $AA(\mathbb{R}, X)$.

Remark 2.3 ([3]). *An almost automorphic function may not be uniformly continuous. Indeed, the real function* $f(t) = \sin\left(\dfrac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right)$ *for* $t \in \mathbb{R}$*, belongs to* $AA(\mathbb{R},\mathbb{R})$*, but is not uniformly continuous. Hence,* f *does not belong to* $AP(\mathbb{R},\mathbb{R})$*.*

Then, we have the following inclusions:

$$AP(\mathbb{R}, X) \subset AA(\mathbb{R}, X) \subset BC(\mathbb{R}, X).$$

Definition 2.4 ([3]). *A bounded continuous function* $f : \mathbb{R}\times X \to Y$ *is called almost automorphic if for each bounded set* $K \subset X$ *and for every sequence of real numbers* $\{\tau'_n\}_{n\geq 0}$*, there exist a subsequence* $\{\tau_n\}_{n\geq 0} \subset \{\tau'_n\}_{n\geq 0}$ *and a mesurable function* $\tilde{f} : \mathbb{R}\times X \to Y$*, such that*

$$\tilde{f}(t,x) = \lim_{n\to\infty} f(t+\tau_n, x) \text{ and } f(t,x) = \lim_{n\to\infty} \tilde{f}(t-\tau_n, x)$$

are well defined in $t \in \mathbb{R}$ *and* $x \in K \subset X$*.*

Definition 2.5 ([3]). *A continuous function* $F : \mathbb{R}\times\mathbb{R} \to X$ *is said to be bi-almost automorphic if for every sequence* $(s'_n)_{n\geq 0}$ *of real numbers, there exist a subsequence* $(s_n)_{n\geq 0} \subset (s'_n)_{n\geq 0}$ *and a measurable function* $G : \mathbb{R}\times\mathbb{R} \to X$*, such that*

$$G(t,s) = \lim_{n\to\infty} F(t+s_n, s+s_n) \quad \text{and} \quad F(t,s) = \lim_{n\to\infty} G(t-s_n, s-s_n) \quad \text{for all } t,s \in \mathbb{R}.$$

The space of all such functions is denoted by $bAA(\mathbb{R}, X)$.

2.1 μ-pseudo almost automorphic functions

This section is devoted to properties of μ-ergodic and μ-pseudo almost automorphic functions. In the sequel, we denote by $\mathcal{B}(\mathbb{R})$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}(\mathbb{R})$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$ for all $a, b \in \mathbb{R}$ with $(a \leq b)$, we denote also by Y any other Banach space. We assume the following hypothesis.

(M) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a+\tau : a \in A\}) \leq \beta\mu(A) \quad \text{where } A \in \mathcal{B}(\mathbb{R}) \text{ and } A \cap I = \emptyset.$$

Definition 2.6 ([6]). *Let* $\mu \in \mathcal{M}$*. A continuous bounded function* $f : \mathbb{R} \longrightarrow X$ *is called* μ*-ergodic, if*

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t)\| d\mu(t) = 0.$$

The space of all such functions is denoted by $\mathcal{E}(\mathbb{R}, X, \mu)$.

Proposition 2.7 ([6]). *Let $\mu \in \mathcal{M}$. Then,*

(i) $(\mathcal{E}(\mathbb{R}, X, \mu), \|\cdot\|_\infty)$ *is a Banach space.*

(ii) If μ satisfies **(M)**, *then $\mathcal{E}(\mathbb{R}, X, \mu)$ is translation invariant.*

Example 2.8. *(1) An ergodic function in the sense of Zhang [25] is a μ-ergodic function in the particular case where the measure μ is the Lebesgue measure.*

(2) Let $\rho : \mathbb{R} \longrightarrow [0, +\infty)$ be a $\mathcal{B}(\mathbb{R})$-measurable function. We define the positive measure μ on $\mathcal{B}(\mathbb{R})$ by

$$\mu(A) = \int_A \rho(t)dt \quad \text{for } A \in \mathcal{B}(\mathbb{R}),$$

where dt denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. The measure μ is absolutely continuous with respect to dt and the function ρ is called the Radon-Nikodym derivative of μ with respect to dt. In this case $\mu \in \mathcal{M}$ if and only if the function ρ is locally Lebesgue-integrable on $\mathbb{R}$ and it satisfies

$$\int_{\mathbb{R}} \rho(t)dt = +\infty.$$

(3) In [18], the authors considered the space of bounded continuous functions $f : \mathbb{R} \longrightarrow X$ satisfying

$$\lim_{r\to+\infty} \frac{1}{2r} \int_{[-r,r]} \|f(t)\|dt = 0 \quad \text{and} \quad \lim_{N\to+\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \|f(n)\| = 0.$$

This space coincides with the space of μ-ergodic functions where μ is defined in $\mathcal{B}(\mathbb{R})$ by the sum $\mu(A) = \mu_1(A) + \mu_2(A)$ with μ_1 is the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and

$$\mu_2(A) = \begin{cases} card(A \cap \mathbb{Z}) & \text{if } A \cap \mathbb{Z} \text{ is finite,} \\ \infty & \text{if } A \cap \mathbb{Z} \text{ is infinite.} \end{cases}$$

Definition 2.9 ([5]). *Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \longrightarrow X$ is said to be μ-pseudo almost automorphic if f is written in the form:*

$$f = g + \varphi,$$

where $g \in AA(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$.

The space of all such functions is denoted by $PAA(\mathbb{R}, X, \mu)$.

Proposition 2.10 ([5]). *Let $\mu \in \mathcal{M}$ satisfy* (**M**). *Then the following are true:*

(i) The decomposition of a μ-pseudo almost automorphic in the form $f = g + \varphi$ where $g \in AA(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$, is unique.

(ii) $PAA(\mathbb{R}, X, \mu)$ equipped with the supnorm is a Banach space.

Definition 2.11 ([7]). *A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is said to be almost automorphic in t uniformly with respect to $x \in X$ if the following two conditions hold:*

(i) For all $x \in X$, $f(\cdot, x) \in AA(\mathbb{R}, Y)$,

(ii) f is uniformly continuous on each compact $K \subset X$ with respect to the second variable x, namely, for each compact $K \subset X$, for all $\epsilon > 0$, there exists $\delta > 0$ such that all x_1,$x_2 \in K$, one has $\|x_1 - x_2\| \leq \delta \Rightarrow \sup_{t\in\mathbb{R}} \|f(t, x_1) - f(t, x_2)\| \leq \epsilon$.

Denote by $AAU(\mathbb{R} \times X, Y)$ the set of all such functions.

Definition 2.12. *Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is said to be μ-ergodic in t uniformly with with respect to $x \in X$, if the following two conditions hold:*

(i) For all $x \in X$, $f(\cdot, x) \in \mathcal{E}(\mathbb{R}, Y, \mu)$,

(ii) f is uniformly continuous on each compact $K \subset X$ with respect to the second variable x.

Denote by $\mathcal{E}U(\mathbb{R} \times X, Y, \mu)$ the set of all such functions.

Definition 2.13. *Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is said to be μ-pseudo almost automorphic in t uniformly with with respect to $x \in X$, if f is written in the form:*

$$f = g + h$$

where $g \in AAU(\mathbb{R} \times X, Y)$ and $h \in \mathcal{E}U(\mathbb{R} \times X, Y, \mu)$.

$PAAU(\mathbb{R} \times X, Y)$ denotes the set of such functions. We have

$$AAU(\mathbb{R} \times X, Y) \subset PAAU(\mathbb{R} \times X, Y).$$

Proposition 2.14 ([5]). *Let $\mu \in \mathcal{M}$ and $f : \mathbb{R} \times X \longrightarrow Y$ be a μ-pseudo almost automorphic in t uniformly with with respect to $x \in X$. Then*

(i) For all $x \in X$, $f(\cdot, x) \in PAA(\mathbb{R}, Y, \mu)$,

(ii) f is uniformly continuous on each compact $K \subset X$ with respect to the second variable x.

Theorem 2.15 ([5]). *Let $\mu \in \mathcal{M}$, $f \in PAAU(\mathbb{R} \times X, Y, \mu)$ and $x \in PAA(\mathbb{R}, X, \mu)$. Assume that the following hypothesis holds:*

(C) *For all bounded subset K of X, f is bounded on $\mathbb{R} \times K$.*

Then $[t \mapsto f(t, x(t))] \in PAA(\mathbb{R}, Y, \mu)$.

2.2 Integrable dichotomy

Let X and Y be any Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_Y$ respectively. Throughout this work we will assume that Y is densely and continuously imbedded in X *i.e.*, Y is a dense subspace of X and there is a constant C such that

$$\|\xi\| \leq C\|\xi\|_Y \quad \text{for } \xi \in Y.$$

Consider the following linear evolution equation:

$$\begin{cases} u'(t) = A(t)u(t), & t \geq s, \\ u(s) = x \in X, \end{cases} \tag{2.1}$$

The associated inhomogeneous equation is given by:

$$\frac{d}{dt}u(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{2.2}$$

where $f : \mathbb{R} \longrightarrow X$ is continuous and bounded.

Definition 2.16 ([20]). *Let X be a Banach space. The family $(A(t))_{t\geq 0}$ of infinitesimal generators of C_0-semigroup on X is called stable if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$(\omega, \infty) \subset \rho(A(t)) \quad \text{for } t \geq 0$$

and

$$\left\| \prod_{j=1}^{k} R(\lambda, A(t_j)) \right\| \leq M(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and for every finite sequence $\{t\}_{j=1}^{k}$ with $0 \leq t_1 \leq \cdots \leq t_k < \infty$ and $k = 1, 2, \ldots$

Definition 2.17. *For each $t \in \mathbb{R}$, let $A(t)$ be the infinitesimal generator of a C_0 semigroup $T_t(s)$, $s \in \mathbb{R}$, on X. A subspace Y of X is called $A(t)$-admissible if it is an invariant subspace of $T_t(s)$, $s \in \mathbb{R}$, and the restriction of $T_t(s)$ to Y is a C_0 semigroup in Y (i.e. it is strongly continuous in the norm $\|\cdot\|_Y$).*

We will make the following assumptions.

($\mathbf{A}_1$) $(A(t))_{t\in\mathbb{R}}$ is a stable family with stability constants M, ω.

($\mathbf{A}_2$) Y is $A(t)$-admissible for $t \in \mathbb{R}$ and the family $(\tilde{A}(t))_{t\in\mathbb{R}}$ of parts $\tilde{A}(t)$ of $A(t)$ in Y, is a stable family in Y with stability constants $\tilde{M}$, $\tilde{\omega}$.

($\mathbf{A}_3$) For each $t \in \mathbb{R}$, $D(A(t)) \supset Y$, $A(t)$ is a bounded operator from Y into X and $t \to A(t)$ is continuous in the $B(Y, X)$ norm $\|\cdot\|_{Y\to X}$.

It is well known that if a family $(A(t))_{t\in\mathbb{R}}$ satisfies conditions ($\mathbf{A}_1$)-($\mathbf{A}_3$), then one can associate a unique evolution family $(U(t,s))_{s\le t}$ with the equation (2.1), (see [15, 20]). Throughout this work $(A(t), D(A(t)))$, $t \in \mathbb{R}$ satisfies conditions ($\mathbf{A}_1$)-($\mathbf{A}_3$).

Definition 2.18 ([15, 20])**.** *An evolution family $(U(t,s))_{s\le t}$ on a Banach space X is said to have an exponential dichotomy (or hyperbolic) in $\mathbb{R}$ if there exists a family of projections $P(t) \in \mathcal{L}(X)$, $t \in \mathbb{R}$, being strongly continuous with respect to t, and constants $\delta, M > 0$ such that*

(i) $U(t,s)P(s) = P(t)U(t,s)$,

(ii) $U(t,s) : Q(s)X \to Q(t)X$ *is invertible with the inverse* $\tilde{U}(t,s)$,

(iii) $\|U(t,s)P(s)\| \le Me^{-\delta(t-s)}$ *and* $\|\tilde{U}(t,s)Q(t)\| \le Me^{-\delta(t-s)}$,

for all $t, s \in \mathbb{R}$ with $s \le t$, where, $Q(t) = I - P(t)$.

Definition 2.19. *Let $(U(t,s))_{s\le t}$ have an exponential dichotomy. We define the Green function by:*

$$G(t,s) = \begin{cases} U(t,s)P(s), & t,s \in \mathbb{R}, \quad s \le t \\ -\tilde{U}(t,s)Q(s), & t,s \in \mathbb{R}, \quad s > t. \end{cases}$$

For a given evolution family $(U(t,s))_{s\le t}$ associated to equation (2.1), that has an dichotomy exponential, the Green function associated to the evolution family satisfies

$$\|G(t,s)\| = \begin{cases} Me^{-\delta(t-s)}, & \text{if } t \ge s \\ Me^{-\delta(s-t)}, & \text{if } s > t. \end{cases}$$

where $M > 0$ and $\delta > 0$ are positive constant.

Definition 2.20 ([22])**.** *We say that equation* (2.1) *has an integrable dichotomy with data* (λ, P) *if there are projections $P(t)$, $t \in \mathbb{R}$, uniformly bounded and strongly continuous in t satisfying* (*i*) *and* (*ii*), *with $Q(t) = I - P(t)$ and there exists a function $\lambda : \mathbb{R}^2 \to (0, \infty)$ such that*

$$\|G(t,s)\| \le \lambda(t,s), \quad \text{for all } t,s \in \mathbb{R}, \tag{2.3}$$

and

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}} \lambda(t,s)ds \le L < \infty. \tag{2.4}$$

In the pseudo almost automorphic context, we will make the following additional assumption for the function $\lambda(t,s)$ in Definition 2.20.

(A) Let $\lambda_1 : (-\infty, -T) \to (0,\infty)$ and $\lambda_2 : (T,\infty) \to (0,\infty)$ defined by $\lambda_1(s) = \int_{-T}^{T} \lambda(t,s)d\mu(t)$, $\lambda_2(s) = \int_{-T}^{T} \lambda(t,s)d\mu(t)$ for all $T > 0$. We assume that there exists a constant $C > 0$ such that for all $T > 0$,

$$\int_s^T \lambda(t,s)d\mu(t) \le C, \text{ and } \int_{-T}^s \lambda(t,s)d\mu(t) \le C, \tag{2.5}$$

$$\int_{-\infty}^{-T} \lambda_1(s)ds \le C, \text{ and } \int_T^{\infty} \lambda_2(s)ds \le C. \tag{2.6}$$

Remark 2.21. *We notice that some differences between exponential dichotomy and integrable dichotomy. In the case of exponential dichotomy, if we consider the Lebesgue mesure on $\mathcal{B}(\mathbb{R})$, the constante C quoted in* **(A)** *is equal to $\max\{\frac{M}{\delta}, \frac{M}{\delta^2}\}$ and $L = 2\frac{M}{\delta}$. Indeed, for $T > 0$, we have*

$$\int_{\mathbb{R}} G(t,s)ds = M\int_{-\infty}^{t} e^{-\delta(t-s)}ds + M\int_{t}^{\infty} e^{-\delta(s-t)}ds = 2\frac{M}{\delta} = L, \tag{2.7}$$

$$\text{for } t \ge s, \quad M\int_s^T e^{-\delta(t-s)}dt = \frac{M}{\delta}\left[-e^{-\delta(T-s)} + 1\right] \le \frac{M}{\delta}, \tag{2.8}$$

$$\text{for } t \ge s, \quad M\int_{-\infty}^{-T}\int_{-\infty}^{T} e^{-\delta(t-s)}dtds = \frac{M}{\delta}\left(e^{\delta T} - e^{-\delta T}\right)\int_{-\infty}^{-T} e^{\delta s}ds \le \frac{M}{\delta^2}. \tag{2.9}$$

If $t < s$, we obtain the same results. Moreover a system that admits integrable dichotomy is not necessarily exponentially stable what means that integrable dichotomy is more general than exponential dichotomy. For more details, one can see [13, 22].

Theorem 2.22 ([21])**.** *Assume that equation* (2.1) *has an integrable dichotomy and f is a bounded function. Then equation* (2.2) *has a unique bounded integral solution given by*

$$u(t) = \int_{\mathbb{R}} G(t,s)f(s)ds, \quad t \in \mathbb{R}. \tag{2.10}$$

3 Almost automorphic and pseudo almost automorphic solutions in the nonhomogeneous linear case

(H1) We assume that $(A(t))_{t\in\mathbb{R}}$ generates an evolution family $\{U(t,s)\}_{(s\leq t\in\mathbb{R})}$, on X *i.e.* $(A(t), D(A(t)))$, $t\in\mathbb{R}$ satisfy conditions $(\mathbf{A}_1)$-$(\mathbf{A}_3)$.

(H2) The evolution family $U(t,s)$ generated by $A(t)$ has an integrable dichotomy satisfying (2.3) with function λ, dichotomy projections $P(t)$, $t\in\mathbb{R}$, and Green's function $G(t,s)$.

(H3) The Green's function $G(t,s)x$ function is bi-almost automorphic in $t,s\in\mathbb{R}$, for all $x\in X$.

We first consider the nonhomogeneous linear case

$$u'(t) = A(t)u(t) + f(t), \tag{3.1}$$

where $f:\mathbb{R}\to X$ is a function.

3.1 Almost automorphic solutions of equation (3.1)

Theorem 3.1. *Assume that* **(H1)**, **(H2)** *hold and* $f\in AA(\mathbb{R},X)$. *Then equation* (3.1) *has a unique almost automorphic mild solution given by*

$$u(t) = \int_{\mathbb{R}} G(t,s)f(s)ds, \quad t\in\mathbb{R}. \tag{3.2}$$

Proof. By the Theorem 2.22, u is a unique mild solution to equation (3.1). Now, it remains to show that $u\in AA(\mathbb{R},X)$. Let $\{\tau_n'\}$ be a sequence of real numbers. Since $f\in AA(\mathbb{R},X)$, there exists a subsequence $\{\tau_n\}$ of $\{\tau_n'\}$ such that

$$\lim_n G(t+\tau_n, s+\tau_n) = \tilde{G}(t,s), \text{ and } \lim_n \tilde{G}(t-\tau_n, s-\tau_n) = G(t,s),$$

$\tilde{f}(t) = \lim_{n\to\infty} f(t+s_n)$ and $f(t) = \lim_{n\to\infty}\tilde{f}(t-s_n)$ for each $t,s\in\mathbb{R}$. Now, we define

$$\tilde{u}(t) = \int_{\mathbb{R}} \tilde{G}(t,s)\tilde{f}(s)ds, \quad t\in\mathbb{R}.$$

Note that

$$\begin{aligned}\|u(t+\tau_n)-\tilde{u}(t)\| &= \left\|\int_{\mathbb{R}} G(t+\tau_n,s)f(s)ds - \int_{\mathbb{R}}\tilde{G}(t,s)\tilde{f}(s)ds\right\| \\ &= \left\|\int_{\mathbb{R}} G(t+\tau_n,s+\tau_n)f(s+\tau_n)ds - \int_{\mathbb{R}}\tilde{G}(t,s)\tilde{f}(s)ds\right\|\end{aligned}$$

$$\leq \int_{\mathbb{R}} \left\| G(t+\tau_n, s+\tau_n)\left[f(s+\tau_n) - \tilde{f}(s)\right]\right\| ds$$
$$+ \int_{\mathbb{R}} \left\| \left[G(t+\tau_n, s+\tau_n) - \tilde{G}(t,s)\right] \tilde{f}(s)\right\| ds.$$

Let

$$I_{1,n} := \int_{\mathbb{R}} G(t+\tau_n, s+\tau_n)\left[f(s+\tau_n) - \tilde{f}(s)\right] ds$$

and

$$I_{2,n} := \int_{\mathbb{R}} \left[G(t+\tau_n, s+\tau_n) - \tilde{G}(t,s)\right] \tilde{f}(s) ds.$$

We have

$$I_{1,n} \leq \int_{\mathbb{R}} \lambda(t,s)\left[f(s+\tau_n) - \tilde{f}(s)\right] ds.$$

Since $f \in AA(\mathbb{R}, X)$ and by the dominated convergence Theorem, it follows that $I_{1,n} \to 0$ as $n \to \infty$.

For $I_{2,n}$ since $G(t,s)$ is bi-almost automorphic, given $\varepsilon > 0$, there is $N > 0$ such that for $n \geq N$, we have

$$\|G(t+\tau_n, s+\tau_n)\tilde{f}(s) - \tilde{G}(t,s)\tilde{f}(s)\| < \varepsilon \|f\|_\infty, \quad t, s \in \mathbb{R},$$

so for $n \geq N$,

$$I_{2,n} \leq \int_{\mathbb{R}} \|G(t+\tau_n, s+\tau_n)\tilde{f}(s) - \tilde{G}(t,s)\tilde{f}(s)\| ds$$

Thus, by the dominated convergence Theorem we have that $I_{2,n} \to 0$ as $n \to \infty$. Thus $\lim_n u(t+\tau_n) = \tilde{u}(t)$. We can show in a similar way that $\lim_n \tilde{u}(t-\tau_n) = u(t)$. Hence, $\lim_n u(t+\tau_n) = \tilde{u}(t)$ and $\lim_n \tilde{u}(t-\tau_n) = u(t)$, for $t \in \mathbb{R}$. Therefore, we conclude that $u \in AA(\mathbb{R}, X)$. □

Theorem 3.2. *Let $\mu \in \mathcal{M}$. Assume that* **(H1)**-**(H3)** *are satisfied and $f \in PAA(\mathbb{R}, X, \mu)$. Let u be a bounded solution of equation* (3.1). *Then $u \in PAA(\mathbb{R}, X, \mu)$.*

Proof. Let $f = g + h \in PAA(\mathbb{R}, X, \mu)$, where $g \in AA(\mathbb{R}, X)$ and $h \in \mathcal{E}(\mathbb{R}, X, \mu)$. Then u has a unique decomposition:

$$u = u_1 + u_2$$

where, for all $t \in \mathbb{R}$, we have

$$u_1(t) = \int_{\mathbb{R}} G(t,s)g(s)ds$$

and

$$u_2(t) = \int_{\mathbb{R}} G(t,s)h(s)ds$$

Using Theorem 3.1, we obtain that $u_1 \in AA(\mathbb{R}, X)$. It remains to show that $u_2 \in \mathcal{E}(\mathbb{R}, X, \mu)$. Let

$r > 0$. Then,

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t)\|\,d\mu(t) &= \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{\mathbb{R}} G(t,s)h(s)ds\right\| d\mu(t)\\ &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{-\infty}^{t} G(t,s)h(s)ds\right\| d\mu(t)\\ &+ \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{t}^{\infty} G(t,s)h(s)ds\right\| d\mu(t).\end{aligned}$$

For any fixed $r > 0$, we have

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{-\infty}^{t} G(t,s)h(s)ds\right\| d\mu(t) &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{-\infty}^{-r}\|G(t,s)h(s)\|ds\,d\mu(t)\\ &+ \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{-r}^{t}\|G(t,s)h(s)\|ds\,d\mu(t)\\ &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{-\infty}^{-r}\lambda(t,s)\|h(s)\|ds\,d\mu(t)\\ &+ \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{-r}^{t}\lambda(t,s)\|h(s)\|ds\,d\mu(t).\end{aligned}$$

By assumption (**H3**) and by changing the order of integration, we have

$$\int_{-r}^{r}\int_{-\infty}^{-r}\lambda(t,s)\|h(s)\|dsd\mu(t) := \int_{-\infty}^{-r}\left(\int_{-r}^{r}\lambda(t,s)d\mu(t)\right)\|h(s)\|ds \leq \|h\|_\infty\int_{-\infty}^{-r}\lambda_1(s)ds \leq C\|h\|_\infty,$$

and

$$\int_{-r}^{r}\int_{-r}^{t}\lambda(t,s)\|h(s)\|ds\,d\mu(t) := \int_{-r}^{r}\left(\int_{t}^{r}\lambda(t,s)d\mu(t)\right)\|h(s)\|ds \leq C\int_{-r}^{r}\|h(s)\|ds.$$

By a similary way, we have

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{t}^{\infty} G(t,s)h(s)ds\right\| d\mu(t) &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{t}^{\infty}\|G(t,s)h(s)\|ds\,d\mu(t)\\ &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{t}^{\infty}\lambda(t,s)\|h(s)\|ds\,d\mu(t)\\ &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{t}^{r}\lambda(t,s)\|h(s)\|ds\,d\mu(t)\\ &+ \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{r}^{\infty}\lambda(t,s)\|h(s)\|ds\,d\mu(t).\end{aligned}$$

By assumption (**H3**) and by changing the order of integration, we have

$$\int_{-r}^{r}\int_{t}^{r}\lambda(t,s)\|h(s)\|ds\,d\mu(t) := \int_{-r}^{r}\left(\int_{-r}^{s}\lambda(t,s)d\mu(t)\right)\|h(s)\|ds \leq C\int_{-r}^{r}\|h(s)\|ds,$$

and

$$\int_{-r}^{r}\int_{r}^{\infty}\lambda(t,s)\|h(s)\|ds\,d\mu(t)=\int_{r}^{\infty}\left(\int_{-r}^{s}\lambda(t,s)d\mu(t)\right)\|h(s)\|ds\le\|h\|_\infty\int_{r}^{\infty}\lambda_2(s)ds\le C\|h\|_\infty.$$

Thus, we have

$$\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t)\|d\mu(t)\le\frac{2C}{\mu([-r,r])}\left(\|h\|_\infty+\int_{-r}^{r}\|h(s)\|ds\right). \tag{3.3}$$

From (3.3), we claim that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t)\|d\mu(t)=0.$$

Hence, $u_2\in PAA(\mathbb{R},X,\mu)$. We obtain the proof of the theorem. □

4 μ-pseudo almost automorphic solutions of equations (4.1) and (4.2)

Let X and Y be Banach spaces and $BC(\mathbb{R}\times X,Y)$ be the Banach space of all bounded continuous functions from $\mathbb{R}\times X$ in Y with the supremum norm of $\|\cdot\|_\infty$. In this section, we consider the nonlinear differential equation (4.1), where $f:\mathbb{R}\times X\to X$ is a function under convenient conditions,

$$u'(t)=A(t)u(t)+f(t,u(t)),\quad t\in\mathbb{R}, \tag{4.1}$$

and we analyze the delay case, were $\tau>0$ is fixed,

$$u'(t)=A(t)u(t)+f(t,u(t),u(t-\tau)),\quad t\in\mathbb{R}. \tag{4.2}$$

Definition 4.1. *A bounded continuous function $u:\mathbb{R}\to X$ is called a mild solution of equation* (4.1) *if*

$$u(t)=\int_{\mathbb{R}}G(t,s)f(s,u(s),u(s-\tau))ds,\quad t\in\mathbb{R}. \tag{4.3}$$

Definition 4.2. *A bounded continuous function $u:\mathbb{R}\to X$ is called a mild solution of equation* (4.2) *if*

$$u(t)=\int_{\mathbb{R}}G(t,s)f(s,u(s))ds,\quad t\in\mathbb{R}. \tag{4.4}$$

4.1 Existence of almost automorphic solutions to equation (4.1)

We need the following additional assumption:

(H4) There exists $\kappa > 0$ constant such that

$$\|f(t,u_1) - f(t,u_2)\| \leq \kappa \|u_1 - u_2\|, \quad \text{for all } t \in \mathbb{R},\, u_1, u_2 \in X. \tag{4.5}$$

Theorem 4.3. *Let $\mu \in \mathcal{M}$ satisfy* **(M)**. *Asumme that* **(H1)**-**(H4)** *hold and $f \in PAA(\mathbb{R}\times X, X, \mu)$ with*

$$\kappa < \frac{1}{L}$$

Then, equation (4.1) *has a unique mild solution $u \in PAA(\mathbb{R}, X, \mu)$ given by*

$$u(t) = \int_{\mathbb{R}} G(t,s) f(s, u(s)) ds, \quad t \in \mathbb{R}.$$

Proof. Let define the functional Λ on $PAA(\mathbb{R}, X, \mu)$ by

$$(\Lambda\phi)(t) = \int_{\mathbb{R}} G(t,s) f(s, \phi(s)) ds, \quad t \in \mathbb{R}.$$

By the composition Theorem 2.15 and Theorem 3.2, one has $\Lambda(PAA(\mathbb{R}, X, \mu)) \subset PAA(\mathbb{R}, X, \mu)$. Moreover we prove existence and uniqueness of solution to equation (4.1). Considering the fact that $\|f\|_\infty < \infty$, for all $t \in \mathbb{R}$, we have

$$\|(\Lambda\phi)(t)\| \leq \int_{-\infty}^{\infty} \|G(t,s) f(s,\phi(s))\| ds \leq \int_{-\infty}^{\infty} \lambda(t,s) \|f(s,\phi(s))\| ds \leq \|f\|_\infty \int_{-\infty}^{\infty} \lambda(t,s) ds \leq L\|f\|_\infty.$$

This proves that $\Lambda\phi$ is bounded. Now, we will prove that Λ is a contraction.

$$\begin{aligned}
\|(\Lambda\phi)(t) - (\Lambda\varphi)(t)\| &\leq \int_{-\infty}^{\infty} \|G(t,s)\| \|f(s,\phi(s)) - f(s,\varphi(s))\| ds \\
&\leq \int_{-\infty}^{\infty} \lambda(t,s) \|f(s,\phi(s)) - f(s,\varphi(s))\| ds \\
&\leq \kappa \|\phi - \varphi\|_\infty \int_{\mathbb{R}} \lambda(t,s) ds \leq \kappa L \|\phi - \varphi\|_\infty.
\end{aligned}$$

Therefore, by the Banach fixed point theorem, Λ has a unique fixed point such that $\Lambda\phi = \phi$, which is a μ-pseudo almost automorphic mild solution of equation (4.1). □

4.2 Existence of almost automorphic solutions to equation (4.2)

We need the following additional assumption:

(H5) The function $f(t,u,v)$ is locally Lipschitz in $u,v \in X$ *i.e.* for each positive number θ, for all, u_1,u_2,v_1,v_2 with $\|u_i\| \le \theta$, $\|v_i\| \le \theta$, $i=1,2$

$$\|f(t,u_1,v_1)-f(t,u_2,v_2)\| \le k_1(\theta)\|u_1-u_2\| + k_2(\theta)\|v_1-v_2\|, \tag{4.6}$$

where $k_1,k_2 : [0,\infty) \to [0,\infty)$ are functions and there is a positive constant ρ, such that $2\max(k_1(\rho),k_2(\rho)) < \frac{1}{L}$ and $\sup_{t\in\mathbb{R}} \|f(t,0,0)\| \le \frac{\rho}{L}\left[1-2L\max(k_1(\rho),k_2(\rho))\right]$.

Theorem 4.4. *Assume that* **(H1)**-**(H3)** *and f hold* **(H5)**. *Then, equation* (4.2) *has a unique bounded solution $u(t)$, $t\in\mathbb{R}$, with $\|u\|_\infty \le \rho$.*

Proof. Let $G(t,s)$ be the Green's function associated with the equation (4.2) and we define the functional on X by

$$(\Gamma\phi)(t) = \int_{-\infty}^{\infty} G(t,s)f(s,\phi(s),\phi(s-\tau))ds, \quad t\in\mathbb{R}.$$

We show that Γ has a fixed point. First, we prove that Γ is bounded. There are ρ constant positive and a ball $\overline{\mathcal{B}}(0,\rho)$ which satisfies assumption (**H5**). Thus, we have,

$$\begin{aligned}
\|(\Gamma\phi)(t)\| &\le \int_{-\infty}^{\infty} \|G(t,s)f(s,\phi(s),\phi(s-\tau))\|ds \le \int_{-\infty}^{\infty} \lambda(t,s)\|f(s,\phi(s),\phi(s-\tau))\|ds \\
&\le (k_1(\rho)+k_2(\rho))\int_{-\infty}^{\infty}\lambda(t,s)\|\phi(s)\|ds + \int_{-\infty}^{\infty}\lambda(t,s)\|f(s,0,0)\|ds \\
&\le L(k_1(\rho)+k_2(\rho))\|\phi\|_\infty + L\sup_{t\in\mathbb{R}}\|f(t,0,0)\| \\
&\le 2L\max(k_1(\rho),k_2(\rho))\rho + \rho\left[1-2L\max(k_1(\rho),k_2(\rho))\right] \le \rho
\end{aligned}$$

This proves that $\Gamma\phi \in \overline{\mathcal{B}}(0,\rho)$ for all $\phi\in\overline{\mathcal{B}}(0,\rho)$. Finally, we prove that Γ is a contraction in $\overline{\mathcal{B}}(0,\rho)$. In fact,

$$\begin{aligned}
\|(\Gamma\phi)(t)-(\Gamma\varphi)(t)\| &\le \int_{-\infty}^{\infty}\|G(t,s)\|\|f(s,\phi(s),\phi(s-\tau))-f(s,\varphi(s),\varphi(s-\tau))\|ds \\
&\le \int_{-\infty}^{\infty}\lambda(t,s)\|f(s,\phi(s),\phi(s-\tau))-f(s,\varphi(s),\varphi(s-\tau))\|ds \\
&\le L\int_{-\infty}^{\infty}\|k_1(\rho)\|\phi(s)-\varphi(s)\| + k_2(\rho)\|\phi(s-\tau)-\varphi(s-\tau)\|ds \\
&\le L(k_1(\rho)+k_2(\rho))\|\phi-\varphi\|_\infty.
\end{aligned}$$

Using Banach fixed point Theorem, we deduce by **(H5)** that Γ has a fixed point ϕ. $\square$

Now, we will prove that equation (4.2) has an almost automorphic solution.

Theorem 4.5. *Assume that* **(H1)-(H3)** *and* **(H5)** *hold and* $f \in AA(\mathbb{R} \times X \times X, X)$*. Then, equation* (4.2) *has a unique almost automorphic mild solution* $u(t)$*,* $t \in \mathbb{R}$*, with* $\|u\|_\infty \leq \rho$*.*

Proof. We define the functional on X as in Theorem 4.4 by

$$(\Gamma\phi)(t) = \int_{-\infty}^{\infty} G(t,s)f(s,\phi(s),\phi(s-\tau))ds, \quad t \in \mathbb{R}.$$

We show that $\Gamma(AA(\mathbb{R},X)) \subset AA(\mathbb{R},X)$. Since $f \in AA(\mathbb{R} \times X \times X, X)$, and for each $u \in \overline{\mathcal{B}}(0,\rho)$ there exists a subsequence $\{\tau_n\}$ of $\{\tau_n'\}$ such that

$$\lim_n G(t+\tau_n, s+\tau_n)x - \tilde{G}(t,s)x = 0, \quad \text{and} \quad \lim_n \tilde{G}(t-\tau_n, s-\tau_n)x - G(t,s)x = 0,$$

$$\tilde{f}(t,u(t),u(t-\tau)) = \lim_{n\to\infty} f(t+s_n, u(t+s_n), u(t+s_n-\tau))$$

and

$$f(t) = \lim_{n\to\infty} \tilde{f}(t-s_n, u(t-s_n), u(t-s_n-\tau))$$

for each $t, s \in \mathbb{R}$, $x \in K$. Thus, we have

$$\tilde{\Gamma u}(t) = \int_{\mathbb{R}} \tilde{G}(t,s)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))ds, \quad t \in \mathbb{R}.$$

Note that

$$\begin{aligned}
\|\Gamma u(t+\tau_n) - \tilde{\Gamma u}(t)\| &= \left\| \int_{\mathbb{R}} G(t+\tau_n,s)f(s,u(s),u(s-\tau))ds - \int_{\mathbb{R}} \tilde{G}(t,s)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))ds \right\| \\
&= \left\| \int_{\mathbb{R}} G(t+\tau_n, s+\tau_n)f(s+\tau_n, u(s+s_n), u(s+s_n-\tau))ds \right. \\
&\left. - \int_{\mathbb{R}} \tilde{G}(t,s)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))ds \right\| \\
&\leq \int_{\mathbb{R}} \left\| G(t+\tau_n, s+\tau_n) \left[f(s+\tau_n, u(s+s_n), u(s+s_n-\tau)) - \tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau)) \right] \right\| ds \\
&+ \int_{\mathbb{R}} \left\| \left[G(t+\tau_n, s+\tau_n) - \tilde{G}(t,s) \right] \tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau)) \right\| ds.
\end{aligned}$$

Let

$$J_{1,n} := \int_{\mathbb{R}} G(t+\tau_n, s+\tau_n) \left[f(s+\tau_n, u(s+s_n), u(s+s_n-\tau)) - \tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau)) \right] ds$$

and

$$J_{2,n} := \int_{\mathbb{R}} \left[G(t+\tau_n, s+\tau_n) - \tilde{G}(t,s) \right] \tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))ds.$$

We have

$$J_{1,n} \leq \int_{\mathbb{R}} \lambda(t,s)\left[f(s+\tau_n, u(s+s_n), u(s+s_n-\tau)) - \tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))\right] ds.$$

Since $f \in AA(\mathbb{R}\times X\times X, X)$ and by the dominated convergence theorem, it follows that $J_{1,n} \to 0$ as $n \to \infty$.

For $J_{2,n}$ since $G(t,s)$ is bi-almost automorphic, given $\varepsilon > 0$, there is $N > 0$ such that for $n \geq N$, we have

$$\|G(t+\tau_n, s+\tau_n)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau) - \tilde{G}(t,s)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))\| < \varepsilon\|f\|_\infty, \quad t,s \in \mathbb{R},$$

so for $n \geq N$,

$$J_{2,n} \leq \int_{\mathbb{R}} \|G(t+\tau_n, s+\tau_n)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau) - \tilde{G}(t,s)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))\|ds.$$

Thus, by the dominated convergence theorem we have that $J_{2,n} \to 0$ as $n \to \infty$. Thus $\lim_n \Gamma u(t+\tau_n) = \tilde{\Gamma u}(t)$. We can show in a similar way that $\lim_n \tilde{\Gamma u}(t-\tau_n) = \Gamma u(t)$. Hence, $\lim_n \Gamma u(t+\tau_n) = \tilde{\Gamma u}(t)$ and $\lim_n \tilde{\Gamma u}(t-\tau_n) = \Gamma u(t)$, for $t \in \mathbb{R}$. By Theorem 3.2, equation (4.2) has a unique bounded mild solution $u(t)$, $t \in \mathbb{R}$, with $\|u\|_\infty \leq \rho$ and $u \in AA(\mathbb{R}, X)$. □

Theorem 4.6. *Let $\mu \in \mathcal{M}$ and μ satisfy* **(M)**. *Assume that* **(H1)**-**(H3)** *and* **(H5)** *hold and $f \in PAA(\mathbb{R}\times X\times X, X, \mu)$. Then, equation (4.2) has a unique μ-pseudo almost automorphic mild solution $u(t)$, $t \in \mathbb{R}$, with $\|u\|_\infty \leq \rho$.*

Proof. We define the functional on X as in Theorem 4.4 by

$$\Gamma\phi(t) = \int_{-\infty}^{\infty} G(t,s)f(s,\phi(s),\phi(s-\tau))ds, \quad t \in \mathbb{R}.$$

By Theorem 4.4, equation (4.2) has a unique bounded mild solution $u(t)$, $t \in \mathbb{R}$, with $\|u\|_\infty \leq \rho$. Let $f = g + h \in PAA(\mathbb{R}\times X\times X, X, \mu)$ where $g \in AA(\mathbb{R}\times X\times X, X)$ and $h \in \mathcal{E}(\mathbb{R}\times X\times X, X, \mu)$. Thus, $\Gamma\phi$ has a unique decomposition:

$$\Gamma\phi(t) = u_1(t) + u_2(t)$$

where, for all $t \in \mathbb{R}$, we have

$$u_1(t) = \int_{\mathbb{R}} G(t,s)g(s,u(s),u(s-\tau))ds$$

and

$$u_2(t) = \int_{\mathbb{R}} G(t,s)h(s,u(s),u(s-\tau))ds.$$

Using Theorem 4.5, we obtain that $u_1 \in AA(\mathbb{R}, X)$. It remains to show that $u_2 \in \mathcal{E}(\mathbb{R}, X, \mu)$. Let $r > 0$. Then,

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t)\|\,d\mu(t) &= \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{\mathbb{R}} G(t,s)h(s,u(s),u(s-\tau))ds\right\| d\mu(t)\\ &\le \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{-\infty}^{t} G(t,s)h(s,u(s),u(s-\tau))ds\right\| d\mu(t)\\ &+ \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{t}^{\infty} G(t,s)h(s,u(s),u(s-\tau))ds\right\| d\mu(t)\end{aligned}$$

For any fixed $r > 0$, by calculations similar as to the Theorem 3.2, we have

$$\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t)\|d\mu(t) \le \frac{2C}{\mu([-r,r])}\left(\|h\|_\infty + \int_{-r}^{r}\|h(s)\|ds\right) \tag{4.7}$$

From (4.7), we claim that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t)\|d\mu(t) = 0$$

Hence, $u_2 \in PAA(\mathbb{R}, X, \mu)$. We obtain the proof of the Theorem. □

5 Applications

In the next example, we show that integrable dichotomy is a generalization of exponential dichotomy.

Example 5.1. *We give an example of family of operators $(A(t))_{t\in\mathbb{R}}$ that generates an evolution family with an integrable dichotomy. Let $\{b_k\}_{k\in\mathbb{Z}}$ be a positive Riemamn sequence such that $b_k = \frac{1}{k^2+1}$. Let $J_k := [k - b_k^2, k + b_k^2]$, for $k \in \mathbb{Z}$. Let $\ell : \mathbb{R} \to (0,\infty)$ be continuously differentiable function given by $\ell(t) = 1$, if $t \notin J_k$ and in J_k, $\ell(t) \in \left[\frac{1}{k^2+1}, 1\right]$ where $\ell(k) = b_k$. We have*

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\int_{J_k}\ell^{-1}(s)ds &= \sum_{k\in\mathbb{Z}}\int_{k-\frac{1}{(k^2+1)^2}}^{k+\frac{1}{(k^2+1)^2}}(k^2+1)ds = 2\sum_{k\in\mathbb{Z}}\frac{k^2+1}{(k^2+1)^2}\\ &\le 2\left(1 + 2\sum_{k=1}^{\infty}\frac{1}{k^2}\right) \le 2\left(\frac{\pi^2}{3}+1\right) < \infty.\end{aligned}$$

Consider the scalar differential equation

$$u'(t) = a(t)u(t), \quad a(t) = -\alpha + \ell'(t)\ell(t)^{-1}, \quad \alpha > 0, \tag{5.1}$$

one has

$$u(t) = u_0 e^{-\alpha t}\ell(t) \text{ where } u_0 \text{ is the initial data .}$$

It is well-known that the evolution family of the equation (5.1) *with projections* $P(t) = I$, $t \in \mathbb{R}$ *is given by* $U(t,s) = e^{-\alpha(t-s)}\frac{\ell(t)}{\ell(s)}$. *We have Green's function* $G(t,s) = U(t,s)$ *has an integrable dichotomy. Indeed,*

$$\int_{-\infty}^{t} U(t,s)ds \leq \int_{-\infty}^{t} e^{-\alpha(t-s)} + \sum_{k=-\infty}^{[t]+2} \int_{J_k} \ell^{-1}(s)ds \leq \frac{1}{\alpha} + 2\left(\frac{\pi^2}{3} + 1\right) < \infty.$$

Condition (2.4) *is satisfied with* $L = \frac{1}{\alpha} + 2\left(\frac{\pi^2}{3} + 1\right)$. *The equation* (5.1) *is not exponentialy stable. In fact,*

$$U(k + b_k^2, k) = (k^2+1)e^{-\frac{\alpha}{(k^2+1)^2}} \to \infty, \quad \text{as } k \to \infty.$$

Thus integrable dichotomy is more general than the exponential dichotomy. Note that

$$| U(t,s) | \leq e^{-\alpha(t-s)} + \lambda_0(s), \quad s \leq t$$

with

$$\lambda_0(s) = \sum_{k\in\mathbb{Z}} \ell^{-1}(s)\chi_{J_k}(s),$$

where χ_{J_k} *is the characteristic function on* J_k. *It is clear that* $\lambda_0 \in L^1(\mathbb{R})$. *Then equation* (5.1) *has an integrable dichotomy with* $\lambda(t,s) = e^{-\alpha(t-s)} + \lambda_0(s)$, $s \leq t$ *satisfying*

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \lambda(t,s)ds \leq L. \tag{5.2}$$

In a similar way, we can prove that

$$\sup_{t\in\mathbb{R}} \int_{t}^{\infty} U(t,s)ds \leq L, \tag{5.3}$$

but the evolution family is not exponentially stable at $-\infty$. *Let the diagonal matrix*

$$A(t) = diag(b_1(t), b_2(t), \dots, b_n(t))$$

with each b_i *satisfying* (5.2) *for* $i = 1, \dots, k$ *and satisfying* (5.3) *for* $i = k+1, \dots, n$ *(*$k > 0$*). Then, this construction yields the linear system*

$$x' = A(t)x$$

which has an integrable dichotomy with

$$\lambda(t,s) = e^{-|t-s|} + \lambda_0(s), \quad t, s \in \mathbb{R},$$

λ_0 *integrable in* $\mathbb{R}$. *We consider the projections*

$$P(t) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad Q(t) = I - P(t) = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix},$$

where I_r *and* I_{n-r} *are identity matrix of order respectively* r *and* $n-r$. *Finally, one extend the diagonal and integrable caracter of the dichotomy of* $A(t)$ *to a diagonal infinite dimensional.*

Example 5.2. *Let* μ *be a mesure with a Radon-Nikodym derivative* ρ *defined by:*

$$\rho(t) = \begin{cases} e^t, & t \leq 0 \\ 1, & t > 1. \end{cases} \tag{5.4}$$

We consider the existence and uniqueness of a μ*-pseudo almost automorphic solutions for the following system:*

$$\begin{cases} \dfrac{\partial u(t,\xi)}{\partial t} = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + \alpha(t)u(t,\xi) + g(t,u(t,\xi)), & t \in \mathbb{R}, \quad \xi \in [0,\pi], \\ u'(t,0) = u'(t,\pi) = 0, & t \in \mathbb{R}, \end{cases} \tag{5.5}$$

where $\alpha(t) = \frac{1}{2}\sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right) \in AA(\mathbb{R}, X)$. *Take* $X = L^2[0,\pi]$ *with norm* $\|\cdot\|$ *and inner product* $(\cdot,\cdot)_2$. $g : \mathbb{R} \times L^2[0,\pi] \to L^2[0,\pi]$ *is* μ*-pseudo almost automorphic with*

$$g(t,\xi) = e^{-|t|}\psi(\xi),$$

where $t \mapsto e^{-|t|}$ *belongs to* $\mathcal{E}(\mathbb{R},\mathbb{R},\mu)$. *The function* ψ *is Lipschitzian. Let* $\kappa > 0$

$$|\psi(x) - \psi(y)| \leq \kappa|x-y|.$$

Let $f : \mathbb{R} \times L^2[0,\pi] \to L^2[0,\pi]$ *be a function defined by*

$$f(t,v)(x) = e^{-|t|}\psi(v(x)).$$

We define $A : D(A) \subset X \to X$ *by*

$$A\phi = \phi'' \ \ \textit{for} \ \ \phi(\cdot) \in D(A),$$

with domain

$$D(A) = \{u \in H^2(0,\pi) : u'(0) = u'(\pi) = 0\}.$$

It is well-known that the operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X such that $\|T(t)\| \leq 1$ for $t \geq 0$. Moreover, we have

$$T(t)\phi = \sum_{n=0}^{\infty} e^{-n^2 t}(\phi, e_n)_2 e_n, \quad \textit{for all} \quad t \geq 0,\ \phi \in X,$$

with $e_n(t) = \sqrt{\frac{2}{\pi}}\cos(nt)$ for each $n \in \mathbb{N}$. Define a family of linear operators $A(t)$ by:

$$A(t) = \frac{\partial^2}{\partial x^2} + \alpha(t)I = A + \alpha(t)I \quad \textit{for } t \in \mathbb{R},$$

with domain

$$D(A(t)) = D(A) = \{u \in H^2(0,\pi) : u'(0) = u'(\pi) = 0\}.$$

It is easy to see that the family of linear operators $A(t)$ satisfy assumptions **($\mathbf{A}_1$)-($\mathbf{A}_3$)**. *Indeed, just take $Y = X$, $M = 1$ and $\omega = \frac{1}{2}$.*

Let $v(t) = u(t,\cdot)$. Then (5.5) becomes

$$\frac{d}{dt}v(t) = A(t)v(t) + f(t, v(t)).$$

The operators $A(t)$ generate an evolution family $(U(t,s))_{t\geq s}$ given by:

$$U(t,s)\phi = \sum_{n=0}^{\infty} e^{\int_s^t [\alpha(\tau)-n^2]d\tau}(\phi, e_n)_2 e_n, \quad \textit{for all } t \geq s,\ \phi \in X.$$

Lemma 5.3. *The evolution family has an integrable dichotomy with data (λ, P).*

Proof. We divide the series in two parts *i.e.*, thus

$$U(t,s)\phi = e^{\int_s^t [\alpha(\tau)-1]d\tau}(\phi, e_0)_2 e_0 + \sum_{n=1}^{\infty} e^{\int_s^t [\alpha(\tau)-n^2]d\tau}(\phi, e_n)_2 e_n, \quad \text{for all } \ t \geq s,\ \phi \in X.$$

For $t \geq s$ and $\phi \in Vect\{e_0\}$,

$$|U(t,s)\phi| = |e^{\int_s^t \alpha(\tau)d\tau}(\phi, e_0)_2 e_0| \leq e^{\frac{1}{2}(t-s)}|\phi|.$$

Let $\phi \in Vect\{e_n; n = 1, 2, \dots\}$,

$$|U(t,s)\phi| = \left|\sum_{n=1}^{\infty} e^{\int_s^t [\alpha(\tau)-n^2]d\tau}(\phi, e_n)_2 e_n\right| \leq e^{\int_s^t [\alpha(\tau)-1]d\tau}\left|\sum_{n=1}^{\infty}(\phi, e_n)_2 e_n\right| \leq e^{-\int_s^t [1-\alpha(\tau)]d\tau}|\phi|.$$

Let $I-P = diag(1,0,\dots,0,0,0,\dots)$ and $P = diag(0,1,1,\dots)$ be projections with $Rank(I-P) = 1$ and $Rank(P) = \infty$. Thus, the Green function is defined by

$$G(t,s) = \begin{cases} U(t,s)P = \sum_{n=1}^{\infty} e^{\int_s^t [\alpha(\tau)-n^2]d\tau} e_n, & \text{if } t \geq s, \\ -\tilde{U}(t,s)(I-P) = -e^{-\int_s^t \alpha(\tau)d\tau} e_0, & \text{if } t < s. \end{cases}$$

Then, $u'(t) = A(t)u(t)$ has an integrable dichotomy with data (λ, P), where λ is given by:

$$\lambda(t,s) = \begin{cases} e^{-\int_s^t [1-\alpha(\tau)]d\tau}, & \text{if } t \geq s, \\ e^{-\int_s^t \alpha(\tau)d\tau}, & \text{if } t < s. \end{cases}$$

Let us calculate L and C as mentioned in Definition 2.20. Let $t \in \mathbb{R}$, using the fact that $-\frac{1}{2} \leq \alpha(\tau) \leq \frac{1}{2}$, one obtain

$$\begin{aligned} \sup_{t\in\mathbb{R}} \int_{\mathbb{R}} \lambda(t,s)ds &= \sup_{t\in\mathbb{R}} \left(\int_{-\infty}^{t} e^{-\int_s^t [1-\alpha(\tau)]d\tau} ds + \int_t^{\infty} e^{-\int_s^t \alpha(\tau)d\tau} ds \right) \\ &\leq \left(\int_{-\infty}^{t} e^{-\frac{1}{2}(t-s)} ds + \int_t^{\infty} e^{\frac{1}{2}(t-s)} ds \right) = 4 = L. \end{aligned}$$

Now, let us verify hypothesis (**A**). Let $T > 0$, we have

$$\begin{aligned} \int_T^{\infty} \int_{-T}^{T} \lambda(t,s)d\mu(t)ds &= \int_T^{\infty} \left(\int_{-T}^{0} e^t e^{\frac{1}{2}(t-s)} dt + \int_0^T e^{\frac{1}{2}(t-s)} dt \right) ds \\ &\leq \left(\frac{2}{3} + 2e^{\frac{1}{2}T} \right) \int_T^{\infty} e^{-\frac{1}{2}s} ds \leq \frac{16}{3} = C. \end{aligned}$$

In a similar way, we can show that

$$\int_s^T \lambda(t,s)d\mu(t) \leq C, \int_{-T}^{s} \lambda(t,s)d\mu(t) \leq C, \quad \text{and} \quad \int_{-\infty}^{-T} \int_{-T}^{T} \lambda(t,s)d\mu(t)ds \leq C.$$ □

Hence, **(H1)** *and* **(H2)** *hold.*

Lemma 5.4. *The Green's function is bi-almost automorphic.*

Proof. Let $\alpha \in AA(\mathbb{R}, X)$, then, for every sequence $(s'_k)_{k\geq 0}$ of real numbers, there exists a subsequence $(s_k)_{k\geq 0} \subset (s'_k)_{k\geq 0}$ and a measurable function $\tilde{\alpha}$, such that

$$\lim_k \alpha(\tau + s_k) = \tilde{\alpha}(\tau) \quad \text{and} \quad \lim_k \tilde{\alpha}(\tau - s_k) = \alpha(\tau) \quad \text{for all } \tau \in \mathbb{R}.$$

Let us define, $\tilde{U}(t,s)\phi = T(t-s)e^{\int_s^t \tilde{\alpha}(\tau)d\tau}\phi,$ for all $t \geq s$, $\phi \in X$. Since U is bi-almost automorphic, we have

$$\begin{aligned}
\lim_k U(t+s_k, s+s_k)\phi - \tilde{U}(t,s)\phi &\leq \lim_k \left\| T(t-s)e^{\int_{s+s_k}^{t+s_k} \alpha(\tau)d\tau}\phi - T(t-s)e^{\int_s^t \tilde{\alpha}(\tau)d\tau}\phi \right\| \\
&\leq \lim_k \left\| T(t-s)\left(e^{\int_{s+s_k}^{t+s_k} \alpha(\tau)d\tau} - e^{\int_s^t \tilde{\alpha}(\tau)d\tau} \right)\phi \right\| \\
&\leq \lim_k \left\| T(t-s)\left(e^{\int_s^t \alpha(\tau-s_k)d\tau} - e^{\int_s^t \tilde{\alpha}(\tau)d\tau} \right)\phi \right\| \\
&\leq \lim_k \left\| T(t-s)e^{\int_s^t \tilde{\alpha}(\tau)d\tau}\left(e^{\int_s^t |\alpha(\tau-s_k)-\tilde{\alpha}(\tau)|d\tau} - 1 \right)\phi \right\|
\end{aligned}$$

As $\alpha \in AA(\mathbb{R}, X)$, we have

$$\left| e^{\int_s^t |\alpha(\tau-s_k)-\tilde{\alpha}(\tau)|d\tau} - 1 \right| \to 0 \quad \text{as} \quad k \to \infty.$$

Then

$$\lim_k U(t+s_k, s+s_k)\phi - \tilde{U}(t,s)\phi = 0.$$

In a similar way, we can prove that $\lim_k \tilde{U}(t-s_k, s-s_k)\phi - U(t,s)\phi = 0$. Then, U is bi-almost automorphic. □

Consequently, all assumptions of the Theorem 4.3 are satisfied. We can deduce by the Theorem 4.3 that the problem (4.1) *has an unique μ-pseudo almost automorphic mild solution on $\mathbb{R}$, under the condition κ small enough.*

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