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Compactness of the difference of weighted composition operators between weighted l^p spaces

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ABSTRACT

This paper investigates the properties of weighted composition operators acting between different weighted l^p spaces. Inspired by recent advancements in the field, we explore criteria for the continuity and compactness of these operators. Specifically, we provide simple conditions, in terms of normalized canonical sequences, for the continuity and compactness of the difference between two weighted composition operators, $W_{\varphi,u}$ and $W_{\psi,v}$. Furthermore, we calculate the essential norm of these operators. Our results extend and generalize previous works, offering new insights into the behavior of weighted composition operators in Banach sequence spaces. The findings contribute to the understanding of these operators' topological properties, particularly their applications in sequence spaces and functional analysis.

RESUMEN

Este artículo investiga las propiedades de operadores de composición con peso actuando entre diferentes espacios l^p con pesos. Inspirados por avances recientes en el área, exploramos criterios para la continuidad y compacidad de estos operadores. Específicamente, entregamos condiciones simples, en términos de sucesiones canónicas normalizadas, para la continuidad y compacidad de la diferencia entre dos operadores de composición con peso, $W_{\varphi,u}$ y $W_{\psi,v}$. Más aún, calculamos la norma esencial de estos operadores. Nuestros resultados extienden y generalizan trabajos previos, ofreciendo nuevas formas de entender el comportamiento de operadores de composición con peso en espacios de Banach de sucesiones. Los hallazgos contribuyen a la comprensión de las propiedades topológicas de estos operadores, particularmente sus aplicaciones a espacios de sucesiones y análisis funcional.

Keywords and Phrases: Banach sequence spaces, weighted composition operators, compactness.

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1 Introduction

The study of the properties of weighted composition operators has captivated numerous researchers worldwide. These operators play a significant role in various areas of functional analysis and have applications in sequence spaces and spaces of analytic functions. Specifically, in the context of Banach sequence spaces, weighted composition operators are useful for studying processes where the inputs are infinite collections of data $\{x(k)\}$ that undergo an organization and selection process, and are finally assigned a weight to obtain an output, similar to creating frequency tables in statistics. Organizing a sequence $\boldsymbol{x} = \{x(k)\}$ involves defining a function $\varphi : \mathbb{N} \to \mathbb{N}$, while assigning weights involves multiplying by a sequence $\boldsymbol{u} = \{u(k)\}$. This leads to the definition of the weighted composition operator $W_{\varphi,u}$ by

$$W_{\varphi,u}(\boldsymbol{x}) := \boldsymbol{u} \cdot (\boldsymbol{x} \circ \varphi).$$

The operator $W_{\varphi,u}$ can be seen as a composition of two important classical transformations: the multiplication operator M_u and the composition operator C_φ. In fact, when φ is the identity, $W_{\varphi,u}$ becomes M_u, and when $u(n) = 1$ for all n, it becomes C_φ. The properties of these operators have been widely studied in various contexts, including weighted sequence spaces [5, 9, 14, 15], which we define in the next paragraph.

Throughout the development of this document, p represents a fixed parameter in $[1, \infty)$. A numerical sequence $\boldsymbol{x} = \{x(k)\}$ is said to belong to the weighted l^p space, denoted as $\boldsymbol{x} \in l^p(\boldsymbol{r})$, if

$$\|\boldsymbol{x}\|_{l^p(\boldsymbol{r})} = \left(\sum_{k=1}^{\infty} |x(k)|^p \, r(k)^p \right)^{1/p} < \infty, \tag{1.1}$$

where $\boldsymbol{r} = \{r(k)\}$ is a weight, that is, $r(k) > 0$ for all $k \in \mathbb{N}$. The pair $\left(l^p(\boldsymbol{r}), \|\cdot\|_{l^p(\boldsymbol{r})}\right)$ constitutes a Banach space. These kinds of spaces naturally appear in the literature when studying properties of some operators in sequence spaces. For instance, for $p > 1$, the Cesàro space ces_p is contained in $l^p\left(k^{1-p}\right)$, indicating that every evaluation functional on ces_p is continuous.

Inspired by the work of Carpintero *et al.* [5], where they explored in detail the properties of weighted composition operators acting on weighted $\ell^\infty(\boldsymbol{r})$ sequence spaces, and as a continuation of the recent work of Cardona-Gutierrez *et al.* [4], which characterized the functions u and φ that define weighted composition operators with closed ranges when acting between two different weighted l^p spaces and analyzed when this operator is upper or lower semi-Fredholm, we aim to give simple criteria in terms of the normalized canonical sequences for the continuity and compactness of the difference of two weighted composition operators $W_{\varphi,u} - W_{v,\psi}$ acting between two different weighted l^p spaces. An important consequence of our results is the computation of the essential norm of the weighted composition operators $W_{\varphi,u}$ acting between two distinct weighted l^p spaces.

Our findings significantly extend and generalize previous works, such as those by [8, 10], which analyzed the case of weighted ℓ^2, and more recently, the work by Albanese and Mele [2], where the continuity and compactness of $W_{\varphi,u}$ between two different weighted l^p spaces were characterized.

In this article, we are particularly interested in knowing when $W_{\varphi,u}(\boldsymbol{x}) \in l^p(\boldsymbol{s})$ for all $\boldsymbol{x} \in l^p(\boldsymbol{r})$ (continuity problem) and in establishing other topological properties such as the compactness of the difference of two weighted composition operators (compactness problem). These problems have been widely studied in the context of holomorphic function spaces (see [7, 11, 13] and references therein), but in the context of Banach sequence spaces, they are still under development. Specifically, we shall prove the following properties:

(1) The operator $W_{\varphi,u}$ is continuous from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ if and only if

$$L_{\varphi,u} = \sup_{n\in\mathbb{N}} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} < \infty.$$

In this case, $\|W_{\varphi,u}\| = L_{\varphi,u}$.

(2) The difference of weighted composition operators $W_{\varphi,u} - W_{\psi,v}$ from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ is compact if and only if

$$\lim_{n\to\infty} \frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = 0.$$

(3) The essential norm of $W_{\varphi,u}$ from $l^p(\boldsymbol{r})$ to $l^p(\boldsymbol{s})$ is computed by

$$\|W_{\varphi,u}\|_e = \limsup_{n\to\infty} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}}.$$

This last result extends a result by Castillo *et al.* in [6].

The problem (1), was recently solved by Albanese and Mele [2]; however, in Section 2, for the sake of completeness and to benefit the reader, we provide a simple proof.

Additionally, in Section 3, we establish a very general criteria for the compactness of pointwise continuous operators acting between different weighted l^p spaces, which allows us to characterize the compactness of the difference of two weighted composition operators (see Theorem 3.1).

Finally, in this article, we use $\boldsymbol{x} = \{x(k)\}$ to denote a numerical complex sequence, while $(\boldsymbol{x}_n)$ denotes a sequence of sequences. Also, for a fixed $n \in \mathbb{N}$, we consider the canonical sequence $\boldsymbol{e}_n$, defined as $e_n(k) = 1$ if $k = n$ and $e_n(k) = 0$ otherwise.

2 Continuity of the weighted composition operators on $l^p(\boldsymbol{r})$

In this section we characterize all continuous weighted composition operators between two different weighted l^p spaces in terms of the norm of the images of the normalized canonical sequence. From now, for a function $\varphi : \mathbb{N} \to \mathbb{N}$, it is convenient to define

$$\text{Ran}\,(\varphi) = \{n \in \mathbb{N} : n = \varphi(k) \text{ for some } k \in \mathbb{N}\}\,.$$

We can see that

$$\|W_{\varphi,u}(\boldsymbol{e}_n)\|^p_{l^p(\boldsymbol{s})} = \sum_{k:\varphi(k)=n} |u(k)|^p\, s(k)^p$$

and $\|W_{\varphi,u}\,(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})} = 0$ whenever $n \notin \text{Ran}\,(\varphi)$. The following result is due to Albanese and Mele [2], and we include a brief proof for the benefit of the reader.

Theorem 2.1 ([2])**.** *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights. Suppose that $\boldsymbol{u} = \{u(k)\}$ is a complex sequence and let $\varphi : \mathbb{N} \to \mathbb{N}$ be a function. The operator $W_{\varphi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is continuous if and only if*

$$L_{\varphi,u} = \sup_{n\in\mathbb{N}} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} < \infty. \tag{2.1}$$

In this case, $\|W_{\varphi,u}\|_{op} = L_{\varphi,u}$.

Proof. Since $\boldsymbol{e}_n \in l^p(\boldsymbol{r})$ for all $n \in \mathbb{N}$, the condition (2.1) holds when we suppose that the operator $W_{\varphi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is continuous. Conversely, if there exists $L_{\varphi,u} > 0$ such that

$$\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})} \leq L_{\varphi,u}\, \|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})} = L_{\varphi,u} r(n),$$

and we fix any $\boldsymbol{x} = \{x(k)\} \in l^p(\boldsymbol{r})$, then we have

$$\begin{aligned}\|W_{\varphi,u}\,(\boldsymbol{x})\|^p_{l^p(\boldsymbol{s})} &= \sum_{k=1}^{\infty} |u(k)|^p\, |x\,(\varphi(k))\,|^p s(k)^p \leq \sum_{n\in\varphi(\mathbb{N})} |x(n)|^p L^p_{\varphi,u} r(n)^p \\ &\leq L^p_{\varphi,u} \sum_{n=1}^{\infty} |x(n)|^p r(n)^p = L^p_{\varphi,u} \|\boldsymbol{x}\|^p_{l^p(\boldsymbol{r})},\end{aligned}$$

and the operator $W_{\varphi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is continuous. The above argument also proves that

$$\|W_{\varphi,u}\|_{op} = \sup_{n\in\mathbb{N}} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = L_{\varphi,u}.$$

This proves the result. $\square$

Since weighted composition operators generalize multiplication and composition operators, we have the following two important consequences:

Corollary 2.2. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights and suppose that $\boldsymbol{u} = \{u(k)\}$ is a complex sequence. The multiplication operator $M_u : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is continuous if and only if*

$$\sup_{n\in\mathbb{N}} \frac{\|M_u(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = \sup_{n\in\mathbb{N}} \frac{s(n)}{r(n)}|u(n)| < \infty.$$

Proof. It follows from Theorem 2.1 with $\varphi = \mathrm{Id}$, the identity function on $\mathbb{N}$, and by recalling that $W_{\mathrm{Id},u} = M_u$. □

Corollary 2.3. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights and let $\varphi : \mathbb{N} \to \mathbb{N}$ be a function. The composition operator $C_\varphi : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is continuous if and only if*

$$\sup_{n\in\varphi(\mathbb{N})} \frac{\|C_\varphi(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = \sup_{n\in\varphi(\mathbb{N})} \frac{1}{r(n)} \left(\sum_{k:\varphi(k)=n} s(k)^p \right)^{1/p} < \infty.$$

Proof. It follows from Theorem 2.1 with $\boldsymbol{u}(n) = 1$ for all $n \in \mathbb{N}$, a constant function on $\mathbb{N}$, and by recalling that, in this case, $W_{\varphi,u} = C_\varphi$. □

Similar results were obtained in [3] in the context of analytic functions (see also [12]).

3 On the compactness

In this section we shall obtain a characterization for the compactness of the difference operator $W_{\varphi,u} - W_{\psi,v} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ in terms of the canonical sequences. We said that a linear operator $K : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is pointwise continuous if for each sequence $(\boldsymbol{x}_n) \subseteq l^p(\boldsymbol{r})$ such that $\boldsymbol{x}_n \to 0$ pointwise ($\lim_{n\to\infty} x_n(m) = 0$ for all $m \in \mathbb{N}$), we have

$$\lim_{n\to\infty} (K(\boldsymbol{x}_n))(m) = 0$$

for all $m \in \mathbb{N}$. Clearly, the difference between two weighted composition operators is pointwise continuous. For this kind of operators, we have the following result which could have some interest by itself. A much more general result can be found in [1]. We include a proof for benefit of the reader.

Theorem 3.1. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights and suppose that $K : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is a pointwise continuous operator. The operator $K : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is compact if and only if for each norm-bounded sequence $(\boldsymbol{x}_n) \subseteq l^p(\boldsymbol{r})$ such that $\boldsymbol{x}_n \to 0$ pointwise, we have*

$$\lim_{n\to\infty} \|K(\boldsymbol{x}_n)\|_{l^p(\boldsymbol{s})} = 0. \tag{3.1}$$

Proof. Let us suppose first that $K : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is a compact operator. Let $(\boldsymbol{x}_n) \subseteq l^p(\boldsymbol{r})$ be any norm-bounded sequence such that $\boldsymbol{x}_n \to 0$ pointwise and suppose that the condition (3.1) is false. Then, there exists an $\epsilon > 0$ and a subsequence $(\boldsymbol{x}_{n_k})$ of $(\boldsymbol{x}_n)$ such that

$$\|K(\boldsymbol{x}_{n_k})\|_{l^p(\boldsymbol{s})} \geq \epsilon \tag{3.2}$$

for all $k \in \mathbb{N}$. Thus, the compactness of K implies that, by passing to a subsequence, if necessary, we can suppose that $(K(\boldsymbol{x}_{n_k}))$ converges to $\boldsymbol{y} \in l^p(\boldsymbol{s})$. That is,

$$\lim_{k\to\infty} \|K(\boldsymbol{x}_{n_k}) - \boldsymbol{y}\|_{l^p(\boldsymbol{s})} = 0. \tag{3.3}$$

We shall prove that $\boldsymbol{y} = \boldsymbol{0}$ (the null sequence). Indeed, for $m \in \mathbb{N}$ arbitrary but fixed, we have

$$|y_{n_k}(m) - y(m)|^p \leq \frac{1}{s(m)^p} \|K(\boldsymbol{x}_{n_k}) - \boldsymbol{y}\|^p_{l^p(\boldsymbol{s})},$$

with $\boldsymbol{y}_{n_k} = K(\boldsymbol{x}_{n_k})$. Thus, since K is pointwise continuous, we can write

$$|y(m)|^p = \lim_{k\to\infty} |y_{n_k}(m) - y(m)|^p \leq \lim_{k\to\infty} \frac{1}{s(m)^p} \|K(\boldsymbol{x}_{n_k}) - \boldsymbol{y}\|^p_{l^p(\boldsymbol{s})} = 0.$$

This last fact produces a contradiction between (3.2) and (3.3). Therefore

$$\lim_{n\to\infty} \|K(\boldsymbol{x}_n)\|_{l^p(\boldsymbol{s})} = 0,$$

and the implication is proved.

Next, we suppose that for all norm-bounded sequence $(\boldsymbol{x}_n) \subseteq l^p(\boldsymbol{r})$ such that $\boldsymbol{x}_n \to 0$ pointwise, we have

$$\lim_{n\to\infty} \|K(\boldsymbol{x}_n)\|_{l^p(\boldsymbol{s})} = 0.$$

We are going to show that the operator $K : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is compact. To see this, we fix any $(\boldsymbol{y}_n) \subseteq l^p(\boldsymbol{r})$ such that $\|\boldsymbol{y}_n\|_{l^p(\boldsymbol{r})} \leq 1$ for all $n \in \mathbb{N}$. The numerical sequence $\{y_n(1)\}$ of all first components is bounded since

$$|y_n(1)|^p\, r(1)^p \leq \sum_{k=1}^{\infty} |y_n(k)|^p\, r(k)^p = \|\boldsymbol{y}_n\|^p_{l^p(\boldsymbol{r})} \leq 1.$$

Hence, the Bolzano–Weierstrass theorem guarantees that there exists a convergent subsequence $\left\{y_n^{(1)}(1)\right\}$ of $\{y_n(1)\}$ and we can find $y(1) \in \mathbb{C}$ such that

$$\lim_{n\to\infty} |y_n^{(1)}(1) - y(1)| = 0.$$

Hence, we obtain a subsequence $\left(\boldsymbol{y}_n^{(1)}\right)$ of $(\boldsymbol{y}_n)$ whose first component is a convergent numerical sequence.

Arguing similarly, we have $\left|y_n^{(1)}(2)\right| r(2)^p \leq 1$, so there is a $y(2) \in \mathbb{C}$ and a subsequence $\left(\boldsymbol{y}_n^{(2)}\right)$ of $\left(\boldsymbol{y}_n^{(1)}\right)$ such that

$$\lim_{n\to\infty} \left|y_n^{(2)}(2) - y(2)\right| = 0.$$

Furthermore, we also have

$$\lim_{n\to\infty} \left|y_n^{(2)}(1) - y(1)\right| = 0.$$

Thus, by repeating this process, we obtain a subsequence $\left(\boldsymbol{y}_{n_k}\right)$ of $(\boldsymbol{y}_n)$ and a numerical sequence $\boldsymbol{y} = \{y(j)\}$ such that $\boldsymbol{y}_{n_k} \to \boldsymbol{y}$ pointwise. Also, for $H \in \mathbb{N}$ fixed we have

$$\sum_{j=1}^{H} |y(j)|^p r(j)^p = \limsup_{k\to\infty} \sum_{j=1}^{H} |y_{n_k}(j)|^p r(j)^p \leq \limsup_{k\to\infty} \left\|\boldsymbol{y}_{n_k}\right\|_{l^p(\boldsymbol{r})}^p \leq 1$$

and $\boldsymbol{y} \in l^p(\boldsymbol{r})$. Thus, applying the hypothesis to the sequence $\boldsymbol{x}_k = \boldsymbol{y}_{n_k} - \boldsymbol{y}$ which converges to zero pointwise, we conclude that

$$\lim_{k\to\infty} \|K(\boldsymbol{x}_k)\|_{l^p(\boldsymbol{s})} = \lim_{k\to\infty} \left\|K\left(\boldsymbol{y}_{n_k}\right) - K(\boldsymbol{y})\right\|_{l^p(\boldsymbol{s})} = 0$$

and the operator $K : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is compact. □

As an important consequence of the above result we have:

Theorem 3.2. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights, suppose that $\boldsymbol{u} = \{u(k)\}$ and $\boldsymbol{v} = \{v(k)\}$ are complex sequences, $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ are functions and $W_{\varphi,u}, W_{\psi,v} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ are continuous operators. The difference $W_{\varphi,u} - W_{\psi,v}$ from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ is compact if and only if*

$$\lim_{n\to\infty} \frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = 0. \tag{3.4}$$

Proof. Let us suppose first that the difference $W_{\varphi,u} - W_{\psi,v} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is a compact operator. For each $n \in \mathbb{N}$, we set

$$\boldsymbol{x}_n = \frac{\boldsymbol{e}_n}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}}.$$

Then $(\boldsymbol{x}_n)$ is a norm-bounded sequence which converges pointwise to the null sequence. Hence, Theorem 3.1 implies that the expression (3.4) holds.

Assume now that (3.4) holds and suppose that $(\boldsymbol{x}_n)$ is any bounded sequence in $l^p(\boldsymbol{r})$ such that

$$\lim_{n\to\infty} x_n(m) = 0$$

for all $m \in \mathbb{N}$. We shall prove that

$$\lim_{n\to\infty} \|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{x}_n)\|_{l^p(\boldsymbol{s})} = 0.$$

We can write

$$\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{x}_n)\|^p_{l^p(\boldsymbol{s})} = S_1(n) + S_2(n),$$

where

$$S_1(n) = \sum_{k:\varphi(k)=\psi(k)} |u(k) - v(k)|^p |x_n(\varphi(k))|^p s(k)^p,$$

$$S_2(n) = \sum_{k:\varphi(k)\neq\psi(k)} |u(k)x_n(\varphi(k)) - v(k)x_n(\psi(k))|^p s(k)^p.$$

For the first sum we have

$$S_1(n) \leq \sum_{m=1}^{\infty} |x_n(m)|^p r(m)^p \frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_m)\|^p_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_m\|^p_{l^p(\boldsymbol{r})}}.$$

While for the second sum we can see that

$$S_2(n) \leq 2^p \sum_{k:\varphi(k)\neq\psi(k)} (|u(k)x_n(\varphi(k))|^p + |v(k)x_n(\psi(k))|^p) s(k)^p \leq S_3(n) + S_4(n)$$

with

$$S_3(n) = 2^p \sum_{m\in\varphi(\mathbb{N})} |x_n(m)|^p \sum_{l\in\psi(\mathbb{N})-\{m\}} \sum_{k\in\varphi^{-1}(\{m\})\cap\psi^{-1}(\{l\})} |u(k)|^p s(k)^p,$$

$$S_4(n) = 2^p \sum_{l\in\psi(\mathbb{N})} |x_n(l)|^p \sum_{m\in\varphi(\mathbb{N})-\{l\}} \sum_{k\in\varphi^{-1}(\{m\})\cap\psi^{-1}(\{l\})} |v(k)|^p s(k)^p.$$

But, if $k \in \varphi^{-1}(\{m\}) \cap \psi^{-1}(\{l\})$, then $\varphi(k) = m$ and $\psi(k) = l \neq m$ and thus $e_m(\varphi(k)) = 1$, $e_m(\psi(k)) = 0$ and the third sum on the right of $S_3(n)$ can be written as

$$\sum_{k\in\varphi^{-1}(\{m\})\cap\psi^{-1}(\{l\})} |u(k)e_m(\varphi(k)) - v(k)e_m(\psi(k))|^p s(k)^p.$$

Thus

$$S_3(n) \leq 2^p \sum_{m=1}^{\infty} |x_n(m)|^p r(m)^p \frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_m)\|^p_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_m\|^p_{l^p(\boldsymbol{r})}}$$

and the same is also true for $S_4(n)$. Therefore,

$$\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{x}_n)\|_{l^p(\boldsymbol{s})}^p \leq 2^{p+2} \sum_{m=1}^{\infty} |x_n(m)|^p \, r(m)^p \frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_m)\|_{l^p(\boldsymbol{s})}^p}{\|\boldsymbol{e}_m\|_{l^p(\boldsymbol{r})}^p}.$$

Finally, by hypothesis, for any $\varepsilon > 0$, we can find $m_0 \in \mathbb{N}$ such that

$$\frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_m)\|_{l^p(\boldsymbol{s})}^p}{\|\boldsymbol{e}_m\|_{l^p(\boldsymbol{r})}^p} < \frac{\varepsilon}{2^{p+2}}$$

for all $m \geq m_0$. Also, there exists $M > 0$ such that $\|\boldsymbol{x}_n\|_{l^p(\boldsymbol{r})} \leq M$. Thus, we can write

$$\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{x}_n)\|_{l^p(\boldsymbol{s})}^p \leq 2^{p+2} \sum_{m=1}^{m_0} |x_n(m)|^p \, r(m)^p \frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_m)\|_{l^p(\boldsymbol{s})}^p}{\|\boldsymbol{e}_m\|_{l^p(\boldsymbol{r})}^p} + \varepsilon M^p$$

and the result follows from Theorem 3.1 since $(\boldsymbol{x}_n)$ converges pointwise to zero as $n \to \infty$. □

As an immediate consequence, we have:

Corollary 3.3. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights, suppose that $\boldsymbol{u} = \{u(k)\}$ is a complex sequence and $\varphi : \mathbb{N} \to \mathbb{N}$ is a function.*

(1) *The operator $W_{\varphi,u}$ from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ is compact if and only if*

$$\lim_{n\to\infty} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = 0.$$

This result was recently obtained by Albanese and Mele in [2, Theorem 3.12].

(2) *The multiplication operator M_u, as defined in the proof of Corollary 2.2, from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ is compact if and only if*

$$\lim_{n\to\infty} \frac{\|M_u(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = \lim_{n\to\infty} \frac{s(n)}{r(n)} |u(n)| = 0.$$

(3) *The composition operator C_φ, as defined in the proof of Corollary 2.3, from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ is compact if and only if*

$$\lim_{n\to\infty} \frac{\|C_\varphi(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = \lim_{n\to\infty} \frac{1}{r(n)} \left(\sum_{k:\varphi(k)=n} s(k)^p \right)^{1/p} = 0.$$

4 On the essential norm of $W_{\phi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$

In this section we calculate the essential norm of weighted composition operators acting between weighted l^p spaces in terms of canonical basis. We recall that if X and Y are Banach spaces and $\mathcal{K}(X, Y)$ denotes the set of all compact operators from X into Y, then the essential norm of T is denoted by $\|T\|_e$ and it is the distance of T to $\mathcal{K}(X, Y)$. That is,

$$\|T\|_e = \inf\{\|T - K\|_{op} : K \in \mathcal{K}(X, Y)\}.$$

It is clear that $T : X \to Y$ is compact if and only if $\|T\|_e = 0$. Hence, in virtue of Corollary 3.3 (1), the following result is expected.

Theorem 4.1. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights, suppose that $\boldsymbol{u} = \{u(k)\}$ is a complex sequence, $\varphi : \mathbb{N} \to \mathbb{N}$ is a function and suppose that the operator $W_{\varphi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is continuous. Then*

$$\|W_{\varphi,u}\|_e = \limsup_{n\to\infty} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}}.$$

Proof. It is convenient to consider, for $\epsilon > 0$ fixed, the following set

$$S_\epsilon = \left\{ n \in \mathbb{N} : \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} \geq \epsilon \right\}.$$

Then

$$\limsup_{n\to\infty} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = \inf\{\epsilon > 0 : S_\epsilon \text{ is finite}\}.$$

Clearly $S_\epsilon \subseteq \varphi(\mathbb{N})$ for all $\epsilon > 0$ and $S_{\epsilon_1} \subseteq S_{\epsilon_2}$ whenever $\epsilon_1 > \epsilon_2$. The set

$$S = \{\epsilon > 0 : S_\epsilon \text{ is finite}\}$$

is bounded from below by zero, hence we can consider the number

$$\eta = \inf\{\epsilon > 0 : S_\epsilon \text{ is finite}\}.$$

We have two case: $\eta = 0$ and $\eta > 0$. We are going to prove that in both of the cases we can conclude $\|W_{\varphi,u}\|_e = \eta$.

Suppose first that $\eta = 0$. Then S_ϵ is finite for all $\epsilon > 0$. We shall prove that the operator $W_{\varphi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is compact. Indeed, if $W_{\varphi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is not a compact operator, then by Corollary 3.3, we can find an $\epsilon_0 > 0$ and an unbounded and increasing sequence $\{n_k\} \subseteq \mathbb{N}$ such that

$$\frac{\|W_{\varphi,u}(\boldsymbol{e}_{n_k})\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}} \geq \epsilon_0$$

for all $k \in \mathbb{N}$. This means that S_{ϵ_0} is an infinite set and it is a contradiction to the fact that $\eta = 0$.

Suppose now that $\eta > 0$. Consider $\epsilon > 0$ such that $\eta - \frac{1}{2}\epsilon > 0$. Then by definition of infimum, $\eta - \frac{1}{2}\epsilon \notin S$, the set

$$S_{\eta-\frac{1}{2}\epsilon} = \left\{ n \in \mathbb{N} : \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} \geq \eta - \frac{\epsilon}{2} \right\}$$

is infinite and we can find an unbounded and increasing sequence $\{n_k\}$ of positive integers contained in $S_{\eta-\frac{1}{2}\epsilon}$. Hence, the sequence $(\boldsymbol{x}_k)$ defined by

$$\boldsymbol{x}_k = \frac{\boldsymbol{e}_{n_k}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}}$$

is bounded in $l^p(\boldsymbol{r})$, it converges pointwise to zero as $k \to \infty$ and therefore, Theorem 3.1 allows us to say that

$$\lim_{k\to\infty} \left\| K\left(\frac{\boldsymbol{e}_{n_k}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}} \right) \right\|_{l^p(\boldsymbol{s})} = 0$$

for any compact operator K from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$. Thus, for any $K \in \mathcal{K}(l^p(\boldsymbol{r}), l^p(\boldsymbol{s}))$ we have

$$\begin{aligned}
\|W_{\varphi,u} - K\| &\geq \left\| (W_{\varphi,u} - K)\left(\frac{\boldsymbol{e}_{n_k}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}} \right) \right\|_{l^p(\boldsymbol{s})} \\
&\geq \left\| W_{\varphi,u}\left(\frac{\boldsymbol{e}_{n_k}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}} \right) \right\|_{l^p(\boldsymbol{s})} - \left\| K\left(\frac{\boldsymbol{e}_{n_k}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}} \right) \right\|_{l^p(\boldsymbol{s})} \\
&\geq \eta - \frac{1}{2}\epsilon - \left\| K\left(\frac{\boldsymbol{e}_{n_k}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}} \right) \right\|_{l^p(\boldsymbol{s})}
\end{aligned}$$

for all $n_k \in S_{\eta-\frac{1}{2}\epsilon}$. Taking $k \to \infty$, we obtain

$$\|W_{\varphi,u} - K\| \geq \eta - \frac{1}{2}\epsilon$$

and since $K \in \mathcal{K}(l^p(\boldsymbol{r}), l^p(\boldsymbol{s}))$ and $\epsilon > 0$ are arbitrary, we really have $\|W_{\varphi,u}\|_e \geq \eta$.

Next, we shall prove that $\|W_{\varphi,u}\|_e \leq \eta$. By definition of infimum, for any $\epsilon > 0$, the number $\eta + \epsilon$ is not a lower bound, hence the set

$$S_{\eta+\epsilon} = \left\{ n \in \mathbb{N} : \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} \geq \eta + \epsilon \right\}$$

is finite. We set the symbol v by

$$v(k) = \begin{cases} u(k), & \text{if } \varphi(k) \in S_{\eta+\epsilon}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $S_{\eta+\epsilon}$ is finite, it is clear that

$$\lim_{n\to\infty}\frac{\|W_{\varphi,v}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}}=0.$$

Indeed, for $n>\max S_{\eta+\epsilon}$, we have

$$\|W_{\varphi,v}(\boldsymbol{e}_n)\|^p_{l^p(\boldsymbol{s})}=\sum_{k=1}^{\infty}|v(k)|^p\,|e_n(\varphi(k))|^p\,s(k)^p=\sum_{k:\varphi(k)\in S_{\eta+\epsilon}}|u(k)|^p\,|e_n(\varphi(k))|^p\,s(k)^p=0.$$

In particular, $W_{\varphi,v}:l^p(\boldsymbol{r})\to l^p(\boldsymbol{s})$ is a compact operator (see Corollary 3.3 (1)). Hence, the definition of essential norm of $W_{\varphi,u}$ allow us to write

$$\begin{aligned}\|W_{\varphi,u}\|_e\le\|W_{\varphi,u}-W_{\varphi,v}\|&=\sup\left\{\frac{\|W_{\varphi,u-v}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}},n\in\mathbb{N}\right\}\\&=\sup\left\{\frac{\|W_{\varphi,u-v}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}},n\in\mathbb{N}\setminus S_{\eta+\epsilon}\right\}\le\eta+\epsilon,\end{aligned}$$

since $u(k)-v(k)=0$ when $\varphi(k)\in S_{\eta+\epsilon}$. Hence, we conclude that $\|W_{\varphi,u}\|_e\le\eta$ and the proof of theorem is complete. □

Remark 4.2. *From the proof of the above theorem, we can see that $\eta=0$ when $\varphi(\mathbb{N})$ is finite. Hence, any weighted composition operator $W_{\varphi,u}$ from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ in which the symbol φ is a bounded function is a compact operator. Furthermore, since a linear operator $K:X\to Y$ is compact if and only if its essential norm is zero, an immediate consequence of our Theorem 4.1 is Theorem 3.12 in [2], which is stated in Corollary 3.3 (1).*

Corollary 4.3. *Let $\boldsymbol{r},\boldsymbol{s}$ be two weights, suppose that $\boldsymbol{u}=\{u(k)\}$ is a complex sequence and $\varphi:\mathbb{N}\to\mathbb{N}$ is a function.*

(1) *Suppose that the multiplication operator $M_u:l^p(\boldsymbol{r})\to l^p(\boldsymbol{s})$, as defined in the proof of Corollary 2.2, is continuous. The essential norm of this operator M_u from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ is computed by*

$$\limsup_{n\to\infty}\frac{\|M_u(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}}=\limsup_{n\to\infty}\frac{s(n)}{r(n)}|u(n)|.$$

(2) *Suppose that the composition operator $C_\varphi:l^p(\boldsymbol{r})\to l^p(\boldsymbol{s})$, as defined in the proof of Corollary 2.3, is continuous. The essential norm of this operator C_φ from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ is computed by*

$$\limsup_{n\to\infty}\frac{\|C_\varphi(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}}=\limsup_{n\to\infty}\frac{1}{r(n)}\left(\sum_{k:\varphi(k)=n}s(k)^p\right)^{1/p}.$$

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