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On the Φ-Hilfer iterative fractional differential equations

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ABSTRACT

To avoid studying iterative differential equations with distinct fractional order derivatives it is essential to treat them with a broad fractional derivative, which leaves other fractional derivatives as a special case. In this way, we study an initial value problem for non-linear iterative fractional differential equations involving Φ-Hilfer fractional derivative. We establish the existence and uniqueness of the solution through fixed point theorems. We prove results concerning the dependence of solution and Ulam-Hyers stability of the problem. Finally, we present an example for illustration to demonstrate our outcome.

RESUMEN

Para evitar estudiar ecuaciones diferenciales iterativas con dereivadas fraccionarias de distintos órdenes, es esencial tratarlas a través de una derivada fraccionaria amplia, que deje otras derivadas fraccionarias como un caso especial. De este modo, estudiamos un problema de valor inicial para ecuaciones diferenciales fraccionarias iterativas no-lineales que involucra la derivada fraccionaria Φ-Hilfer. Establecemos la existencia y unicidad de la solución a través de teoremas de punto fijo. Demostramos resultados relacionados a la dependencia de la solución y la estabilidad de Ulam-Hyers del problema. Finalmente, presentamos un ejemplo para ilustrar lo obtenido.

Keywords and Phrases: Iterative fractional differential equations, Φ-Hilfer derivative, fixed point theorems, existence and uniqueness, data dependency, Ulam-Hyers stability.

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1 Introduction

Fractional calculus is a branch of mathematics in which we obtain definitions of derivatives and integrals with arbitrary positive real order so that the classical derivative can act as a special case. There are many more definitions of fractional derivatives, see the monographs [17,28,29]. It is worth obtaining the most generalized fractional differential operator to unify all these definitions. Later, Sousa and Oliveira (2018) [35] investigated the most generalized Φ-Hilfer fractional derivative. In [14, 21, 22, 27], significant theoretical advancements concerning various forms of nonlinear Φ-Hilfer fractional differential equations and several important properties of their solutions are examined. Development of theory after the proposal of the Φ-Hilfer fractional derivative, other versions of fractional operators were studied. For example, a work that addresses the fractional derivative in variable order with respect to the Φ function [38] and the work on calculus of Φ-Hilfer fractional derivative with an additional parameter $k > 0$ and associated fractional differential equations [15, 20].

We note that fractional calculation has been extensively studied and its theory, although well consolidated, still new versions of fractional operators are presented and, certainly interesting and important applications arising from them, will be discussed in the near future. On the other hand, we can also highlight problems of fractional differential equations with p-Laplacian, which have been attracting the attention of researchers. In 2022, Sousa *et al.* [39] first work on variational problems using the Φ-Hilfer fractional derivative was presented. In the work, the authors presented a new fractional Sobolev space for the Φ-Hilfer fractional derivative, and built a variational structure so that it was possible to investigate the existence of weak solutions to a fractional p-Laplacian problem via Nehari manifold [33, 34, 37].

The differential equations which involves the iterates of unknown function is called as Iterative differential equations (IDEs). IDEs are especially useful for simulating real-world systems where the rate of change is dependent on both the function and the number of times the unknown function is applied. They extend traditional differential equations to capture more complex, nonlinear, and self-referential dynamics, with applications across various fields, including biology, physics, and engineering. Examples include infectious disease models [45], the motion of charged particles with retarded interaction [11], insect population dynamics [2], and Nicholson's blowflies model [16]. Due to their wide range of applications, IDEs are an essential area of study.

Eder [7] studied the IDEs of the form

$$u'(t) = u(u(t)),$$

and showed that every solution either identically vanishes or is strictly monotonic. Feckan [8] investigated the functional differential equation

$$u'(t) = h(u(u(t))), \quad u(0) = 0.$$

Vasile Bernide [1] proved convergence theorems under weaker conditions than those suggested by A. Buica [3] and proved the existence of solutions for first-order iterative differential equations.

Iterative fractional differential equations (FDEs) deals iterative differential equations associated with various types of fractional derivatives. They serve as powerful tools for modeling complex systems that exhibit memory effects, non-local interactions, and long-term dependencies. Here, we highlight a few significant studies on iterative FDEs.

Ibrahim [11] investigated the existence and approximation of solution for the iterative Riemann-Liouville FDEs of the form

$$D^{\xi}u(t) = h(t, u(t), u(u(t))), \quad u(0) = u_0.$$

Damag *et al.* [4] proved the existence of solution for the iterative FDEs

$$D^{\xi}u(t) = h(t, u(t), u(u(t)), u'(t)), \quad u(t_0) = u_0, \quad t_0 \in J,$$

by applying non-expansive operator method and Browder-Ghode-Kirk fixed point theorem. Guerfi and Ardjouni [9] investigated existence, uniqueness, continuous dependence and Ulam-Hyers stability of mild solution for the Caputo iterative FDEs of the form

$$\begin{aligned} &{}^{C}D^{\xi}_{0+}u(t) = h\left(u^{[0]}(t), u^{[1]}(t), \ldots, u^{[n]}(t)\right), \\ &u(0) = u'(0) = 0. \end{aligned}$$

Existence and approximation problems for the iterative differential equations are solved in [5, 6, 12, 24, 44–46, 48]. Also, iterative integro-differential equations are studied [10, 13, 18, 32]. For further development of iterative differential equations see [26, 31, 41, 42] and the references therein.

Vivek *et al.* [40] examined the class of Φ-Riemann-Liouville iterative fractional differential equation with non-local condition

$$\begin{aligned} &D^{\xi;\Phi}u(t) = h(t, u(u(t))), \quad 0 < \xi < 1, \\ &u(0) + f(u) = u_0. \end{aligned}$$

Motivated by interesting work mentioned above on iterative differential equations we consider the non-linear iterative FDEs of the form

$$\begin{aligned} {}^{H}D_{0+}^{\xi,\,\eta;\,\Phi}u(t) = h\Big(&\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[0]}(t), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[1]}(t), \\ &\ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[n]}(t)\Big),\ t\in J, \end{aligned} \tag{1.1}$$

$$I_{0+}^{1-\zeta;\,\Phi}u(0) = u_0, \quad u_0 \geq 0, \quad \zeta = \xi + \eta(1-\xi), \tag{1.2}$$

where $J=[0,T]$, Φ is an increasing function on J such that $\Phi \in C^1(J)$ and $\Phi'(t) \neq 0$, for all $t \in J$, ${}^{H}D_{0+}^{\xi,\,\eta;\,\Phi}(\cdot)$ is the Φ-Hilfer derivative of order $\xi \in (0,1)$ and type $\eta \in [0,1]$. Further,

$$\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(t) = t, \tag{1.3}$$

$$\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[j]}(t) = (\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[j-1]}(t)\right), \quad j=1,\ldots n, \tag{1.4}$$

are the iterates of the function $(\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u$ and $h \in C(J^{n+1},\mathbb{R})$ is a positive non-linear function that fulfills a few other requirements, which are detailed subsequently.

We believe that the main results of this paper are best presented as follows:

(1) Before attacking the main results, it was necessary to discuss some properties for the space with weight $C_{1-\zeta;\Phi}(J,\mathbb{R},M)$.

(2) The first contribution of the paper was to investigate the existence and uniqueness of solutions to the problem (1.1)-(1.2) through the theory of fixed points.

(3) In addition to the above, we investigated the continuous dependence and Ulam-Hyers stability.

(4) Finally, we present an example, in order to elucidate the results discussed.

We analyzed iterative differential equations associated Φ-Hilfer fractional derivative for the existence and qualitative properties of solutions in the space of weighted Lipschitz functions.

The iterates of unknown functions defined by (1.3) and (1.4) that appears in the equations (1.1)-(1.2) make the study challenging as it requires domain and codomain of unknown functions should be same and hence appropriate solutions space is required to deal with the solutions of iterative FDEs (1.1)-(1.2). In this context the two weighted spaces are defined. The weighted space $C_{1-\zeta;\Phi}(J,\mathbb{R},L)$ ensures that that the iterates are well defined and $C_{1-\zeta;\Phi}(J,\mathbb{R},M)$ ensures the existence of solution for the iterative FDEs.

The Φ-Hilfer fractional derivative is the most generalized form of fractional derivatives, encompassing various fractional differential operators described in [35] as special cases for varying values of

η and different choices of the function Φ. In this context, the Φ-Hilfer fractional derivative serves as a powerful tool in fractional calculus that unifies the study of fractional differential equations (FDEs) under a single framework. As a result, it is no longer necessary to conduct independent analyses of FDEs using various fractional derivative operators.

This paper is organized as follows. In Section 2, we discuss about Φ-fractional calculus, define some weighted spaces that required for further calculation. Section 3 deals with the properties of weighted space. In Section 4, we investigate existence via fixed point theorem and uniqueness result. Further Section 5 includes continuous dependence, Ulam-Hyers and generalized Ulam-Hyers stability of solution. In Section 6, example is provided to illustrate our results.

2 Preliminaries

In this section, we provide definitions and few basic results pertaining to Φ-fractional calculus. Further, we provide the suitable weighted space to deal with solutions of iterative FDEs.

2.1 Φ-fractional calculus

Definition 2.1 ([17]). *The Φ-Riemann-Liouville fractional integral of order $\xi > 0$ ($\xi \in \mathbb{R}$) of the function $u \in C([a,b],\mathbb{R})$ is given by*

$$I_{a^+}^{\xi\,;\Phi} u(t) = \frac{1}{\Gamma(\xi)} \int_a^t \Phi'(s) \left(\Phi(t) - \Phi(s)\right)^{\xi-1} u(s)\, ds. \tag{2.1}$$

Definition 2.2 ([35]). *The Φ-Hilfer fractional derivative of a function $u \in C^m([a,b],\mathbb{R})$ of order $m-1 < \xi < m$ and type $\eta \in [0,1]$, is defined by*

$${}^H D_{a^+}^{\xi,\,\eta;\,\Phi} u(t) = I_{a^+}^{\eta(m-\xi);\,\Phi} \left(\frac{1}{\Phi'(t)}\frac{d}{dt}\right)^m I_{a^+}^{(1-\eta)(m-\xi);\,\Phi} u(t), \quad t \in (a,b].$$

Lemma 2.3 ([35]). *Let $m-1 < \xi < m \in \mathbb{N}$, $u \in (C^m[a,b],\mathbb{R})$ and $\eta \in [0,1]$. Then*

(i) $I_{a^+}^{\xi\,;\Phi}\, {}^H D_{a^+}^{\xi,\,\eta;\,\Phi} u(t) = u(t) - \sum_{k=1}^{m} \frac{(\Phi(t)-\Phi(a))^{\zeta-k}}{\Gamma(\xi-k+1)} u_\Phi^{[m-k]} I_{a^+}^{(1-\eta)(m-\xi)\,;\Phi} u(a)$, *where* $u_\Phi^{[m-k]}u(t) = \left(\frac{1}{\Phi'(t)}\frac{d}{dt}\right)^{m-k} u(t)$,

(ii) ${}^H D_{a^+}^{\xi,\,\eta;\,\Phi} I_{a^+}^{\xi\,;\Phi} u(t) = u(t)$,

where $\zeta = \xi + \eta(m-\xi)$.

2.2 Weighted spaces

Consider the weighted space

$$C_{1-\zeta;\,\Phi}(J,\mathbb{R}) = \{u : (0,T] \to \mathbb{R}|\, (\Phi(t)-\Phi(0))^{1-\zeta}\, u(t) \in C[0,T]\}$$

with the norm

$$||u||_{C_{1-\zeta\,;\Phi}(J,\mathbb{R})} = \sup_{t\in J}\left|(\Phi(t)-\Phi(0))^{1-\zeta}\, u(t)\right|, \quad 0<\zeta\leq 1.$$

Then the space $\left(C_{1-\zeta;\,\Phi}(J,\mathbb{R}), ||\cdot||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}\right)$ is Banach space.

For $0 < L \leq T$ and $M > 0$, we define the following sets

$$C_{1-\zeta\,;\Phi}(J,\mathbb{R};L) = \left\{u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R}) : 0 \leq (\Phi(t)-\Phi(0))^{1-\zeta}\, u(t) \leq L\right\},$$

and

$$C_{1-\zeta;\Phi}(J,\mathbb{R};M) = \Big\{u \in C_{1-\zeta;\Phi}(J,\mathbb{R};L) : \\ \left|(\Phi(t_2)-\Phi(0))^{1-\zeta}\, u(t_2) - (\Phi(t_1)-\Phi(0))^{1-\zeta}\, u(t_1)\right| \leq M\, |t_2-t_1|\,, t_1,t_2 \in J\Big\}.$$

If $\zeta = 1$ then above weighted spaces reduces respectively to

$$C(J,\mathbb{R};L) = \{u \in C(J,\mathbb{R}) : 0 \leq u(t) \leq L,\, \forall t \in J\}$$

and

$$C(J,\mathbb{R};M) = \{u \in C(J,\mathbb{R};L) : |u(t_2)-u(t_1)| \leq M|t_2-t_1|,\, \forall t_1,t_2 \in J\}, \quad M>0$$

which are defined in [9].

Lemma 2.4 ([48])**.** *If $u_1, u_2 \in C(J,\mathbb{R};M)$, then*

$$\left\|u_1^{[n]} - u_2^{[n]}\right\|_{C(J)} \leq \sum_{j=0}^{n-1} M^j\, \|u_1-u_2\|_{C(J)}\,, \quad n = 1,2,\ldots$$

where $C(J,\mathbb{R}) = \{u|u : J \to \mathbb{R}\, \text{is continuous}\}$ is Banach space with the supremum norm.

3 Properties of weighted space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$.

To prove existence of solution of iterative FDEs (1.1)-(1.2) we use the following Schauder's fixed point theorem.

Theorem 3.1 (Schauder's fixed point theorem [30])**.** *Let U be a non-empty compact convex subset of Banach space $(B,||\cdot||)$ and $A:U\to U$ is a continuous mapping. Then A has a fixed point.*

In the view of Theorem 3.1, we have to prove that the space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is non-empty, convex and compact subset of a Banach space $C_{1-\zeta;\,\Phi}(J,\mathbb{R})$, and the proof of the same is provided in following theorems.

Theorem 3.2. *For $0<L\le T$ and $M>0$, the weighted space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is non-empty, closed and convex subset of $C_{1-\zeta;\,\Phi}(J,\mathbb{R})$.*

Proof. Define $v:(0,T]\to\mathbb{R}$ by $v(t)=(\Phi(t)-\Phi(0))^{\zeta-1}L$, $t\in(0,T]$. Then $(\Phi(t)-\Phi(0))^{1-\zeta}v(t)=L\in C(J,\mathbb{R})$. Therefore $v\in C_{1-\zeta;\,\Phi}(J,\mathbb{R};L)$. Further for any $t_1,t_2\in J$, we have

$$\begin{aligned}&\left|(\Phi(t_2)-\Phi(0))^{1-\zeta}v(t_2)-(\Phi(t_1)-\Phi(0))^{1-\zeta}v(t_1)\right|\\&\quad=\left|(\Phi(t_2)-\Phi(0))^{1-\zeta}(\Phi(t_2)-\Phi(0))^{\zeta-1}L-(\Phi(t_1)-\Phi(0))^{1-\zeta}(\Phi(t_1)-\Phi(0))^{\zeta-1}L\right|\\&\quad=0\le M|t_2-t_1|.\end{aligned}$$

From above discussion it follows that $v\in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$.

Let $\{u_n\}_{n=1}^{\infty}$ be any sequence in $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ and $u\in C_{1-\zeta;\,\Phi}(J,\mathbb{R})$ is such that

$$\lim_{n\to\infty}||u_n-u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}=0. \tag{3.1}$$

Note that

$$\begin{aligned}0&\le\left|(\Phi(t)-\Phi(0))^{1-\zeta}(u_n(t)-u(t))\right|\\&\le\sup_{t\in J}\left|(\Phi(t)-\Phi(0))^{1-\zeta}(u_n(t)-u(t))\right|=||u_n-u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}.\end{aligned} \tag{3.2}$$

Using squeeze theorem for sequences from (3.1) and (3.2) it follows that

$$\lim_{n\to\infty}\left|(\Phi(t)-\Phi(0))^{1-\zeta}u_n(t)-(\Phi(t)-\Phi(0))^{1-\zeta}u(t)\right|=0. \tag{3.3}$$

Further if $u_n\in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ then $u_n\in C_{1-\zeta;\,\Phi}(J,\mathbb{R};L)$ for all n. Thus

$$0\le(\Phi(t)-\Phi(0))^{1-\zeta}u_n(t)\le L,\quad\text{for all } n \text{ and } t\in J. \tag{3.4}$$

Taking limit as $n \to \infty$ in inequality (3.4) and using the continuity of modulus and the limit (3.3), we have

$$0 \leq (\Phi(t) - \Phi(0))^{1-\zeta} u(t) \leq L, \quad \text{for all } t \in J.$$

Therefore $u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; L)$.

Consider for $t_1, t_2 \in J$,

$$\begin{aligned}
&\left|(\Phi(t_2) - \Phi(0))^{1-\zeta} u(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta} u(t_1)\right| \\
&\quad \leq \left|(\Phi(t_2) - \Phi(0))^{1-\zeta} (u_n(t_2) - u(t_2))\right| + \left|(\Phi(t_1) - \Phi(0))^{1-\zeta} (u_n(t_1) - u(t_1))\right| \\
&\quad + \left|(\Phi(t_2) - \Phi(0))^{1-\zeta} u_n(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta} u_n(t_1)\right| \\
&\quad \leq 2 \left\|u_n - u\right\|_{C_{1-\zeta;\Phi}(J,\mathbb{R})} + M|t_2 - t_1|.
\end{aligned}$$

Letting $n \to \infty$ we get, $\left|(\Phi(t_2) - \Phi(0))^{1-\zeta} u(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta} u(t_1)\right| \leq M|t_2 - t_1|$. Thus $u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}, M)$.

Consider any $v, w \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}, M)$ and $s \in [0, 1]$. Then $(\Phi(t) - \Phi(0))^{1-\zeta} v(t)$ and $(\Phi(t) - \Phi(0))^{1-\zeta} w(t)$ are continuous on J hence $(\Phi(t) - \Phi(0))^{1-\zeta} (sv + (1-s)w)(t)$ is continuous on J. This gives $sv + (1-s)w \in C_{1-\zeta;\,\Phi}(J)$. Since $v, w \in C_{1-\zeta;\Phi}(J, \mathbb{R}, L)$ we have $0 \leq (\Phi(t) - \Phi(0))^{1-\zeta} v(t) \leq L$ and $0 \leq (\Phi(t) - \Phi(0))^{1-\zeta} w(t) \leq L$. Therefore for any $t \in J$, yields that

$$\begin{aligned}
&0 \leq (\Phi(t) - \Phi(0))^{1-\zeta} (sv + (1-s)w)(t) \\
&\quad = s (\Phi(t) - \Phi(0))^{1-\zeta} v(t) + (\Phi(t) - \Phi(0))^{1-\zeta} w(t) - s (\Phi(t) - \Phi(0))^{1-\zeta} w(t) \\
&\quad \leq sL + L - sL = L.
\end{aligned}$$

This proves $sv + (1-s)w \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}, L)$. Consider any $t_1, t_2 \in J$, then

$$\begin{aligned}
&\left|(\Phi(t_2) - \Phi(0))^{1-\zeta} (sv + (1-s)w)(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta} (sv + (1-s)w)(t_1)\right| \\
&\quad = s \left|(\Phi(t_2) - \Phi(0))^{1-\zeta} v(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta} v(t_1)\right| \\
&\quad + (1-s) \left|(\Phi(t_2) - \Phi(0))^{1-\zeta} w(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta} w(t_1)\right| \\
&\quad \leq sM|t_2 - t_1| + (1-s)M|t_2 - t_1| = M|t_2 - t_1|.
\end{aligned}$$

From above discussion it follows that $sv + (1-s)w \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ for any $s \in [0, 1]$. Thus proof of $C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ is non-empty, closed and convex subset of $C_{1-\zeta;\,\Phi}(J, \mathbb{R})$ is completed. □

Theorem 3.3. *For $0 < L \leq T$ and $M > 0$, the weighted space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is uniformly bounded and equicontinuous.*

Proof. Let any $u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ then $u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};L)$. Hence

$$0 \leq (\Phi(t) - \Phi(0))^{1-\zeta}\, u(t) \leq L, \quad \text{for all } t \in J.$$

This gives $||u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} \leq L$, for all $u \in C_{1-\zeta\,;\Phi}(J,\mathbb{R};M)$. This proves $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is uniformly bounded.

Let any $u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$. Then $(\Phi(t) - \Phi(0))^{1-\zeta}u(t)$ is continuous for each $t \in J$. Further, we have

$$\left|(\Phi(t_2) - \Phi(0))^{1-\zeta}\, u(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta}\, u(t_1)\right| \leq M|t_2 - t_1|, \quad \text{for all } t_1, t_2 \in J.$$

Let any $\epsilon > 0$. Define $\delta = \frac{\epsilon}{M}$. Then $t_1, t_2 \in J$, $|t_2 - t_1| < \delta$ implies

$$\left|(\Phi(t_2) - \Phi(0))^{1-\zeta}\, u(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta}\, u(t_1)\right| < \epsilon.$$

This proves $C_{1-\zeta;\Phi}(J,\mathbb{R};M)$ is equicontinuous. This completes the proof of $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is uniformly bounded and equicontinuous. □

Remark 3.4. *From* Theorem 3.3 *and Arzela-Ascoli theorem it follows that $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is relatively compact. But $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is also closed subset of $C_{1-\zeta;\,\Phi}(J,\mathbb{R})$ and hence $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is compact subspace of $C_{1-\zeta;\,\Phi}(J,\mathbb{R})$.*

4 Existence and uniqueness results

Theorem 4.1. *Assume that the function $h : J^{n+1} \to [0,\infty)$ satisfies the Lipschitz type condition*

$$|h(t, u_1, u_2, \ldots, u_n) - h(t, v_1, v_2, \ldots, v_n)| \leq \sum_{i=1}^{n} c_i|u_i - v_i|, \quad \textit{where } c_i > 0. \tag{4.1}$$

Then, the iterative FDEs (1.1)-(1.2) *has at least one solution in the weighted space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$, provided*

$$\frac{u_0}{\Gamma(\zeta)} + \frac{\rho^*}{\Gamma(\xi+1)}(\Phi(T) - \Phi(0))^{\xi-\zeta+1} \leq L, \tag{4.2}$$

and

$$\frac{\rho^*}{\Gamma(\xi+1)}\left|(\xi - \zeta + 1)\,(\Phi(c) - \Phi(0))^{\xi-\zeta}\, \Phi'(c)\right| \leq M, \quad \textit{for some } c \in (0,T), \tag{4.3}$$

where

$$\rho = \sup_{t\in J}\{h(t,0,0,\dots,0)\} \quad \textit{and} \quad \rho^* = \rho + L\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j.$$

Proof. Considering equivalent fractional integral equation [36] to the iterative FDEs (1.1)-(1.2), we define an operator A on $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ by

$$(Au)(t) = \frac{(\Phi(t)-\Phi(0))^{\zeta-1}}{\Gamma(\zeta)}u_0 + \frac{1}{\Gamma(\xi)}\int_0^t \Phi'(\tau)\,(\Phi(t)-\Phi(\tau))^{\xi-1} \times h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[1]}(\tau), \dots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[n]}(\tau)\right) d\tau, \tag{4.4}$$

where $t \in (0,T]$. In the view of Schauder's fixed point theorem, we have to show that the mapping $A : C_{1-\zeta;\,\Phi}(J,\mathbb{R};M) \to C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is well defined and continuous. Proof of the same is given in several steps.

Since h is continuous on J we have $h \in C_{1-\zeta;\,\Phi}(J)$. Further, $I_{0+}^{\xi;\,\Phi}$ is bounded from $C_{1-\zeta;\,\Phi}(J)$ to $C_{1-\zeta;\,\Phi}(J)$ implies $I_{0+}^{\xi;\,\Phi}h \in C_{1-\zeta;\,\Phi}(J)$. This gives $Au \in C_{1-\zeta;\,\Phi}(J)$, for all $u \in C_{1-\zeta;\,\Phi}(J)$. Thus the mapping A is well defined.

Now, we show that the mapping A is continuous. Using Lipschitz condition on h, for any $t \in J$, one has

$$\begin{aligned}
&\left|(\Phi(t)-\Phi(0))^{1-\zeta}(Au-Av)(t)\right| \\
&\le \frac{(\Phi(t)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)}\int_0^t \Phi'(\tau)\,(\Phi(t)-\Phi(\tau))^{\xi-1} \\
&\quad \times \left| h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[1]}(\tau), \dots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[n]}(\tau)\right)\right. \\
&\quad \left. - h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}v\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}v\right)^{[1]}(\tau), \dots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}v\right)^{[n]}(\tau)\right)\right| d\tau \\
&\le \frac{(\Phi(t)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)}\int_0^t \Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\xi-1} \\
&\quad \times \sum_{i=1}^{n} c_i \left|\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[i]}(\tau) - \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[i]}(\tau)\right| d\tau \\
&\le \frac{(\Phi(t)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)} \\
&\quad \times \sum_{i=1}^{n} c_i \left\|\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[i]} - \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[i]}\right\|_{C(J,\mathbb{R})} \int_0^t \Phi'(\tau)\,(\Phi(t)-\Phi(\tau))^{\xi-1}\,d\tau \\
&\le \frac{(\Phi(t)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \left\|(\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u - (\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right\|_{C(J,\mathbb{R})} \\
&= \frac{(\Phi(t)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \left\|(\Phi(\cdot)-\Phi(0))^{1-\zeta}\,(u-v)\right\|_{C(J,\mathbb{R})} \\
&= \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j\,\frac{(\Phi(t)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\,||u-v||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}.
\end{aligned}$$

Therefore, we get

$$\|Au - Av\|_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} \leq \frac{1}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j (\Phi(T)-\Phi(0))^{\xi-\zeta+1} ||u-v||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}. \tag{4.5}$$

Let any $\epsilon > 0$. Define

$$\delta = \frac{\epsilon\Gamma(\xi+1)}{(\Phi(T)-\Phi(0))^{\xi-\zeta+1} \sum\limits_{i=1}^{n} c_i \sum\limits_{j=0}^{i-1} M^j}.$$

Then for any $u, v \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ and $||u-v||_{C_{1-\zeta,\,\Phi}(J,\mathbb{R})} < \delta$ we have $||Au-Av||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} < \epsilon$. This proves A is continuous mapping. Next we prove that

$$A\left(C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)\right) \subseteq C_{1-\zeta;\,\Phi}(J,\mathbb{R};M).$$

Let any $u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$. Then,

$$\begin{aligned} \left|(\Phi(t)-\Phi(0))^{1-\zeta}(Au)(t)\right| &\leq \frac{u_0}{\Gamma(\zeta)} + \frac{(\Phi(t)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)} \int_0^t \Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\xi-1} \\ &\times \left| h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[n]}(\tau)\right)\right| d\tau. \end{aligned} \tag{4.6}$$

Using Lipschitz condition on h for any $u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$, it follows that

$$\begin{aligned} &\left| h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[n]}(\tau)\right)\right| \\ &\leq \left| h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[n]}(\tau)\right) \right. \\ &\left. - h(\tau,0,\ldots,0)\right| + |h(\tau,0,\ldots,0)| \\ &\leq \sum_{i=1}^{n} c_i \left| \left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[i]}(\tau)\right| + \rho \leq \sum_{i=1}^{n} c_i \left\| \left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[i]}\right\|_{C(J,\mathbb{R})} + \rho. \end{aligned}$$

Using the inequality in Lemma 2.4, we obtain

$$\begin{aligned} &\left| h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[n]}(\tau)\right)\right| \\ &\leq \rho + \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \left\|(\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right\|_{C(J,\mathbb{R})}. \end{aligned}$$

Using the definition of space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$, we get

$$\begin{aligned} &\left| h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta} u\right)^{[n]}(\tau)\right)\right| \\ &\leq \rho + \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j L = \rho^*, \quad u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M). \end{aligned} \tag{4.7}$$

Using inequality (4.7) in (4.6) for any $u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$, we obtain

$$\begin{aligned}\left|(\Phi(t)-\Phi(0))^{1-\zeta}(Au)(t)\right| &\leq \frac{u_0}{\Gamma(\zeta)}+\frac{\rho^*}{\Gamma(\xi)}(\Phi(t)-\Phi(0))^{1-\zeta}\int_0^t \Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\xi-1}d\tau\\ &\leq \frac{u_0}{\Gamma(\zeta)}+\frac{\rho^*}{\Gamma(\xi+1)}(\Phi(t)-\Phi(0))^{\xi-\zeta+1} \leq \frac{u_0}{\Gamma(\zeta)}+\frac{\rho^*}{\Gamma(\xi+1)}(\Phi(T)-\Phi(0))^{\xi-\zeta+1}\\ &\leq L.\end{aligned}$$

Therefore

$$0 \leq (\Phi(t)-\Phi(0))^{1-\zeta}(Au)(t) \leq \left|(\Phi(t)-\Phi(0))^{1-\zeta}(Au)(t)\right| \leq L,\ u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M). \tag{4.8}$$

This proves $Au \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};L)$.

Further, for any $t_1,t_2 \in J$ with $t_1<t_2$, using inequality (4.7), we have

$$\begin{aligned}&\left|(\Phi(t_2)-\Phi(0))^{1-\zeta}(Au)(t_2)-(\Phi(t_1)-\Phi(0))^{1-\zeta}(Au)(t_1)\right| = \left|\frac{(\Phi(t_2)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)}\int_0^{t_2}\Phi'(\tau)(\Phi(t_2)-\Phi(\tau))^{\xi-1}\right.\\ &\quad\times\left|h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[0]}(\tau),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[n]}(\tau)\right)\right|\\ &\quad-\frac{(\Phi(t_1)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)}\int_0^{t_1}\Phi'(\tau)(\Phi(t_1)-\Phi(\tau))^{\xi-1}\\ &\quad\left.\times\left|h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[0]}(\tau),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[n]}(\tau)\right)\right|d\tau\right|\\ &\leq\left|\frac{\rho^*}{\Gamma(\xi)}(\Phi(t_2)-\Phi(0))^{1-\zeta}\frac{(\Phi(t_2)-\Phi(0))^{\xi}}{\xi}-\frac{\rho^*}{\Gamma(\xi)}(\Phi(t_1)-\Phi(0))^{1-\zeta}\frac{(\Phi(t_1)-\Phi(0))^{\xi}}{\xi}\right|\\ &=\left|\frac{\rho^*}{\Gamma(\xi+1)}\left[(\Phi(t_2)-\Phi(0))^{\xi-\zeta+1}-(\Phi(t_1)-\Phi(0))^{\xi-\zeta+1}\right]\right|.\end{aligned}$$

Define $g(t)=(\Phi(t)-\Phi(0))^{\xi-\zeta+1}$, $t\in[0,T]$. Then clearly g is continuous on $[t_1,t_2]$ and differentiable on (t_1,t_2) for any $t_1,t_2\in J$ with $t_1<t_2$. Therefore using mean value theorem there exists $c\in(0,T)$ such that

$$g'(c)=\frac{g(t_2)-g(t_1)}{t_2-t_1}.$$

Using definition of function g, it follows that

$$(\Phi(t_2)-\Phi(0))^{\xi-\zeta+1}-(\Phi(t_1)-\Phi(0))^{\xi-\zeta+1}=\{(\xi-\zeta+1)\,(\Phi(c)-\Phi(0))^{\xi-\zeta}\,\Phi'(c)\}(t_2-t_1).$$

Therefore, using condition (4.3), one has

$$\begin{aligned}&\left|(\Phi(t_2)-\Phi(0))^{1-\zeta}(Au)(t_2)-(\Phi(t_1)-\Phi(0))^{1-\zeta}(Au)(t_1)\right|\\ &\qquad\leq\frac{\rho^*}{\Gamma(\xi+1)}\left|\{(\xi-\zeta+1)\,(\Phi(c)-\Phi(0))^{\xi-\zeta}\,\Phi'(c)\}\right|(t_2-t_1)\leq M|t_2-t_1|.\end{aligned}\tag{4.9}$$

From inequalities (4.8) and (4.9) , it follows that $(Au)\in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$. This completes the proof of $A\left(C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)\right)\subseteq C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$.

We have proved that A fulfills all the conditions of Schauder's fixed point theorem. Therefore, A has at least one fixed point which is the solution of the iterative FDEs (1.1)-(1.2). □

Theorem 4.2. *Suppose that all conditions of Theorem 4.1 hold. Then the problem* (1.1)-(1.2) *has a unique solution in* $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ *provided*

$$\frac{(\Phi(T)-\Phi(0))^{\xi+1-\zeta}}{\Gamma(\xi+1)}\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}M^j<1. \tag{4.10}$$

Proof. If possible the iterative FDEs (1.1)-(1.2) has two distinct solution v_1 and v_2 in the weighted space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$. Then in view of equivalent fractional integral equation to the iterative FDEs (1.1)-(1.2) and the operator A defined in (4.4), we have $Av_1=v_1$ and $Av_2=v_2$.

Therefore

$$||v_1-v_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}=||Av_1-Av_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}.$$

Proceeding as in the proof of Theorem 4.1, we obtain the estimation on the line of equation (4.5), as follows

$$\begin{aligned}||v_1-v_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}&=||Av_1-Av_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}\\&\leq\frac{1}{\Gamma(\xi+1)}\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}M^j(\Phi(T)-\Phi(0))^{\xi-\zeta+1}||v_1-v_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}.\end{aligned}$$

Using condition (4.10), in above estimation, we obtain

$$||v_1-v_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}<||v_1-v_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})},$$

which is not possible. Therefore iterative FDEs (1.1)-(1.2) has a unique solution. □

5 Continuous dependence and stability results

5.1 Continuous dependence results

To investigate the data dependency of solution of the nonlinear iterative FDEs (1.1)-(1.2), we consider the another nonlinear iterative FDEs of the form

$${}^{H}D_{0+}^{\xi,\,\eta;\,\Phi}\tilde{u}(t)=\tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\tilde{u}\right)^{[0]}(t),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\tilde{u}\right)^{[1]}(t),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\tilde{u}\right)^{[n]}(t)\right),\quad t\in J, \tag{5.1}$$

$$I_{0+}^{1-\zeta,\,\Phi}\tilde{u}(0)=\tilde{u}_0,\quad \tilde{u}_0\geq 0,\quad \zeta=\xi+\eta(1-\xi), \tag{5.2}$$

where $\tilde{h}$ is a function different from h that satisfies all the assumptions of h.

Theorem 5.1. *Suppose that all the assumptions of Theorem 4.2 hold. Then, solution u of iterative FDEs* (1.1)-(1.2) *and solution $\tilde{u}$ of iterative FDEs* (5.1)-(5.2) *satisfies the inequality*

$$||\tilde{u}-u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} \leq \frac{\frac{1}{\Gamma(\zeta)}}{1-\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\sum\limits_{i=1}^{n} c_i \sum\limits_{j=0}^{i-1} M^j}|\tilde{u}_0-u_0| + \frac{\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}}{1-\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\sum\limits_{i=1}^{n} c_i \sum\limits_{j=0}^{i-1} M^j}\left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})}. \quad (5.3)$$

Proof. Using equivalent fractional integral of iterative FDE (1.1)-(1.2) and (5.1)-(5.2), for any $t \in J$ we have

$$\begin{aligned}&\left|(\Phi(t)-\Phi(0))^{1-\zeta}(\tilde{u}(t)-u(t))\right| \leq \left|\frac{\tilde{u}_0}{\Gamma(\zeta)}-\frac{u_0}{\Gamma(\zeta)}\right| + \frac{(\Phi(t)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)}\int_0^t \Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\xi-1} \\ &\times \left|\tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[n]}(\tau)\right)\right. \\ &\left. - h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)\right| d\tau. \quad (5.4)\end{aligned}$$

Next, using Lipschitz condition on $\tilde{h}$, for any $\tau \in J$, we have

$$\begin{aligned}&\left|\tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[n]}(\tau)\right)\right. \\ &\quad \left. - h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)\right| \\ &\leq \left|\tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[n]}(\tau)\right)\right. \\ &\quad \left. - \tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)\right| \\ &\quad + \left|\tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)\right. \\ &\quad \left. - h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)\right| \\ &\leq \sum_{i=1}^{n} c_i \left|\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[i]}(\tau) - \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[i]}(\tau)\right| + \left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})} \\ &\leq \left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})} + \sum_{i=1}^{n} c_i \left\|\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[i]} - \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[i]}\right\|_{C(J,\mathbb{R})}.\end{aligned}$$

Using Lemma 2.4, for any $\tau \in J$, we obtain

$$\begin{aligned}
&\Big|\tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[0]}(\tau),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[n]}(\tau)\right)\\
&\quad -h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)\Big|\\
&\leq \left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})}+\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}M^j\left\|(\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}-(\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right\|_{C(J,\mathbb{R})}\\
&\leq \left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})}+\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}M^j||\tilde{u}-u||_{C_{1-\zeta,\,\Phi}(J,\mathbb{R})}.
\end{aligned} \tag{5.5}$$

Using estimation (5.5) in the inequality (5.4), for any $t \in J$, we have

$$\begin{aligned}
&\left|(\Phi(t)-\Phi(0))^{1-\zeta}(\tilde{u}(t)-u(t))\right|\\
&\quad\leq \frac{|\tilde{u}_0-u_0|}{\Gamma(\zeta)}+\frac{(\Phi(t)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\left[\left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})}+\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}M^j\left\|\tilde{u}-u\right\|_{C_{1-\zeta;\,\Phi}(J)}\right]\\
&\quad\leq \frac{|\tilde{u}_0-u_0|}{\Gamma(\zeta)}+\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\left[\left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})}+\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}M^j||\tilde{u}-u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}\right]
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
||\tilde{u}-u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}\leq{}& \frac{\frac{1}{\Gamma(\zeta)}}{1-\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}M^j}|\tilde{u}_0-u_0|\\
&+\frac{\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}}{1-\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}M^j}\left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})}. \quad \square
\end{aligned}$$

Remark 5.2. (1) *Theorem 5.1 gives the continuous dependence of the solution of the problem* (1.1)-(1.2) *on the initial condition as well as on the nonlinear functions.*

(2) *If* $h = \tilde{h}$ *in* (5.3) *then* Theorem 5.1 *gives the dependency of the solution of* (1.1)-(1.2) *on initial condition.*

(3) *If* $u_0 = \tilde{u}_0$ *in* (5.3) *then* Theorem 5.1 *gives the dependency of the solution of* (1.1)-(1.2) *on the nonlinear functions.*

(4) *If* $h = \tilde{h}$ *and* $u_0 = \tilde{u}_0$ *in* (5.3), Theorem 5.1 *gives the uniqueness of solution of the problem* (1.1)-(1.2).

5.2 Stability results

To discuss the Ulam-Hyers stablity results we need the following definitions.

Definition 5.3 ([19]). *The iterative FDEs* (1.1)-(1.2) *is said to be Ulam-Hyers stable if there exists a real number $K > 0$ such that for each $\epsilon > 0$ and for each solution $v \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ of the inequality*

$$\Big|{}^{H}D^{\xi,\,\eta;\,\Phi}_{0+}v(t) -h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[0]}(t), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[1]}(t), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[n]}(t)\right)\Big| \leq \epsilon, \quad (5.6)$$

with $I^{1-\zeta;\,\Phi}_{o+}v(0) = u_0$, there exists a solution $u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ of problem (1.1)-(1.2) *that satisfy*

$$||u - v||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)} \leq K\epsilon, \quad t \in J.$$

Definition 5.4 ([19]). *The iterative FDEs* (1.1)-(1.2) *is said to be generalized Ulam-Hyers stable if there exists $\chi \in C(J, \mathbb{R}^+)$ with $\chi(0) = 0$ such that for each $\epsilon > 0$ and for each solution $v \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ of the inequality* (5.6) *with $I^{1-\zeta;\,\Phi}_{o+}v(0) = u_0$, there exists a solution $u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ of the problem* (1.1)-(1.2) *satisfying*

$$||u - v||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)} \leq \chi(\epsilon), \quad t \in J.$$

Theorem 5.5. *Assume all the assumptions of* Theorem 4.2 *hold. Then the problem* (1.1)-(1.2) *is Ulam-Hyers stable.*

Proof. Consider $v \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ be a function such that $I^{1-\zeta;\,\Phi}_{o+}v(0) = u_0$, that satisfy the inequality (5.6). Then integrating it, we obtain

$$\begin{aligned}&\Big|v(t) - \frac{(\Phi(t)-\Phi(0))^{\zeta-1}}{\Gamma(\zeta)}u_0 - \frac{1}{\Gamma(\xi)}\int_0^t \Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\xi-1}\\ &\times h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[n]}(\tau)\right)d\tau\Big|\\ &\leq I^{\xi;\,\Phi}_{0+}\epsilon = \frac{\epsilon}{\Gamma(\xi+1)}(\Phi(t)-\Phi(0))^{\xi}, \quad t \in (0, T].\end{aligned}$$

If $u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ is the solution of the iterative FDEs (1.1)-(1.2) then using Lipschitz condition of h, we obtain

$$\begin{aligned}&|v(t) - u(t)|\\ &= \Big|v(t) - \frac{(\Phi(t)-\Phi(0))^{\zeta-1}}{\Gamma(\zeta)}u_0 - \frac{1}{\Gamma(\xi)}\int_0^t \Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\xi-1}\\ &\quad\times h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)d\tau\Big|\end{aligned}$$

$$\leq \left| v(t) - \frac{(\Phi(t)) - \Phi(0))^{\zeta-1}}{\Gamma(\zeta)} u_0 - \frac{1}{\Gamma(\xi)} \int_0^t \Phi'(\tau)(\Phi(t) - \Phi(\tau))^{\xi-1} \right.$$
$$\left. \times h\left(\left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[0]} (\tau), \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[1]} (\tau), \ldots, \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[n]} (\tau) \right) d\tau \right|$$
$$+ \left| \frac{1}{\Gamma(\xi)} \int_0^t \Phi'(\tau)(\Phi(t) - \Phi(\tau))^{\xi-1} \right.$$
$$\times \left[h\left(\left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[0]} (\tau), \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[1]} (\tau), \ldots, \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[n]} (\tau) \right) \right.$$
$$\left. \left. - h\left(\left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right)^{[0]} (\tau), \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right)^{[1]} (\tau), \ldots, \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right)^{[n]} (\tau) \right) \right] d\tau \right|$$
$$\leq \frac{\epsilon}{\Gamma(\xi+1)} (\Phi(t) - \Phi(0))^{\xi} + \frac{1}{\Gamma(\xi)} \int_0^t \Phi'(\tau)(\Phi(t) - \Phi(\tau))^{\xi-1}$$
$$\times \sum_{i=1}^{n} c_i \left| \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[i]} (\tau) - \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right)^{[i]} (\tau) \right| d\tau.$$

Using the inequality in the Lemma 2.4, we have

$$|v(t) - u(t)|$$
$$\leq \frac{\epsilon}{\Gamma(\xi+1)} (\Phi(t) - \Phi(0))^{\xi} + \frac{(\Phi(t) - \Phi(0))^{\xi}}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \left\| \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[i]} - \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right)^{[i]} \right\|_{C(J)}$$
$$\leq \frac{\epsilon}{\Gamma(\xi+1)} (\Phi(t) - \Phi(0))^{\xi} + \frac{(\Phi(t) - \Phi(0))^{\xi}}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \left\| (\Phi(\cdot) - \Phi(0))^{1-\zeta} v - (\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right\|_{C(J)}$$
$$= \frac{\epsilon}{\Gamma(\xi+1)} (\Phi(t) - \Phi(0))^{\xi} + \frac{(\Phi(t) - \Phi(0))^{\xi}}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|v - u\|_{C_{1-\zeta;\,\Phi}(J)}, \quad t \in J.$$

Therefore consider for all $t \in J$,

$$\|v - u\|_{C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)} = \sup_{t\in J} \left| (\Phi(t) - \Phi(0))^{1-\zeta} (v(t) - u(t)) \right|$$
$$\leq \frac{\epsilon}{\Gamma(\xi+1)} (\Phi(T) - \Phi(0))^{\xi-\zeta+1} + \frac{(\Phi(T) - \Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|v - u\|_{C_{1-\zeta;\,\Phi}(J)}.$$

This gives

$$\|v - u\|_{C_{1-\zeta;\,\Phi}(J)} \leq \frac{\frac{\epsilon}{\Gamma(\xi+1)} (\Phi(T) - \Phi(0))^{\xi-\zeta+1}}{1 - \frac{1}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j (\Phi(T) - \Phi(0))^{\xi-\zeta+1}}.$$

Define $K = \dfrac{\dfrac{(\Phi(T) - \Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}}{1 - \frac{1}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j (\Phi(T) - \Phi(0))^{\xi-\zeta+1}}$. Then $K > 0$ and we have

$$\|v - u\|_{C_{1-\zeta;\,\Phi}(J)} \leq K\epsilon.$$

This proves iterative FDEs (1.1)-(1.2) is Ulam-Hyers stable. □

Corollary 5.6. *Suppose all the assumptions of* Theorem 5.5 *are satisfied then the iterative FDEs* (1.1)-(1.2) *is generalized Ulam-Hyers stable.*

Proof. Follows by taking $\phi(\epsilon) = K\epsilon$. □

6 Examples

Example 6.1. *Consider the following initial value problem for iterative fractional differential equations*

$$\begin{aligned} {}^{H}D_{0+}^{\frac{1}{2},\eta;\Phi}v(t) &= \frac{(\Phi(t)-\Phi(0))^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{1}{100}(\Phi(t)-\Phi(0))^{2-\zeta} \\ &\quad + \frac{1}{200}(\Phi(t)-\Phi(0))^{1-\zeta}\left(\Phi\left(\frac{(\Phi(t)-\Phi(0))^{2-\zeta}}{2}\right)-\Phi(0)\right) - \frac{1}{50}\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}v\right)^{[1]}(t) \\ &\quad - \frac{1}{100}\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}v\right)^{[2]}(t), \qquad (6.1) \\ I_{0+}^{1-\zeta;\Phi}v(0) &= 0, \quad t\in\tilde{J}=[0,1]. \qquad (6.2) \end{aligned}$$

Define the function $h : \tilde{J}^3 \to [0,\infty)$ by,

$$\begin{aligned} h(t,u,v) &= \frac{(\Phi(t)-\Phi(0))^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{1}{100}(\Phi(t)-\Phi(0))^{2-\zeta} \\ &\quad + \frac{1}{200}(\Phi(t)-\Phi(0))^{1-\zeta}\left(\Phi\left(\frac{(\Phi(t)-\Phi(0))^{2-\zeta}}{2}\right)-\Phi(0)\right) - \frac{1}{50}u - \frac{1}{100}v. \end{aligned}$$

Then for any $t\in\tilde{J}$ and $u_i, v_i \in \tilde{J}, (i=1,2)$, we have

$$|h(t,u_1,u_2)-h(t,v_1,v_2)| \le \frac{1}{50}|u_1-v_1| + \frac{1}{100}|u_2-v_2|.$$

This shows h satisfies Lipschitz type condition (4.1) with $c_1 = \frac{1}{50}$ and $c_2 = \frac{1}{100}$. We have $T=1$, choose $L=1$ then the condition $0<L\le T$ hold. Further, in the view of condition (4.2) and (4.3) choose $c=\frac{1}{3}$, $M>0$ and the function Φ such that

$$\frac{2\rho^*}{\sqrt{\pi}}\left|\left(\frac{3}{2}-\zeta\right)\left(\Phi\left(\frac{1}{3}\right)-\Phi(0)\right)^{\frac{1}{2}-\zeta}\Phi'\left(\frac{1}{3}\right)\right| \le M, \qquad (6.3)$$

and

$$\frac{2\rho^*}{\sqrt{\pi}}(\Phi(1)-\Phi(0))^{\frac{3}{2}-\zeta} \le 1, \qquad (6.4)$$

where

$$\begin{aligned}\rho &= \sup_{t\in[0,1]} \{h(t,0,0)\}\\ &= \frac{(\Phi(1)-\Phi(0))^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{(\Phi(1)-\Phi(0))^{2-\zeta}}{100} + \frac{(\Phi(1)-\Phi(0))^{1-\zeta}}{200}\left(\Phi\left(\frac{(\Phi(1)-\Phi(0))^{2-\zeta}}{2}\right)-\Phi(0)\right),\end{aligned} \tag{6.5}$$

$$\rho^* = \rho + c_1 + c_2(1+M) = \rho + \frac{1}{50} + \frac{1}{100}(1+M). \tag{6.6}$$

With the choices of constant M and the function Φ that satisfies conditions (6.3) and (6.4), all the assumptions of Theorem 4.1 are satisfied. Thus Schauder's fixed point Theorem 3.1 guarantee the at least one solution of the iterative FDEs (6.1)-(6.2) in the weighted space $C_{1-\zeta;\Phi}(\tilde{J},\mathbb{R};M)$. By actual substitution one can verify that

$$v(t) = \frac{\Phi(t)-\Phi(0)}{2}, \quad t\in[0,1], \tag{6.7}$$

is the solution of the iterative FDEs (6.1)-(6.2). Further in addition to the conditions (6.3) and (6.4), if the constant M and the function Φ satisfy the condition

$$\frac{2(\Phi(1)-\Phi(0))^{\frac{3}{2}-\zeta}}{\sqrt{\pi}}\left(\frac{1}{50}+\frac{1}{100}(1+M)\right) < 1, \tag{6.8}$$

the problem (6.1)-(6.2) has unique solution in the weighted space $C_{1-\zeta;\Phi}(\tilde{J},\mathbb{R};M)$.

Note that the function v defined in (6.7) is the unique solution of the problem (6.1)-(6.2). If we take $\Phi(t)=t$, $t\in[0,1]$ and $\eta=1$ the problem (6.1)-(6.2) involving Φ-Hilfer fractional derivative reduces to the following initial value problem for iterative FDEs of the form

$$^{C}D^{\frac{1}{2}}_{0+}v(t) = \frac{t^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{1}{100}t + \frac{1}{400}t - \frac{1}{50}v^{[1]}(t) - \frac{1}{100}v^{[2]}(t) \tag{6.9}$$

$$v(0) = 0. \tag{6.10}$$

In this case

$$\rho = \frac{1}{\sqrt{\pi}} + \frac{1}{100} + \frac{1}{400} = 0.5766.$$

If we choose $M=1$ then

$$\rho^* = 0.5766 + \frac{1}{50} + \frac{2}{100} = 0.6166.$$

Further, the conditions (6.3), (6.4) and (6.8) reduce respectively to

$$\frac{2\rho^*}{\sqrt{\pi}}\left|\left(\frac{3}{2}-\zeta\right)\left(\Phi\left(\frac{1}{3}\right)-\Phi(0)\right)^{\frac{1}{2}-\zeta}\Phi'\left(\frac{1}{3}\right)\right| = \frac{0.6166\times 2}{\sqrt{\pi}}\left|\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{-1}{2}}\right| = 0.6025 < 1 \tag{6.11}$$

$$\frac{2\rho^*}{\sqrt{\pi}}\left(\Phi(1)-\Phi(0)\right)^{\frac{3}{2}-\zeta} = \frac{0.6166\times 2}{\sqrt{\pi}} = 0.6957 < 1 \tag{6.12}$$

and

$$\frac{2(\Phi(1)-\Phi(0))^{\frac{3}{2}-\zeta}}{\sqrt{\pi}}\left(\frac{1}{50}+\frac{1}{100}(1+M)\right) = \frac{2}{\sqrt{\pi}}\left(\frac{1}{50}+\frac{2}{100}\right) = 0.0451 < 1. \tag{6.13}$$

Note that all the conditions of Theorem 4.2 are satisfied. Therefore the initial value problem for Caputo iterative FDEs (6.9)-(6.10) has a unique solution in the space $C(\tilde{J},\mathbb{R};1)$. By actual substitution, one can verify that

$$v(t) = \frac{t}{2}, \quad t \in [0,1], \tag{6.14}$$

is the unique solution of the problem (6.9)-(6.10).

We remark that the constants c_1 and c_2 appears naturally as h satisfy Lipschitz condition. $T = 1$ is the end point of the interval on which the problem (6.1)-(6.2) is considered. The constant L $(0 < L \leq T)$, $c \in (0, T)$ and $M > 0$ one choose in the view of condition (4.2) and (4.3). These constants depends on the choice of function Φ.

7 Conclusion

Through analytical approaches we examine the nonlinear iterative FDEs with Φ-Hilfer fractional derivative for existence, uniqueness, stability and dependency of solutions. The conditions (4.2) and (4.3) required to prove the existence and uniqueness results Theorem 4.1 and Theorem 4.2 are strong. Achieving the same kind of outcomes by removing the restrictions in (4.2) and (4.3) will be very interesting. We have given specific examples to demonstrate our findings. Investigating alternative conditions with weaker constraints is essential for ensuring the existence and uniqueness of solutions for iterative Φ-Hilfer fractional differential equations (FDEs). In this context, one can analyze iterative Φ-Hilfer FDEs under various types of initial and boundary conditions to study their existence, uniqueness, different forms of stability, and other qualitative properties. Further, the work explored in [23, 25, 43, 47, 49] can be analyzed by integrating the iterates of unknown function and the fractional calculus.

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The authors have no relevant financial or non-financial interests to disclose.

Author contributions

The author whose name appears on the submission contributed equally to this work, solely responsible for the conception of the work and its final form.

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