CUBO, A Mathematical Journal
Vol. 27, no. 1, pp. 83–92, April 2025
DOI: 10.56754/0719-0646.2701.083

Congruences of infinite semidistributive lattices

George Grätzer[1]
J. B. Nation[2,✉]

[1] *Department of Mathematics, University of Manitoba, Winnipeg, MB R3T 2N2, Canada.*
gratzer@mac.com

[2] *Department of Mathematics, University of Hawaii, Honolulu, HI 96822, USA.*
jb@math.hawaii.edu[✉]

ABSTRACT

Not every finite distributive lattice is isomorphic to the congruence lattice of a finite semidistributive lattice. This note provides a construction showing that many of these finite distributive lattices are isomorphic to congruence lattices of infinite semidistributive lattices.

RESUMEN

No todo reticulado distributivo finito es isomorfo al reticulado de congruencia de un reticulado finito semidistributivo. Esta nota proporciona una construcción mostrando que muchos de estos reticulados finitos distributivos son isomorfos a reticulados de congruencia de reticulados infinitos semidistributivos.

Keywords and Phrases: Distributive lattice, semidistributive lattice, congruence lattice.

2020 AMS Mathematics Subject Classification: 06B10, 06B15.

Published: 28 April, 2025
Accepted: 15 April, 2025
Received: 12 October, 2024

1 Introduction

Congruence lattices of lattices are distributive, and every finite distributive lattice is isomorphic to the congruence lattice of a finite lattice. We would like to know more about: *Which finite distributive lattices are the congruence lattice of some semidistributive lattice?*

Not every finite distributive lattice D is isomorphic to $\operatorname{Con} L$ for a *finite* semidistributive lattice L. There are two known restrictions [2, 9]: if D is the congruence lattice of a finite semidistributive lattice, then considering D as the lattice $\mathcal{O}(P)$ of order ideals of an ordered set, neither $\mathbf{2}$ nor Y (Figure 1) can be an order filter in P. An equivalent formulation is that neither a 3-element chain nor $(B_2)_{++} := \mathbf{2} + \mathbf{2}^2$ can be a filter in D. There may be other restrictions.

This note presents a construction to show that many finite distributive lattices with $\mathbf{3}$ or $(B_2)_{++}$ as a filter are isomorphic to the congruence lattice of an *infinite* semidistributive lattice.

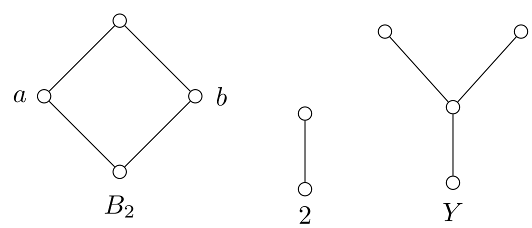

Figure 1: Ordered sets referred to in the text: B_2, $\mathbf{2}$, Y

2 Background

The join-semidistributive law for lattices is

$$\text{(JSD)} \qquad x \vee y = x \vee z \ \text{ implies } \ x \vee y = x \vee (y \wedge z).$$

Its dual is the meet-semidistributive law, (MSD). Lattices that satisfy both are called *semidistributive*, abbreviated SD. The semidistributive laws were found by B. Jónsson as a property of free lattices; see [6–8] and the survey [1].

Finite distributive lattices are isomorphic to the lattice of order ideals (downsets) of an ordered set. In fact, $D \cong \mathcal{O}(P)$, where $P = (\mathrm{J}(D), \leq)$ is the set of join-irreducible elements of D. This reflects the fact that join-irreducible elements in a distributive lattice are join-prime. Our results are formulated in terms of this duality.

Let $\mathbf{n}$ denote an n-element chain, A_n an n-element antichain, and B_n the boolean lattice with n atoms. For the ordered sets, P and Q, the ordered set $P \mathbin{\dot\cup} Q$ has the elements of P and Q incomparable, while the ordered set $P + Q$ has every element of P below every element of Q. For the lattices K and L with 0 and 1, let $K\#L$ denote the glued sum, where $1_K = 0_L$.

Lemma 2.1. *Let P and Q be ordered sets. If $\mathcal{O}(P) = K$ and $\mathcal{O}(Q) = L$, then*

(1) $\mathcal{O}(P \dot\cup Q) = K \times L$,

(2) $\mathcal{O}(P + Q) = K\#L$.

If L is a lattice, then L_+ denotes the lattice obtained by adjoining a new zero element, that is, $L_+ = \mathbf{1} + L$. Thus $L_{++} = \mathbf{2} + L$. Likewise, L^+ is the lattice obtained by adjoining a new top element, that is, $L^+ = L + \mathbf{1}$.

The congruence lattice of a finite lattice is a finite distributive lattice. There are two restrictions mentioned in the introduction: if $\mathcal{O}(P) \cong \operatorname{Con} K$ for a *finite semidistributive* lattice, then neither $\mathbf{2}$ nor Y can be an order filter (upset) of P. Note that $\mathcal{O}(\mathbf{2}) = \mathbf{3}$ and $\mathcal{O}(Y) = (B_2)_{++}$; remember to include the empty order ideal. The following elementary technical observation [9] then shows that neither $\mathbf{3}$ nor $(B_2)_{++}$ is a filter of $\mathcal{O}(P)$.

Lemma 2.2. *Let S and P be finite ordered sets. Then $\mathcal{O}(S)$ is isomorphic to a filter of $\mathcal{O}(P)$ if and only if S is an order filter of P.*

Now $\mathbf{2}$ is the only finite simple SD lattice. Indeed, if L is JSD and has a largest element 1, then it has a prime ideal, and hence L has $\mathbf{2}$ as a homomorphic image. There are however *infinite* simple SD lattices [4].

The original lattices in [4] contained no completely doubly irreducible (c.d.i.) elements, that is, elements that are completely join-irreducible and completely meet-irreducible. A straightforward modification of the construction yields infinite simple semidistributive lattices containing infinitely many c.d.i. elements; see [3]. (Replace the defining relations (7) and (8) in [4] by $b_i < b_{i+1}$ and $d_i < d_{i+1}$; these are slightly stronger.)

The infinite simple SD lattices constructed in [3, 4], containing an infinite chain of c.d.i. elements, are called FN lattices. The letter F will denote an arbitrary FN lattice with c.d.i. elements. An infinite simple semidistributive lattice necessarily has neither 0 nor 1. We will use FN lattices as the building blocks for our constructions.

The least congruence on a lattice is denoted by Δ, and the greatest congruence ∇. In this note we are dealing with infinite lattices that have finite congruence lattices. Of course, that is not always the case.

3 Direct products

The first operation for building new representations from existing ones is the direct product.

Lemma 3.1. *If K and L are lattices, then* $\mathrm{Con}(K \times L) \cong \mathrm{Con}\ K \times \mathrm{Con}\ L$*. For ordered sets, this translates to the disjoint union, that is, if* $\mathrm{Con}\ K \cong \mathcal{O}(P)$ *and* $\mathrm{Con}\ L \cong \mathcal{O}(Q)$*, then*

$$\mathrm{Con}(K \times L) \cong \mathcal{O}(P) \times \mathcal{O}(Q) = \mathcal{O}(P \dot{\cup} Q).$$

The lemma allows us to represent $B_m = \mathcal{O}(A_m)$ as $\mathrm{Con}\ 2^m$ or $\mathrm{Con}\ F^m$ where F is an FN lattice. The following properties will play a role later.

IGD(K) The congruence generated by collapsing any nonempty ideal of K is ∇.

FGD(K) The congruence generated by collapsing any nonempty filter of K is ∇.

A lattice satisfying both IGD and FGD is called *half-simple*, and FN lattices (being simple) clearly are half-simple. Half-simple lattices can have neither 0 nor 1.

Lemma 3.2. *A finite direct product of lattices with* FGD *has* FGD*. Likewise, for* IGD *and half-simple.*

Proof. Let $L = K_1 \times \cdots \times K_n$, with each K_j having FGD, and let G be a nonempty filter of L. Let θ denote the congruence on L obtained by collapsing G. We want to show that $\theta = \nabla_L$.

Lattices have factorable congruences, as a consequence of congruence distributivity. This means that there exist congruences $\theta_i \in \mathrm{Con}\ K_i$ such that, for $x, y \in L$, we have $x\,\theta\,y$ iff $x_i\,\theta_i\,y_i$ for all $1 \leq i \leq n$. But each θ_i is the congruence generated on K_i by the projection of the filter G onto K_i, which is a nonempty filter. Since K_i has FGD, this implies that $\theta_i = \nabla_{K_i}$, whence $\theta = \nabla_L$. □

4 Replacing a c.d.i. element with a half-simple lattice

Let d be a c.d.i. element in a lattice K, and let H be half-simple. The lattice $K(d \hookrightarrow H)$ is the set $(K - \{d\}) \dot{\cup} H$ with the natural order, that is, for $k \in K - \{d\}$ and $h \in H$, $k \leq h$ iff $k \leq d$, and $k \geq h$ iff $k \geq d$. Joins and meets are well-defined in $K(d \hookrightarrow H)$, because d is doubly irreducible. Indeed, $K - \{d\}$ and H are sublattices, while

$$k \vee h = \begin{cases} h, & \text{if } k \leq d; \\ k \vee d, & \text{otherwise;} \end{cases}$$

and dually. Thus, $K(d \hookrightarrow H)$ is semidistributive, if both K and H are semidistributive. The construction is illustrated schematically in Figure 2.

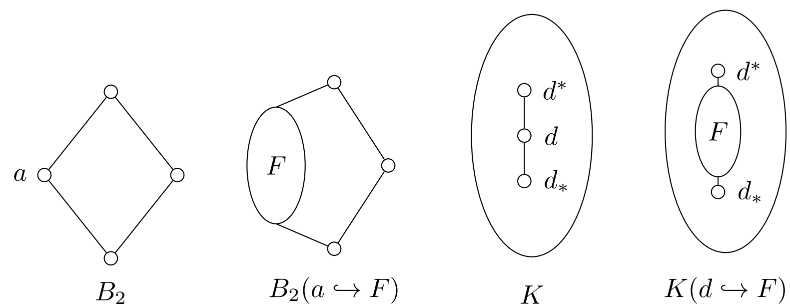

Figure 2: Schematic representation of the construction, replacing a c.d.i. element with an FN lattice F

One can also replace multiple c.d.i. elements independently, forming $K(d_1 \hookrightarrow H_1, \dots, d_n \hookrightarrow H_n)$.

Let us now analyze $\mathrm{Con}\,(K(d \hookrightarrow H))$.

For any element $u \in K$, considering how joins of congruences work, there is a unique largest congruence ζ_u in $\mathrm{Con}\ K$ such that the congruence class $[u]_\theta$ is a singleton, that is, $[u]_\theta = \{u\}$ iff $\theta \leq \zeta_u$. Note that when $\mathrm{Con}\ K \cong \mathcal{O}(P)$, the congruence ζ_u corresponds to an order ideal of P, which we also denote ζ_u.

Theorem 4.1. *Let K be a lattice with a c.d.i. element d, and let H be a half-simple lattice. Form $L = K(d \hookrightarrow H)$. Then*

$$\mathrm{Con}\ L \cong \{(\theta, \alpha) \in \mathrm{Con}\ K \times \mathrm{Con}\ H \colon \theta \nleq \zeta_d \ \to\ \alpha = \nabla_H\}.$$

In terms of ordered sets, if $\mathrm{Con}\ K \cong \mathcal{O}(P)$ *and* $\mathrm{Con}\ H \cong \mathcal{O}(Q)$*, then* $\mathrm{Con}\ L \cong \mathcal{O}(R)$ *where $R = Q \cup P$ with the order $q \leq p$ iff $p \notin \zeta_d$ for $p \in P$, $q \in Q$.*

Figure 3 illustrates how Theorem 4.1 applies to $N_5(c \hookrightarrow F)$ and $\zeta_c > \Delta$.

Proof. Let φ be the congruence on $L = K(d \hookrightarrow H)$ that collapses H back to a single point, so that $L/\varphi \cong K$. By the isomorphism theorems, $\uparrow\varphi$ in $\mathrm{Con}\ L$ is isomorphic to $\mathrm{Con}\ K$. Explicitly, if $f\colon L \twoheadrightarrow K$ with $\ker f = \varphi$ and $\psi \geq \varphi$, then $k\, f(\psi)\, k'$ if and only if there exist x, x' in L with $k = f(x)$, $k' = f(x')$, and $x\,\psi\,x'$. Equivalently, in view of $\psi \geq \varphi$, for all x, x' in L, we have that $f(x)\, f(\psi)\, f(x')$ if and only if $x\,\psi\,x'$.

Let $\mathcal{S}$ be the sublattice of $\mathrm{Con}\ K \times \mathrm{Con}\ H$ given in the theorem. We establish inverse lattice homomorphisms $\sigma\colon\ \mathrm{Con}\ L \to \mathcal{S}$ and $\tau\colon \mathcal{S} \to \mathrm{Con}\ L$.

For $\psi \in \operatorname{Con} L$, let $\sigma(\psi) = (f(\psi \vee \varphi), \psi|_H)$. For $(\theta, \alpha) \in \mathcal{S}$ and k, $k' \in K - \{d\}$, h, $h' \in H$, let

$$k \, \tau(\theta, \alpha) \, k' \text{ iff } k \, \theta \, k',$$
$$h \, \tau(\theta, \alpha) \, h' \text{ iff } h \, \alpha \, h',$$
$$k \, \tau(\theta, \alpha) \, h \text{ iff } k \, \theta \, d.$$

The crucial observations are these.

- If $f(\psi \vee \varphi) \nleq \zeta_d$, then $k \, f(\psi \vee \varphi) \, d$ for some $k \in K - \{d\}$. Hence $k \, \psi \, h$ for some $h \in H$ (as $f(h) = d$).
- If $k \, \psi \, h$ for some $k \in K - \{d\}$ and $h \in H$, then ψ collapses either an ideal or a filter of H (or both). Because H is half-simple, this implies $\psi|_H = \nabla_H$.
- The condition $\psi|_H = \nabla_H$ is equivalent to $\psi \geq \varphi$.

On the other hand, if $\theta \in \operatorname{Con} K$ with $\theta \leq \zeta_d$, and $\alpha \in \operatorname{Con} H$, let ξ be the relation on L such that $\xi|_{K-\{d\}} = \theta|_{K-\{d\}}$, $\xi|_H = \alpha$, and ξ contains no pairs of the form (k, h) or (h, k). Then ξ is a congruence on L and $\xi = \tau(\theta, \alpha)$. The remaining details are left as an exercise to the reader. □

Corollary 4.2. *Let K be a lattice with a c.d.i. element d, and let H be a half-simple lattice. If $\zeta_d = \Delta_K$, then*

$$\operatorname{Con} K(d \hookrightarrow H) \cong \operatorname{Con} H \,\#\, \operatorname{Con} K.$$

In particular, with an FN lattice,

$$\operatorname{Con} K(d \hookrightarrow F) \cong \mathbf{1} + \operatorname{Con} K = (\operatorname{Con} K)_+ \text{ when } \zeta_d = \Delta_K,$$
$$\operatorname{Con} F(d \hookrightarrow H) \cong \operatorname{Con} H + \mathbf{1} = (\operatorname{Con} H)^+ \text{ when } H \text{ is half-simple.}$$

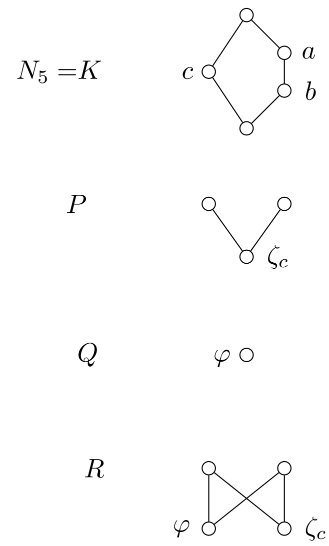

Figure 3: Example $N_5(c \hookrightarrow F)$ for Theorem 4.1. Note $\zeta_c = \mathrm{Cg}(a, b)$.

Recall that for $n \geq 3$, the n-element chain is not the congruence lattice of a finite semidistributive lattice (or even a finite join-semidistributive lattice [2]).

Corollary 4.3. *For every $n \geq 2$, the n-element chain $\mathbf{n}$ can be represented as the congruence lattice of an infinite semidistributive lattice.*

$$
\begin{aligned}
\mathbf{2} = \mathcal{O}(\mathbf{1}) &\qquad F \\
\mathbf{3} = \mathcal{O}(\mathbf{2}) &\qquad F\langle d_1 \hookrightarrow F\rangle \\
\mathbf{4} = \mathcal{O}(\mathbf{3}) &\qquad F\langle d_1 \hookrightarrow F\langle d_2 \hookrightarrow F\rangle\rangle \\
\mathbf{5} = \mathcal{O}(\mathbf{4}) &\qquad F\langle d_1 \hookrightarrow F\langle d_2 \hookrightarrow F\langle d_3 \hookrightarrow F\rangle\rangle\rangle
\end{aligned}
$$

etc.

As an application of direct products (Lemma 3.1):

Corollary 4.4. *For positive integers $n_1, \ldots, n_k$, the lattices $\mathbf{n}_1 \times \cdots \times \mathbf{n}_k$ are congruence lattices of infinite SD lattices.*

If any $n_j \geq 3$, then $\mathbf{n}_1 \times \cdots \times \mathbf{n}_k$ is not the congruence lattice of a finite SD lattice.

The lattice $\mathcal{O}(Y) = (B_2)_{++}$ is the other lattice minimally not representable as the congruence lattice of a finite SD lattice. However, $\mathcal{O}(Y) \cong \mathrm{Con}\, K$ for both of the following infinite SD lattices:

- $K_1 = B_2(a \hookrightarrow F(d \hookrightarrow F))$

- $K_2 = N_5(a_0 \hookrightarrow F)$ where $N_5 = B_2[a]$, doubling an atom.

These lattices are drawn schematically in Figure 4.

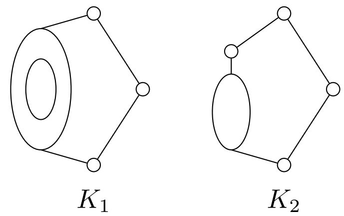

Figure 4: Schematic representation of lattices K_j with Con $K_j = \mathcal{O}(Y)$.

One can just as easily use $K = B_n$ and one of its atoms to represent $(B_n)_{++}$ for any $n \geq 2$ as the congruence lattice of an infinite SD lattice, generalizing either of the representations K_1 or K_2.

A *dual tree* is a connected finite ordered set such that every element has at most one cover. A *dual forest* is a disjoint union of finitely many dual trees. When P is a dual forest, there is a straightforward way to represent $\mathcal{O}(P)$ as a congruence lattice. For branching in the dual tree, we replace multiple c.d.i. elements.

Theorem 4.5. *If P is a finite dual forest, then $\mathcal{O}(P)$ is the congruence lattice of an infinite SD lattice.*

Proof. Without loss of generality P is a dual tree, as we can use direct products for a dual forest.

Let $u \succ v_1, \ldots, v_n$ in P, and assume inductively that each $\mathcal{O}(\downarrow v_j) \cong$ Con H_j for a half-simple SD lattice H_j. Let F be an FN lattice, and choose distinct c.d.i. elements $d_1, \ldots, d_n \in F$. Form $L = F(d_1 \hookrightarrow H_1, \ldots, d_n \hookrightarrow H_n)$. Then L is half-simple, and Con $L \cong \mathcal{O}(\downarrow u)$ by the straightforward extension of Corollary 4.2 for multiple substitutions. □

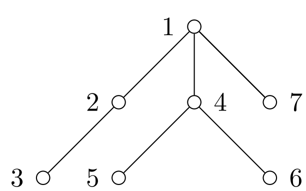

Figure 5: Dual tree example

The method is best illustrated by an example. Let P be the dual tree in Figure 5. To find an infinite SD lattice K with Con $K \cong \mathcal{O}(P)$, we use $K = F_1\langle b_1 \hookrightarrow H_1, b_2 \hookrightarrow H_2, b_3 \hookrightarrow H_3\rangle$ where

$$H_1 = F_2\langle b_4 \hookrightarrow F_3\rangle$$
$$H_2 = F_4\langle b_5 \hookrightarrow F_5, b_6 \hookrightarrow F_6\rangle$$
$$H_3 = F_7.$$

Also observe that Theorem 4.5 includes $\mathcal{O}(A_n + \mathbf{k}) = B_n^{+\cdots+}$ with k "+" signs.

5 Conclusion

We have shown that many finite distributive lattices that are not the congruence lattice of a *finite* semidistributive lattice, are the congruence lattice of an *infinite* semidistributive lattice. Some of these examples were included in an earlier version of this note [5].

This suggests two problems.

Question 1. *Are there additional restrictions on congruence lattices of finite SD lattices?*

Question 2. *Is every finite distributive lattice the congruence lattice of an infinite SD lattice?*

We conjecture that the answers are NO and YES, respectively.

References

[1] K. Adaricheva and J. B. Nation, "Classes of semidistributive lattices," in *Lattice theory: special topics and applications. Vol. 2.* Birkhäuser/Springer, Cham, 2016, pp. 59–101.

[2] K. Adaricheva, R. Freese, and J. B. Nation, "Notes on join semidistributive lattices," *Internat. J. Algebra Comput.*, vol. 32, no. 2, pp. 347–356, 2022, doi: 10.1142/S0218196722500175.

[3] R. Freese and J. B. Nation, "A simple semidistributive lattice with doubly irreducible elements," 2024, https://math.hawaii.edu/~jb/simple_semidistributive_DI.pdf.

[4] R. Freese and J. B. Nation, "A simple semidistributive lattice," *Internat. J. Algebra Comput.*, vol. 31, no. 2, pp. 219–224, 2021, doi: 10.1142/S0218196721500119.

[5] G. Grätzer and J. B. Nation, "Congruence lattices of infinite semidistributive lattices," 2024, https://math.hawaii.edu/~jb/ConSDtrees.pdf.

[6] B. Jónsson, "Sublattices of a free lattice," *Canadian J. Math.*, vol. 13, pp. 256–264, 1961, doi: 10.4153/CJM-1961-021-0.

[7] B. Jónsson and J. E. Kiefer, "Finite sublattices of a free lattice," *Canadian J. Math.*, vol. 14, pp. 487–497, 1962, doi: 10.4153/CJM-1962-040-1.

[8] B. Jónsson and I. Rival, "Lattice varieties covering the smallest nonmodular variety," *Pacific J. Math.*, vol. 82, no. 2, pp. 463–478, 1979.

[9] J. B. Nation, "Congruences of finite semidistributive lattices," *Cubo*, vol. 26, no. 3, pp. 443–473, 2024, doi: 10.56754/0719-0646.2603.443.