CUBO, A Mathematical Journal
Vol. 27, no. 1, pp. 75–82, April 2025
DOI: 10.56754/0719-0646.2701.075

Estimating the remainder of an alternating p-series revisited

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ABSTRACT

For the nth remainder $R_n(p) := \sum_{k=n+1}^{\infty}(-1)^{k+1}k^{-p}$ of an alternating p-series, several asymptotic estimates are presented. For example, for any integer $n \geq 3$, and $p \in \mathbb{R}^+$, we have

$$R_n(p) = \frac{(-1)^n}{2\big(2\lfloor\frac{n+1}{2}\rfloor\big)^p} - \frac{p}{4\big(2\lfloor\frac{n+1}{2}\rfloor\big)^{p+1}} + \varepsilon_n^*(p)$$

and

$$|\varepsilon_n^*(p)| < \frac{p(p+1)}{5\,(n-2)^{p+2}},$$

where $\lfloor x\rfloor$ denotes the integer part (the floor) of x.

RESUMEN

Para el residuo n-ésimo $R_n(p) := \sum_{k=n+1}^{\infty}(-1)^{k+1}k^{-p}$ de una p-serie alternante, se presentan diversas estimaciones asintóticas. Por ejemplo, para cualquier entero $n \geq 3$ y $p \in \mathbb{R}^+$, tenemos

$$R_n(p) = \frac{(-1)^n}{2\big(2\lfloor\frac{n+1}{2}\rfloor\big)^p} - \frac{p}{4\big(2\lfloor\frac{n+1}{2}\rfloor\big)^{p+1}} + \varepsilon_n^*(p)$$

y

$$|\varepsilon_n^*(p)| < \frac{p(p+1)}{5\,(n-2)^{p+2}},$$

donde $\lfloor x\rfloor$ denota la parte entera (el piso) de x.

Keywords and Phrases: Alternating generalized harmonic number, alternating p-series, approximation, Dirichlet's eta function, estimate, remainder, slow convergence.

2020 AMS Mathematics Subject Classification: 41A60, 65B10, 33E20, 33F05, 40A25.

Published: 28 April, 2025
Accepted: 01 April, 2025
Received: 11 June, 2024

1 Introduction

In [5] it was shown that the best constants a and b such that inequalities

$$\frac{1}{2n+a} \le \left| \sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{1}{k} \right| < \frac{1}{2n+b} \tag{1.1}$$

hold for every $n \ge 1$ are $a = \frac{1}{1-\ln 2} - 2 \approx 1.25\,8891$ and $b = 1$.

In the paper [1] it was proved for the nth remainder $R_n(p)$,

$$R_n(p) := \sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}}{k^p}, \tag{1.2}$$

for alternating p-series (for Dirichlet eta function $\eta(p)$, *i.e.* for the Riemann alternating zeta function),

$$\eta(p) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^p}\,, \tag{1.3}$$

the relations

$$a(n,p) := \frac{1}{2(n+1)^p - \rho(p)} \le |R_n(p)| \le \frac{1}{2n^p + \sigma(p)} =: b(n,p), \tag{1.4}$$

true for integers $n \ge 1$ and $p \ge 2$ and with (the best) constants

$$\rho(p) := 2^{p+1} - \frac{1}{1-(1-2^{1-p})\zeta(p)} \quad \text{and} \quad \sigma(p) := \frac{1}{1-(1-2^{1-p})\zeta(p)} - 2. \tag{1.5}$$

Accuracy or sharpness of the double inequality $A(x) \le F(x) \le B(x)$ at the point x we define as the difference $B(x) - A(x)$, *i.e.* as the width of the interval $[A(x), B(x)]$. For example the double inequality (1.1) has the sharpness equal to $\frac{a-b}{(2n+a)(2n+b)}$, *i.e.* $\mathcal{O}(\frac{1}{n^2})$, using the Landau big O notation. Similarly, the double inequality (1.4) has the sharpness $\mathcal{O}(\frac{1}{n^{p+1}})$.

Motivated by [3, 4] and [5], and especially by [1], where the validity of (1.4) is based on the supposition that p is a positive integer different from 1, we shall provide some estimates of the remainder $R_n(p)$, which are close to the relation (1.4) and are valid for any $p \in \mathbb{R}^+$.

2 Background

We shall use the results from the paper [2], where appear special sums[1]

$$\sigma_q^*(x,p) := \sum_{i=1}^{\lfloor q/2 \rfloor} \left(4^i - 1\right) \frac{B_{2i} \cdot p^{(2i-1)}}{x^{p+2i-1} \cdot (2i)!} \quad (q \in \mathbb{N},\, p, x \in \mathbb{R}), \tag{2.1}$$

[1]By definition, $\sum_{i=m}^{n} x_i = 0$ if $m > n$.

where B_k denotes the kth Bernoulli coefficient (or Bernoulli number)[2], defined by the identity $\frac{t}{e^t-1} \equiv \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$ ($|t| < 2\pi$), where the symbol $x^{(k)}$ designates the upper (rising) Pochhammer product defined as

$$x^{(0)} := 1, \quad x^{(k)} := \prod_{i=0}^{k-1} (x+i) = x(x+1)\cdots(x+k-1) \quad (x \in \mathbb{R},\ k \in \mathbb{N}), \tag{2.2}$$

and where the symbol $\lfloor x \rfloor$ denotes the integer part (the floor) of any $x \in \mathbb{R}^+$.

We will use the following lemma.

Lemma 2.1 ([2, Theorem 1]). *For $p \in \mathbb{R}^+$ and every $k, n, q \in \mathbb{N}$, with $n \geq 2k+1 \geq 3$, the nth remainder $R_n(p) := \eta(p) - \sum_{j=1}^{n} (-1)^{j+1} \frac{1}{j^p}$ is given in the form*

$$R_n(p) = \Delta_q(n,p) + \delta_q(k,p),$$

with

$$\Delta_q(n,p) = \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \sigma_q^*\left(2\lfloor\tfrac{n+1}{2}\rfloor, p\right),$$

and

$$|\delta_q(k,p)| < \frac{5p^{(q-1)}}{3\pi^{q-1}} \cdot \frac{1}{(2k)^{p+q-1}}\,.$$

3 Asymptotic estimates of the remainder $R_n(p)$

Now, for any integer $n \geq 3$, the floor (the integer part) $\nu := \left\lfloor \frac{n-1}{2} \right\rfloor$ is a positive integer estimated as $\frac{n-1}{2} - 1 < \nu \leq \frac{n-1}{2}$. Consequently $n-3 < 2\nu \leq n-1$, that is

$$n-2 \leq 2\nu \leq n-1. \tag{3.1}$$

Therefore, using $k = \nu$ in Lemma 2.1, together with the new naming $\varepsilon_n(p,q) := \delta_q(\nu,p)$, we obtain the next result.

Proposition 3.1. *For integers $n \geq 3$ and $q \geq 1$, and for $p \in \mathbb{R}^+$, we have*

$$R_n(p) = \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \sum_{i=1}^{\lfloor q/2 \rfloor} \left(4^i - 1\right) \frac{B_{2i} \cdot p^{(2i-1)}}{\left(2\left\lfloor\frac{n+1}{2}\right\rfloor\right)^{p+2i-1} \cdot (2i)!} + \varepsilon_n(p,q),$$

[2] $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = B_5 = B_7 = \cdots = 0$, $B_4 = B_8 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, ...

where

$$|\varepsilon_n(p,q)| < \frac{5p^{(q-1)}}{3\pi^{q-1}} \cdot \frac{1}{\left(2\lfloor\frac{n-1}{2}\rfloor\right)^{p+q-1}} \leq \frac{5p^{(q-1)}}{3\pi^{q-1}} \cdot \frac{1}{(n-2)^{p+q-1}}\,.$$

Here q is a parameter controlling the magnitude of the error term $\varepsilon_n(p,q)$.

Using $q=1$ in Proposition 3.1, we obtain the first corollary.

Corollary 3.2. *For an integer $n \geq 3$ and $p \in \mathbb{R}^+$ there hold the following estimates:*

$$\frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{5}{3\left(n-2\right)^p} < R_n(p) < \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} + \frac{5}{3\left(n-2\right)^p}$$

and

$$|R_n(p)| < \frac{1}{2\left(n-1\right)^p} + \frac{5}{3\left(n-2\right)^p}\,.$$

Putting $q=3$ in Proposition 3.1, we get the following corollary.

Corollary 3.3. *For $p \in \mathbb{R}^+$ and every integer $n \geq 3$, the formulas*

$$R_n(p) = \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{p}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+1}} + \varepsilon_n(p,3),$$

hold true, where

$$|\varepsilon_n(p,3)| < \frac{5}{3\pi^2} \cdot \frac{p(p+1)}{\left(2\lfloor\frac{n-1}{2}\rfloor\right)^{p+2}} < \frac{p(p+1)}{5\,(n-2)^{p+2}}$$

and

$$\left|\left|R_n(p)\right| - \left|\frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{p}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+1}}\right|\right| < \frac{p(p+1)}{5\,(n-2)^{p+2}}\,.$$

Setting $q=5$ in Proposition 3.1, we provide the following result.

Corollary 3.4. *For every integer $n \geq 3$ and $p \in \mathbb{R}^+$, there holds the equality*

$$R_n(p) = \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{p}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+1}} + \frac{p(p+1)(p+2)}{48\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+3}} + \varepsilon_n(p,5)$$

with the estimate

$$|\varepsilon_n(p,5)| < \frac{5}{3\pi^4} \cdot \frac{p(p+1)(p+2)(p+3)}{\left(2\lfloor\frac{n-1}{2}\rfloor\right)^{p+4}} < \frac{p(p+1)(p+2)(p+3)}{58\,(n-2)^{p+4}}\,.$$

4 Approximations of $\left|R_n(p)\right|$

Using the Landau big O notation, the relation (1.4) means that $\left|R_n(p)\right| = \mathcal{O}\left(\frac{1}{n^p}\right)$ as $n \to \infty$, for integers $p \geq 2$. However, the next Proposition 4.1 improves this result.

Proposition 4.1. *For integers $n \geq 3$ and $q \geq 1$, for any $p \in \mathbb{R}^+$, and for* $S_n(p,q) := \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \sum_{i=1}^{\lfloor q/2\rfloor} (4^i - 1)\frac{B_{2i}\cdot p^{(2i-1)}}{\left(2\left\lfloor\frac{n+1}{2}\right\rfloor\right)^{p+2i-1}\cdot(2i)!}$, *we have* $\left|R_n(p)\right| = \mathcal{O}\left(\frac{1}{n^{p+q-1}}\right)$ *as $n \to \infty$; more precisely*[3]

$$\left|S_n(p,q)\right| - \frac{5p^{(q-1)}}{3\pi^{q-1}(n-2)^{p+q-1}} \leq \left|R_n(p)\right| \leq \left|S_n(p,q)\right| + \frac{5p^{(q-1)}}{3\pi^{q-1}(n-2)^{p+q-1}}.$$

Proof. Thanks to Proposition 3.1, using the triangle inequalities, we have

$$\begin{aligned}\left|R_n(p)\right| &= \Big|\left|S_n(p,q)\right| - \left(\left|S_n(p,q)\right| - \left|R_n(p)\right|\right)\Big| \geq \left|S_n(p,q)\right| - \Big|\left|S_n(p,q)\right| - \left|R_n(p)\right|\Big| \\ &\geq \left|S_n(p,q)\right| - \left|S_n(p,q) - R_n(p)\right| = \left|S_n(p,q)\right| - \left|\varepsilon_n(p,q)\right|\end{aligned}$$

and

$$\begin{aligned}\left|R_n(p)\right| &= \Big|\left|S_n(p,q)\right| - \left(\left|S_n(p,q)\right| - \left|R_n(p)\right|\right)\Big| \leq \left|S_n(p,q)\right| + \Big|\left|S_n(p,q)\right| - \left|R_n(p)\right|\Big| \\ &\leq \left|S_n(p,q)\right| + \left|S_n(p,q) - R_n(p)\right| = \left|S_n(p,q)\right| + \left|\varepsilon_n(p,q)\right|.\end{aligned}$$

□

Numerical experiment. *Using the Mathematica computer system [6] and considering* (1.4), *together with Proposition 3.1, we obtain for functions*

$$A(n,p,q) := \left|\frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \sum_{i=1}^{\lfloor q/2\rfloor}\frac{(4^i-1)B_{2i}\cdot p^{(2i-1)}}{\left(2\left\lfloor\frac{n+1}{2}\right\rfloor\right)^{p+2i-1}\cdot(2i)!}\right| - \frac{5p^{(q-1)}}{3\pi^{q-1}n^{p+q-1}}$$

and

$$B(n,p,q) := \left|\frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \sum_{i=1}^{\lfloor q/2\rfloor}\frac{(4^i-1)B_{2i}\cdot p^{(2i-1)}}{\left(2\left\lfloor\frac{n+1}{2}\right\rfloor\right)^{p+2i-1}\cdot(2i)!}\right| + \frac{5p^{(q-1)}}{3\pi^{q-1}n^{p+q-1}}$$

the following estimates:

$$A(n,3,3) > a(n,3), \quad \textit{for } 6 \leq n \leq 100,$$
$$B(n,3,3) < b(n,3), \quad \textit{for } 4 \leq n \leq 100,$$
$$B(n,3,3) - A(n,3,3) < b(n,3) - a(n,3), \quad \textit{for } 5 \leq n \leq 100.$$

[3] At $q = 1$, the given lower bound for $\left|R_n(p)\right|$ is negative.

Similarly, we get

$$A(n,3,5) > a(n,3), \quad \textit{for } 4 \le n \le 100,$$
$$B(n,3,5) < b(n,3), \quad \textit{for } 3 \le n \le 100,$$
$$B(n,3,5) - A(n,3,5) < b(n,3) - a(n,3), \quad \textit{for } 3 \le n \le 100.$$

These inequalities are illustrated in Figures 1–3, where the graphs of the functions $n \mapsto A(n,3,q)/a(n,3)$*,* $n \mapsto B(n,3,q)/b(n,3)$ *and* $n \mapsto \big(B(n,3,q) - A(n,3,q)\big)/\big(b(n,3) - a(n,3)\big)$*, having* $q \in \{3,5\}$*, are plotted using the Mathematica software [6]. Thus, numerical examples confirm that our estimates of* $|R_n(p)|$*, given in Proposition 4.1, are more accurate, for* $n \ge 5$ *and* $q \ge 3$*, than that given in* (1.4)*. This is consistent with the fact that the sharpness of the estimates for* $|R_n(p)|$ *given in Proposition 4.1, is equal to* $\mathcal{O}\left(\frac{1}{n^{p+q-1}}\right)$*.*

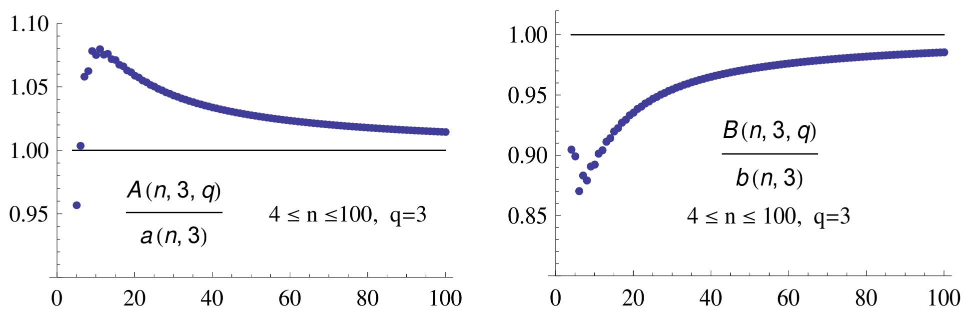

Figure 1: The graphs of the sequences $n \mapsto A(n,3,3)/a(n,3)$ (left) and $n \mapsto B(n,3,3)/b(n,3)$, (right).

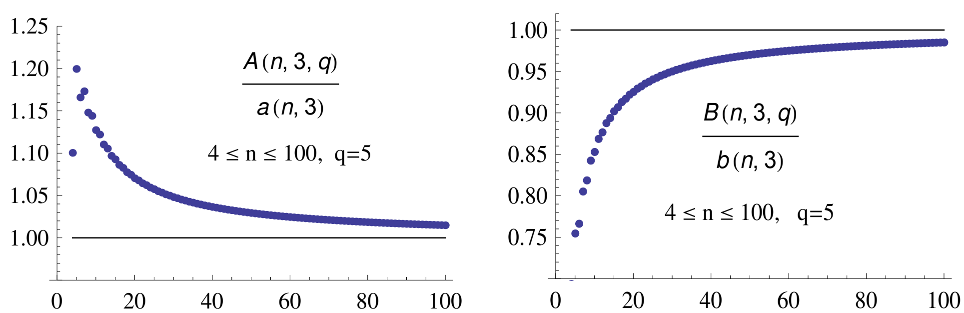

Figure 2: The graphs of the sequences $n \mapsto A(n,3,5)/a(n,3)$ (left) and $n \mapsto B(n,3,5)/b(n,3)$, (right).

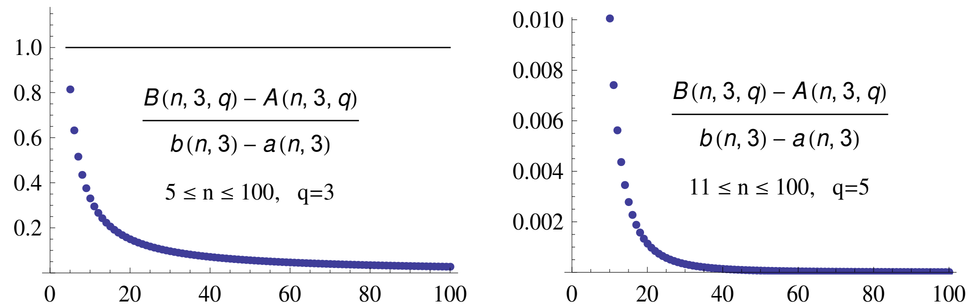

Figure 3: The graphs of the sequences $n \mapsto \frac{B(n,3,3)-A(n,3,3)}{b(n,3)-a(n,3)}$ (left) and $n \mapsto \frac{B(n,3,5)-A(n,3,5)}{b(n,3)-a(n,3)}$ (right).

4.1 Conclusion

The paper easily provides several asymptotic estimates of a remainder of an alternating p-series for all $p \in \mathbb{R}^+$. The presented relations supplement the double inequality for a remainder, given in the paper [1], which works only for integers $p \geq 2$. In addition, the derived estimates are very useful even in the case of $p \approx 0$.

Acknowledgment. The author would like to thank the reviewer for carefully reading the paper and correcting several slips.

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