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Canonical metrics and ambiKähler structures on 4-manifolds with $U(2)$ symmetry

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ABSTRACT

For $U(2)$-invariant 4-metrics, we show that the B^t-flat metrics are very different from the other canonical metrics (Bach-flat, Einstein, extremal Kähler, etc.) We show every $U(2)$-invariant metric is conformal to two separate Kähler metrics, leading to ambiKähler structures. Using this observation we find new complete extremal Kähler metrics on the total spaces of $\mathcal{O}(-1)$ and $\mathcal{O}(+1)$ that are conformal to the Taub-bolt metric. In addition to its usual hyperKähler structure, the Taub-NUT's conformal class contains two additional complete Kähler metrics that make up an ambi-Kähler pair, making five independent compatible complex structures for the Taub-NUT, each of which is conformally Kähler.

RESUMEN

Para 4-métricas $U(2)$-invariantes, mostramos que las métricas B^t-planas son muy diferentes de las otras métricas canónicas (Bach-planas, Einstein, Kähler extremas, etc.) Mostramos que toda métrica $U(2)$-invariante es conforme a dos métricas Kähler separadas, lo que nos lleva a estructuras ambiKähler. Usando esta observación encontramos nuevas métricas Kähler extremas completas en los espacios totales de $\mathcal{O}(-1)$ y $\mathcal{O}(+1)$ que son conformes a la métrica Taub-bolt. Adicionalmente a su estructura usual hiperKähler, la clase conforme de Taub-NUT contiene dos métricas Kähler completas adicionales que hacen un par ambi-Kähler, lo que genera cinco estructuras complejas compatibles independientes para el Taub-NUT, cada una de las cuales es conformemente Kähler.

Keywords and Phrases: Cohomogeneity-1 metric, canonical metric, Taub-NUT, Bach tensor

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1 Introduction

Cohomogeneity-1 metrics with $U(2)$ symmetry have the form

$$g = A(r)\,dr^2 + B(r)\,(\eta^1)^2 + C(r)\left((\eta^2)^2 + (\eta^3)^2\right) \tag{1.1}$$

where η^1, η^2, η^3 are the usual left-invariant covector fields on $\mathbb{S}^3$. Naively the topology is $\mathbb{R}\times\mathbb{S}^3$, but there could be a quotient on the $\mathbb{S}^2$ factor, and topological changes occur at locations where B or C reach zero. We classify canonical metrics of this form including the B^t-flat metrics, and create new explicit examples of canonical metrics using the ambiKähler techniques of [2]. This project began as a way to develop supporting examples for other work, and treads such familiar ground that we expected few surprises. But we did find surprises, two of which we feel worth reporting to the wider community.

The first is how the B^t-flat metrics fit among the other canonical metrics. The space of $U(2)$-invariant extremal Kählers is rather small—up to homothety the moduli space is 3-dimensional—and except for the B^t flat metrics there are basically no other canonical metrics. Up to a choice of conformal factor, the Bach-flat metrics are a 2-parameter subspace of the extremal[1] metrics. The Einstein and harmonic-curvature metrics [14] are identical, and up to conformal factors are exactly the Bach-flat metrics. Half-conformally flat metrics are conformally extremal, and up to conformal factors the metrics with $W^+=0$ (or $W^-=0$) form a 1-parameter subspace of the Bach-flat metrics. The KE metrics and the Ricci-flat metrics are each a 1-parameter subclass of the Bach-flats. Up to homothety there are exactly three complete Ricci-flat KE metrics: flat $\mathbb{R}^4$, the Eguchi-Hanson, and the Taub-NUT. The Taub-NUT is extraordinary; see Proposition 2.5 and Section 4.

The B^t-flat metrics of [25] are exceptions to this framework. A B^t-flat metric is a metric satisfying the Euler-Lagrange equations of the functional

$$B^t = \int |W|^2 + t\int s^2 \tag{1.2}$$

where $t\in(-\infty,\infty]$, and we set $B^\infty=\int s^2$. The B^0 extremals are the Bach-flat metrics, and the B^∞ extremals are either scalar-flat or Einstein (see [5] for stable points of the $\int s^2$ functional). For $t\neq 0,\infty$ the B^t Euler-Lagrange equations are an overdetermined 8^{th} order system. After an appropriate reduction we find a 5-dimensional moduli space of B^t-flat metrics up to homothety. If the constant scalar curvature (CSC) condition is imposed, the CSC B^t-flat metrics constitute a 4-parameter family up to homothety. Intuitively, as t varies in $[0,\infty]$, the B^t-flat metrics would seem to interpolate between the Bach-flat metrics at $t=0$ and the Einstein metrics at $t=\infty$. As we pointed out, up to conformal factors these are exactly the same class, so it would stand

[1] We will use *extremal* to mean *extremal Kähler*, and *KE* to mean *Kähler-Einstein*.

to reason that the B^t-flat metrics would stay within this class. We find this is not the case; see Theorem 1.4.

The second surprise has to do with the global nature of certain complete ambiKähler pairs. Any metric (1.1) is automatically compatible with two complex structures which give opposite orientations that are both conformally Kähler—in short, each Kähler metric of the form (1.1) is a partner in an ambiKähler pair [2]. In Section 4 we consider four examples: an ambiKähler pair conformal to the classic Taub-NUT, and an ambiKähler pair conformal to the classic Taub-bolt. The two metrics conformal to the Taub-NUT are complete extremal Kähler metrics, one of which has zero scalar curvature (ZSC) and is 2-ended, and the other of which is one-ended and strictly extremal. The two metrics conformal to the Taub-bolt are complete extremal metrics, and exist on two different underlying complex surfaces, $\mathcal{O}(-1)$ and $\mathcal{O}(+1) \approx \mathbb{C}P^2 \setminus \{pt\}$. The metric on $\mathcal{O}(+1)$ is the only complete extremal Kähler metric, known to the authors, with a curve of positive self intersection. For instance the Eguchi-Hanson [17] and LeBrun metrics [29] lie on the total spaces of various $\mathcal{O}(k)$ with $k < 0$.

Placing the metric (1.1) in a more useful form, we solve $dz = \frac{2\sqrt{AB}}{C}dr$ for z to obtain

$$g = C\left(\frac{1}{4F}dz^2 + F(\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2\right) \tag{1.3}$$

where we have abbreviated $F = \frac{B}{C}$, now a function of z. If $f = f(z)$ is any function and $\{e_1, e_1, e_3\}$ is the $\mathbb{S}^3$ frame dual to $\{\eta^1, \eta^2, \eta^3\}$, then

$$J_f = -2f\frac{\partial}{\partial z} \otimes \eta^1 + \frac{1}{2f}e_1 \otimes dz - e_2 \otimes \eta^3 + e_3 \otimes \eta^2 \tag{1.4}$$

is a complex structure; see Lemma 2.1. Setting $f = \pm F$, the two complex structures $J^\pm = J_{\pm F}$ are compatible with g, and produce opposite orientations. The (1,1) forms are

$$\omega^\pm = g(J^\pm\cdot, \cdot) = \pm\frac{1}{2}Cdz \wedge \eta^1 + C\eta^2 \wedge \eta^3. \tag{1.5}$$

From $d\eta^i = -\epsilon^i{}_{jk}\eta^j \wedge \eta^k$ we have $d\omega^\pm = (\pm C + C_z)dz \wedge \eta^2 \wedge \eta^3$, so a $U(2)$-invariant metric g is always conformally Kähler, and is Kähler when the conformal factor is $C = C_0 e^{\mp z}$, respectively.

The following linear operators appear frequently:

$$\mathcal{L}^+ = \left(-\frac{1}{2}\frac{d}{dz} + 1\right)\left(-\frac{d}{dz} + 1\right), \quad \mathcal{L}^- = \left(\frac{1}{2}\frac{d}{dz} + 1\right)\left(\frac{d}{dz} + 1\right) \tag{1.6}$$

as does the 4^{th} order linear operator $\mathcal{L}^+ \circ \mathcal{L}^- = \frac{1}{4}\frac{\partial^4}{\partial z^4} - \frac{5}{4}\frac{\partial^2}{\partial z^2} + 1$. The third-order nonlinear operator $\mathcal{B}$ also appears:

$$\mathcal{B}(F,F) = \left(-\frac{1}{2}F_{zz} + \frac{3}{2}F_z + F - 1\right)(\mathcal{L}^+(F) - 1) + F_z\big(\mathcal{L}^+(F)\big)_z. \tag{1.7}$$

This is a bit messy, but $\mathcal{B}$ can be understood as a first integral of the inhomogeneous operator $F \mapsto \mathcal{L}^+(\mathcal{L}^-(F)) - 1$; see equation (3.15). We will often use $\{\sigma^0, \sigma^1, \sigma^2, \sigma^3\}$, where $\sigma^0 = \frac{1}{|dz|}dz$ and $\sigma^i = \frac{1}{|\eta^i|}\eta^i$, to mean the orthonormal frame found by normalizing orthogonal frame $\{dz, \eta^1, \eta^2, \eta^3\}$.

Proposition 1.1. *The metric* (1.3) *has scalar curvature*

$$s = -4C^{-1}\left(\frac{\partial^2 F}{\partial z^2} + \frac{1}{2}F - 2\right) - 24C^{-\frac{3}{2}}\frac{\partial}{\partial z}\left(F\,\frac{\partial C^{\frac{1}{2}}}{\partial z}\right) \tag{1.8}$$

and trace-free Ricci tensor

$$\begin{aligned}\mathring{\mathrm{Ric}} = &\frac{4F}{\sqrt{C}}\left(\frac{\partial^2}{\partial z^2}\frac{1}{\sqrt{C}} - \frac{1}{4}\frac{1}{\sqrt{C}}\right)\cdot\left((\sigma^0)^2 - (\sigma^1)^2\right)\\ &+ 2\left(\frac{1}{\sqrt{C}}\frac{\partial}{\partial z}\left(F\frac{\partial}{\partial z}\frac{1}{\sqrt{C}}\right) - \frac{1}{C}\left(\frac{1}{2}\frac{\partial^2 F}{\partial z^2} - \frac{3}{4}F + 1\right)\right)\cdot\left((\sigma^0)^2 + (\sigma^1)^2 - (\sigma^2)^2 - (\sigma^3)^2\right).\end{aligned} \tag{1.9}$$

The Weyl curvatures and their divergences are

$$\begin{aligned}W^{\pm} &= -\frac{1}{C}(\mathcal{L}^{\pm}(F) - 1)\left(\omega^{\pm}\otimes\omega^{\pm} - \frac{2}{3}Id_{\bigwedge^{\pm}}\right)\\ \delta W^{\pm} &= W^{\pm}\left(\nabla \log\left|e^{\pm\frac{3}{2}z}(\mathcal{L}^{\pm}(F) - 1)\sqrt{C}\right|, \cdot, \cdot, \cdot\right).\end{aligned} \tag{1.10}$$

The Bach tensor is

$$\begin{aligned}Bach = &\frac{16}{3C^2}\cdot F\cdot\left(\mathcal{L}^-(\mathcal{L}^+(F)) - 1\right)\cdot\Big(-2(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\Big)\\ &+ \frac{8}{3C^2}\cdot\mathcal{B}(F,F)\cdot\Big(-(\sigma^0)^2 - (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\Big).\end{aligned} \tag{1.11}$$

If the metric is Kähler with respect to J^+, then the scalar curvature and Ricci form are

$$\begin{aligned}s &= -\frac{8}{C}\left(\mathcal{L}^+(F) - 1\right), \quad \textit{and}\\ \rho &= -\frac{2}{C}\left(\mathcal{L}^+(F) - 1\right)\omega^+ - \frac{2}{C}\left(\left(-\frac{1}{2}\frac{\partial}{\partial z} + 1\right)\left(\frac{\partial}{\partial z} + 1\right)F - 1\right)\omega^-.\end{aligned} \tag{1.12}$$

We remark that the $U(2)$-ansatz linearizes the Bach-flat equations $Bach = 0$, reducing them to $\mathcal{L}^+\circ\mathcal{L}^-(F) - 1 = 0$. The equation $\mathcal{B}(F,F) = 0$ is then an algebraic restriction on initial conditions.

When studying metrics—rather than just solutions of ODEs—it is useful to reduce the metrics by homothetic equivalence. In our setting this reduces the dimension of the solution space by two: one dimension for translation in z and one for multiplication of g by a positive constant.

Proposition 1.2 (Extremal and Bach-flat metrics)**.** *The metric* (1.3) *is extremal with complex structure J^+ if and only if $C = C_0 e^{-z}$ and $\mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$, meaning*

$$F(z) = 1 + \frac{1}{2}C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2}C_4 e^{2z}. \tag{1.13}$$

Such a metric is Bach-flat if and only if, in addition to (1.13)*, also $C_1C_4 - C_2C_3 = 0$.*

Consequently, up to homothety, the extremal metrics form a 3-parameter family, and up to homothety and conformal factors the Bach-flat metrics constitute a 2-parameter subfamily of the extremal metrics.

A metric is said to have *harmonic curvature* if $\delta\,\mathrm{Rm} = 0$, which is equivalent to $\delta W = 0$ and $s = const$; see [7, 14]. In the $U(2)$-invariant case $\delta W = 0$ actually implies $s = const$.

Proposition 1.3 (Einstein and harmonic-curvature metrics)**.** *For the metric* (1.3) *the following are equivalent: 1) $\delta W = 0$, 2) $\delta\,\mathrm{Rm} = 0$, 3) the metric is Einstein: $\mathring{\mathrm{Ric}} = 0$, 4) F and C satisfy*

$$F = 1 + \frac{1}{2}C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2}C_4 e^{2z}, \quad C = \frac{e^{-z}}{(C_5 + C_6 e^{-z})^2}, \tag{1.14}$$

with the two relations $C_1C_5 - C_2C_6 = 0$ and $C_3C_5 - C_4C_6 = 0$. Given (1.14)*, scalar curvature is the constant $s = -24(C_2C_5^2 - 2C_5C_6 + C_3C_6^2)$.*

A $U(2)$-invariant metric is Bach-flat if and only if it is conformally Einstein. The metric (1.14) is KE with respect to J^+ if and only if $C_6 = 0$ (so also $C_1 = C_3 = 0$), and KE with respect to J^- if and only if $C_5 = 0$ (so also $C_2 = C_4 = 0$). Up to homothety, there is a 1-parameter family of Ricci-flat metrics, and exactly three complete Ricci-flat KE metrics: the flat metric, the Taub-NUT metric, and the Eguchi-Hanson metric. See Propositions 3.2 and 3.5.

Theorem 1.4. *In the $U(2)$-invariant case, the space of solutions to the B^t-flat equations is 7-dimensional. Up to homothety, these constitute a 5-parameter family of metrics and the CSC B^t-flat metrics constitute a 4-parameter family. When $t \neq 0, \infty$, there exist CSC B^t-flat metrics that are not conformal to any extremal metric.*

The overdetermined 8^{th} order B^t-flat system is complicated, but appears explicitly in Lemma 3.8. In Section 4 we discuss the ambiKähler transform, and examine complete extremal metrics conformal to the classic Taub-NUT and -bolt metrics.

2 Complex structures, metrics, and topology

The metric (1.3), complex structures $J^{\pm}$, and $(1,1)$ forms $\omega^{\pm} = g(J^{\pm}\cdot,\,\cdot)$ are

$$\begin{aligned} g &= C\Big(\frac{1}{4F}dz^2 + F(\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2\Big) \\ J^{\pm} &= \mp 2F\frac{\partial}{\partial z}\otimes\eta^1 \pm \frac{1}{2F}e_1\otimes dz - e_2\otimes\eta^3 + e_3\otimes\eta^2 \\ \omega^{\pm} &= \pm\frac{1}{2}C dz\wedge\eta^1 \,+\, C\eta^2\wedge\eta^3. \end{aligned} \tag{2.1}$$

In Section 2.1 we study the complex structures. In Section 2.2 we compute curvature quantities up through the Bach tensor. In Section 2.3 we examine the topology and asymptotics which the $U(2)$ ansatz may produce.

2.1 The complex structures

Here we check the integrability of the left-invariant almost complex structures J_f. We also study the *right*-invariant compatible complex structures that we call $I^{\pm}$.

Lemma 2.1. *Given any $f = f(z) \neq 0$, the complex structure J_f is integrable.*

Proof. The splitting $\bigwedge^1_{\mathbb{C}} = \bigwedge^{1,0}\oplus\bigwedge^{0,1}$ into $\pm\sqrt{-1}$ eigenspaces of J_f gives

$$\bigwedge{}^{0,1} \;=\; span_{\mathbb{C}}\left\{\frac{1}{2f}dz - \sqrt{-1}\eta^1,\;\; \eta^2 - \sqrt{-1}\eta^3\right\}. \tag{2.2}$$

On bases we compute

$$\begin{aligned} &d\left(\frac{1}{2f}dz - \sqrt{-1}\eta^1\right) = -2\sqrt{-1}\eta^2\wedge\eta^3 = 2\eta^2\wedge\left(\eta^2 - \sqrt{-1}\eta^3\right), \\ &d\left(\eta^2 - \sqrt{-1}\eta^3\right) = 2\eta^1\wedge\eta^3 + 2\sqrt{-1}\eta^1\wedge\eta^2 = 2\sqrt{-1}\eta^1\wedge\left(\eta^2 - \sqrt{-1}\eta^3\right). \end{aligned} \tag{2.3}$$

Therefore $d\bigwedge^{0,1}\subset\bigwedge^1\wedge\bigwedge^{0,1} = \bigwedge^{1,1}\oplus\bigwedge^{0,2}$ so we conclude that J_f is integrable. □

Lemma 2.2. *The complex structures $J^{\pm}$ are metric compatible. Their $(1,1)$ forms $\omega^{\pm} = g(J^{\pm}\cdot,\cdot)$ are closed if and only if $C = C_0e^{\mp z}$, respectively.*

Proof. Checking compatibility with the metric is an elementary computation which we omit. From (1.5), $d\omega^{\pm} = 0$ if and only if $C = C_0e^{\mp z}$. □

To create right-invariant complex structures and relate them to the metric (which is left-invariant) we require background coordinates. Polar coordinates on $\mathbb{R}^4 \approx \mathbb{C}^2$ are

$$(r,\psi,\theta,\varphi) \;\longmapsto\; \Big(r\cos(\theta/2)e^{-\frac{i}{2}(\psi+\varphi)},\; r\sin(\theta/2)e^{-\frac{i}{2}(\psi-\varphi)}\Big). \tag{2.4}$$

The three "Euler coordinates" (ψ, θ, φ) have ranges $|\psi \pm \varphi| < 2\pi$ and $\theta \in [0, \pi]$. The transitions between the coordinate framing and the left-invariant framing are

$$\begin{aligned}
&\eta^0 = dz \;=\; \tfrac{\sqrt{F}}{2\sqrt{C}}dr && e_0 = \tfrac{\partial}{dz} \;=\; \tfrac{\sqrt{F}}{2\sqrt{C}}\tfrac{\partial}{\partial r} \\
&\eta^1 = \tfrac{1}{2}(d\psi + \cos\theta\, d\varphi) && e_1 = 2\tfrac{\partial}{\partial\psi} \\
&\eta^2 = \tfrac{1}{2}(\sin\psi\, d\theta - \cos\psi\sin\theta\, d\varphi) && e_2 = 2\left(\cos\psi\cot\theta\tfrac{\partial}{\partial\psi} + \sin\psi\tfrac{\partial}{\partial\theta} - \cos\psi\csc\theta\tfrac{\partial}{\partial\varphi}\right) \\
&\eta^3 = \tfrac{1}{2}\left(\cos\psi\, d\theta + \sin\psi\sin\theta\, d\varphi\right) && e_3 = 2\left(-\sin\psi\cot\theta\tfrac{\partial}{\partial\psi} + \cos\psi\tfrac{\partial}{\partial\theta} + \sin\psi\csc\theta\tfrac{\partial}{\partial\varphi}\right).
\end{aligned} \tag{2.5}$$

To create the right-invariant frames we apply quaterionic conjugation $T(z,w) = (\bar{z}, -w)$ to $\mathbb{C}^2$, which changes the parameterization of $\mathbb{C}^2$ to

$$(r, \psi, \theta, \varphi) \;\longmapsto\; \left(r\cos(\theta/2)e^{\frac{i}{2}(\varphi+\psi)},\; -r\sin(\theta/2)e^{\frac{i}{2}(\varphi-\psi)}\right). \tag{2.6}$$

In coordinates, T is $T(r,\psi,\theta,\varphi) = (r, -\varphi, -\theta, -\psi)$. The left-invariant forms η^i pull back to right-invariant forms $\bar{\eta}^i = T^*(\eta^i)$. In the bases $\{\eta^i\}$, $\{\bar{\eta}^i\}$, the linear map $T^* : \bigwedge^1 \to \bigwedge^1$ is

$$T^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos\theta & \cos\psi\sin\theta & -\sin\psi\sin\theta \\ 0 & -\sin\theta\cos\varphi & -\cos\psi\cos\theta\cos\varphi + \sin\psi\sin\varphi & \sin\psi\cos\theta\cos\varphi + \cos\psi\sin\varphi \\ 0 & -\sin\theta\sin\varphi & -\cos\psi\cos\theta\sin\varphi - \sin\psi\cos\varphi & \sin\psi\cos\theta\sin\varphi - \cos\psi\cos\varphi \end{pmatrix}. \tag{2.7}$$

In the bases $\{e_i\}$, $\{\bar{e}_i\}$ we have that $T_* : TM \to TM$ is the transpose $T_* = (T^*)^T$. One can check directly that $T^*, T_* \in O(4)$.

Let $\sigma^i = \frac{1}{|\eta^i|}\eta^i$ be the unit length forms

$$\sigma^0 = \sqrt{\frac{C}{4F}}dz, \quad \sigma^1 = \sqrt{CF}\eta^1, \quad \sigma^2 = \sqrt{C}\eta^2, \quad \sigma^3 = \sqrt{C}\eta^3 \tag{2.8}$$

and let $\{f_i\} = \frac{1}{|e_i|}e_i$ be the corresponding frame. Then the complex structures $J^\pm$ are

$$J^\pm \;=\; \mp f_0 \otimes \sigma^1 \pm f_1 \otimes \sigma^0 - f_2 \otimes \sigma^3 + f_3 \otimes \sigma^2. \tag{2.9}$$

Under T, $J^\pm$ are conjugate to *right*-invariant complex structures $I^\mp$, given by $T_* \circ I^\pm \circ T_* = J^\mp$. Because $I^\mp$ are isomorphic to $J^\pm$ under a diffeomorphism on M^4 (the $\mathbb{S}^3$ antipodal map), I^+ and I^- are integrable. We summarize this in the following lemma.

Lemma 2.3. *The structures $I^\pm$ are integrable, right-invariant, and g-compatible. The structures J^+, I^+ produce a common orientation, with corresponding $(1,1)$-forms $\omega^+, \omega_I^+ \in \bigwedge^+$. Similarly J^-, I^- produce a common orientation, and $\omega^-, \omega_I^- \in \bigwedge^-$.*

The Hermitian structures $(g, J^{\pm})$ produce a very flexible array of Kähler metrics, as F may be chosen freely. By contrast, the Kähler conditions for $(g, I^{\pm})$ are far more restrictive. This is because the left-action of $SU(2)$ fixes g but permutes $I^{\pm}$ among an $\mathbb{S}^2$ worth of complex structures; therefore if $\omega_I^{\pm}$ is Kähler, it is part of a hyperKähler structure. In particular $d\omega_I^{\pm} = 0$ forces $\mathrm{Ric}_g = 0$.

Proposition 2.4. *Letting $\omega_I^- = g(I^- \cdot, \cdot)$, then $d\omega_I^- = 0$ if and only if*

$$F \;=\; (1 + C_1 e^z)^2 \quad \textit{and} \quad C \;=\; \frac{C_0 e^z}{(1 + C_1 e^z)^2}. \tag{2.10}$$

Any such metric is Ricci-flat. The same holds for ω_I^+ after replacing z by $-z$ in (2.10).

Proof. We may compute $d\omega_I^-$ explicitly using the matrices for T^* in (2.7) and its transpose T_*. The computation is tedious but completely elementary, and works out to be

$$\begin{aligned} *d\omega_I^- = \frac{2}{\sqrt{C}} \Bigg(& \cos\theta \left((-2 + F^{\frac{1}{2}}) + F^{\frac{1}{2}} \frac{\partial}{\partial z} \log C \right) \eta^1 \\ & - F^{-\frac{1}{2}} \sin\theta \cos\psi \left(2F^{\frac{1}{2}} - 2F \frac{\partial}{\partial z} \log C - \frac{\partial}{\partial z} F \right) \eta^2 \\ & - F^{-\frac{1}{2}} \sin\theta \sin\psi \left(2F^{\frac{1}{2}} - 2F \frac{\partial}{\partial z} \log C - \frac{\partial}{\partial z} F \right) \eta^3 \Bigg). \end{aligned} \tag{2.11}$$

Setting this to zero gives the partially decoupled system

$$\frac{\partial}{\partial z} F^{\frac{1}{2}} = \left(-1 + F^{\frac{1}{2}} \right), \quad \frac{\partial}{\partial z} \log C \;=\; \left(-1 + 2F^{-\frac{1}{2}} \right) \tag{2.12}$$

which has general solution $F = (1 + C_1 e^z)^2$, $C = \frac{C_0 e^z}{(1 + C_1 e^z)^2}$. Ricci-flatness follows from the general fact that any hyperKähler metric is Ricci flat [5], or from Proposition 3.2 below. □

Proposition 2.4 gives a two parameter family of solutions. Up to homothety we have two metrics.

Proposition 2.5. *Up to homothety, there are exactly two metrics g of the form* (1.3) *for which I^- is a Kähler structure. The first is*

$$F = (1 - e^z)^2 \quad \textit{and} \quad C = \frac{e^z}{(1 - e^z)^2}. \tag{2.13}$$

This hyperKähler metric has an ALF end at $z = 0$ a nut at $z = +\infty$. The second is

$$F = (1 + e^z)^2 \quad \textit{and} \quad C = \frac{e^z}{(1 + e^z)^2}. \tag{2.14}$$

This metric is incomplete, with a nut at $z = -\infty$ and a curvature singularity at $z = +\infty$.

For an analysis of the nut-like topology see Section 2.3.1 and for ALF ends see Section 2.3.2. To verify the claim that (2.14) has a curvature singularity as $z \to +\infty$, we may use (2.27) below to find $|W^+|^2 = 384(-1+e^z)^6$. The metric (2.13) is the Euclidean Taub-NUT; see Section 4.

2.2 Curvature quantities

It is useful to place the metric (2.1) into LeBrun ansatz form [30]. Referring to the polar coordinates of (2.4), from $(r, \varphi, \theta, \psi)$ we change to (Z, τ, x, y) where $x = \log\tan\frac{\theta}{2}$, $y = \varphi$, $\tau = \psi$, and Z solves $dZ = \frac{1}{4}C dz$. Then $(\eta^2)^2 + (\eta^3)^2 = \frac{1}{4}(d\theta^2 + \sin^2\theta\, d\varphi^2) = \frac{1}{4\cosh^2 x}(dx^2 + dy^2)$ and the metric is

$$g = \frac{C}{4\cosh^2 x}(dx^2 + dy^2) \;+\; \frac{FC}{4}\left(d\tau - \tanh(x)dy\right)^2 + \frac{4}{FC}dZ^2. \tag{2.15}$$

Written this way, the metric (2.15) is precisely in the form of Proposition 1 of [30]—the LeBrun ansatz—where $w = \frac{4}{FC}$ and $e^u = \frac{FC^2}{16\cosh^2 x}$. The complex structures in these coordinates are

$$J^{\pm}(dZ) = \mp 2FC\eta^1, \quad J^{\pm}(dx) = -dy. \tag{2.16}$$

We record the useful fact that $\eta^2 \wedge \eta^3 = \frac{1}{4\cosh^2(x)}dx \wedge dy$.

Proposition 2.6 (Ricci Curvature in the Kähler case)**.** *If g is Kähler with respect to J^+, its Ricci form $\rho = Ric(J\cdot,\cdot)$ and scalar curvature are*

$$\rho = -\frac{2}{C}\left(\mathcal{L}^+(F) - 1\right)\omega^+ - \frac{2}{C}\left[\left(-\frac{1}{2}\frac{\partial}{\partial z} + 1\right)\left(\frac{\partial}{\partial z} + 1\right)F - 1\right]\omega^-, \tag{2.17}$$

$$s = -\frac{8}{C}\left(\mathcal{L}^+(F) - 1\right). \tag{2.18}$$

Proof. Setting $C = C_0 e^{-z}$ we follow the computation in [30]. From that paper, the Ricci form is $\rho = -i\partial\bar{\partial}u = \frac{1}{2}d(Jdu)$ where in our case $u = \log(FC^2) - \log(16\cosh^2(x))$, as we found in (2.15). Using coordinates (z, τ, x, y) (specifically using z, not Z from (2.15)), we have $J(dz) = -2F\eta^1$ and $J(dx) = -dy$ from (1.4) and (2.16). Using also $dx \wedge dy = 4\cosh^2(x)\eta^2 \wedge \eta^3$ and $d\eta^1 = -2\eta^2 \wedge \eta^3$,

$$\begin{aligned}
u &= \log F - 2z + 2\log C_0 - 2\log(4\cosh x) \\
du &= (F_z F^{-1} - 2)dz \;-\; 2\tanh(x)dx \\
Jdu &= (-2F_z + 4F)\eta^1 + 2\tanh(x)dy \\
dJdu &= (-2F_{zz} + 4F_z)dz \wedge \eta^1 + (-4F_z - 8F + 8)\eta^2 \wedge \eta^3
\end{aligned} \tag{2.19}$$

From (2.1), $dz \wedge \eta^1 = C^{-1}(\omega^+ - \omega^-)$ and $\eta^2 \wedge \eta^3 = \frac{1}{2}C^{-1}(\omega^+ + \omega^-)$. Therefore

$$\rho = \frac{2}{C}\left(-\frac{1}{2}F_{zz} + \frac{3}{2}F_z - F + 1\right)\omega^+ + \frac{2}{C}\left(\frac{1}{2}F_{zz} - \frac{1}{2}F_z - F + 1\right)\omega^- \tag{2.20}$$

as claimed. Scalar curvature for any Kähler metric is $s = 2*(\omega^+ \wedge \rho)$, so (2.17) along with the facts $\omega^+ \wedge \omega^- = 0$ and $*(\omega^+ \wedge \omega^+) = 2$ gives (2.18). □

Proposition 2.7 (Ricci curvature, general case)**.** *Scalar curvature is*

$$s \ = \ -4C^{-1}\left(\frac{\partial^2 F}{\partial z^2} + \frac{1}{2}F - 2\right) - 24C^{-\frac{3}{2}}\frac{\partial}{\partial z}\left(F\frac{\partial}{\partial z}C^{\frac{1}{2}}\right). \tag{2.21}$$

Using the unit frames σ^i of (2.8) *the trace-free Ricci curvature is*

$$\begin{aligned}\mathring{\mathrm{Ric}} = {} & 4FC^{-\frac{1}{2}}\left(\frac{\partial^2}{\partial z^2}C^{-\frac{1}{2}} - \frac{1}{4}C^{-\frac{1}{2}}\right)\cdot\left((\sigma^0)^2 - (\sigma^1)^2\right) \\ & + 2\left(C^{-\frac{1}{2}}\frac{\partial}{\partial z}\left(F\frac{\partial}{\partial z}C^{-\frac{1}{2}}\right) - C^{-1}\left(\frac{1}{2}\frac{\partial^2 F}{\partial z^2} - \frac{3}{4}F + 1\right)\right) \\ & \cdot\left((\sigma^0)^2 + (\sigma^1)^2 - (\sigma^2)^2 - (\sigma^3)^2\right).\end{aligned} \tag{2.22}$$

Proof. We use the conformal change formulas from [5]. The scalar curvature (2.21) follows from (2.18) along with the formula $\tilde{s} = U^{-2}(s - 6U^{-1}\triangle_g U)$ when $\tilde{g} = U^2 g$. In the Kähler metric where $C = e^{-z}$, the Laplacian $\triangle_g$ acting on any $U = U(z)$ is $\triangle_g U = 4e^{2z}\frac{\partial}{\partial z}\left(e^{-z}F\frac{\partial U}{\partial z}\right)$. To obtain (2.21), use $U = e^{\frac{1}{2}z}C^{\frac{1}{2}}$.

To compute $\mathring{\mathrm{Ric}}$, again we start with the Kähler case; (2.17) gives

$$\mathring{\mathrm{Ric}}_g = 2e^z\left(\frac{1}{2}F_{zz} - \frac{1}{2}F_z - F + 1\right)\left(-(\sigma^0)^2 - (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\right) \tag{2.23}$$

The trace-free Ricci conformally changes by $\mathring{\mathrm{Ric}}_{\tilde{g}} = \mathring{\mathrm{Ric}}_g + 2U(\nabla_g^2 U^{-1} - \frac{1}{4}(\triangle_g U^{-1})g)$. Then using

$$\begin{aligned}2U\Big(\nabla_g^2 U^{-1} - \frac{1}{4}(\triangle_g U^{-1})g\Big) = {} & -4UF(e^z(U^{-1})_z)_z\left(-(\sigma^0)^2 + (\sigma^1)^2\right) \\ & - 2U(e^z F(U^{-1})_z)_z\big(-(\sigma^0)^2 - (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\big)\end{aligned} \tag{2.24}$$

and $U = e^{\frac{1}{2}z}C^{\frac{1}{2}}$, we add (2.24) to (2.23) to give (2.22). □

Proposition 2.8. *The metric* (2.1) *has Weyl curvatures*

$$W^{\pm} \ = \ -C^{-1}\left(\mathcal{L}^{\pm}(F) - 1\right)\left(\omega^{\pm}\otimes\omega^{\pm} - \frac{2}{3}Id_{\bigwedge^{\pm}}\right). \tag{2.25}$$

Proof. We use Derdzinski's Theorem (see [15, Section 3, Proposition 2]) to find W^+ in the Kähler case, then conformally change to the arbitrary case. By Derdzinski's Theorem $W^+ = \frac{s}{12}\left(\frac{3}{2}\omega\otimes\omega - Id_{\bigwedge^+}\right)$ where ω is a Kähler form. When $C = e^{-z}$, ω^+ is Kähler and Proposition 2.6 gives

$$W^+ = -\frac{2}{3}\,e^z\left(\mathcal{L}^+(F) - 1\right)\left(\frac{3}{2}\omega^+\otimes\omega^+ - Id_{\bigwedge^+}\right). \tag{2.26}$$

Conformally changing from $C = e^{-z}$ to any $C = C(z)$ gives (2.25). Computing W^- is the same, after setting $C = e^z$ to make ω^- rather than ω^+ into a Kähler form. □

From Proposition 2.8, $|W^\pm|^2$ and $|W^\pm|^2 dVol$ are

$$\begin{aligned} |W^\pm|^2 &= \frac{32}{3C^2}\left(\mathcal{L}^\pm(F) - 1\right)^2 \qquad \text{and} \\ |W^\pm|^2 dVol &= \frac{16}{3}\left(\mathcal{L}^\pm(F) - 1\right)^2 dz \wedge \eta^1 \wedge \eta^2 \wedge \eta^3. \end{aligned} \tag{2.27}$$

We compute the divergences $\delta W^\pm$ and the Bach tensor.

Proposition 2.9. *For the metric* (2.1),

$$\delta W^\pm \;=\; W^\pm\left(\nabla \log\left| e^{\pm\frac{3}{2}z}(\mathcal{L}^\pm(F) - 1)\sqrt{C}\right|,\;\cdot,\;\cdot,\;\cdot\right). \tag{2.28}$$

Proof. Again we first conformally change the metric so it is Kähler. By Lemma (2.4) the metric $\tilde{g} = e^{-z}C^{-1}g$ is Kähler and the form $\widetilde{\omega} = \tilde{g}(J^+\cdot,\cdot)$ is closed. Then $\tilde{\delta}\widetilde{\omega} = -*d\widetilde{\omega} = 0$ so also $\tilde{\delta}(\widetilde{\omega}\otimes\widetilde{\omega}) = 0$, and $\tilde{\delta}(Id_{\bigwedge^+}) = 0$ because $Id_{\bigwedge^+}$ is covariant-constant. Therefore (2.25) gives

$$\begin{aligned} \tilde{\delta}\,\widetilde{W}^+(\cdot,\cdot,\cdot) &= \tilde{\delta}\left(-e^z(\mathcal{L}^+(F) - 1)\left(\widetilde{\omega}\otimes\widetilde{\omega} - \frac{2}{3}Id_{\bigwedge^+}\right)\right)(\cdot,\cdot,\cdot) \\ &= -\left(\widetilde{\omega}\otimes\widetilde{\omega} - \frac{2}{3}Id_{\bigwedge^+}\right)\left(\widetilde{\nabla}(e^z(\mathcal{L}^+(F) - 1)),\;\cdot,\;\cdot,\;\cdot\right) \\ &= \widetilde{W}^+\left(\widetilde{\nabla}\log\left|e^z(\mathcal{L}^+(F) - 1)\right|,\;\cdot,\;\cdot,\;\cdot\right) \\ &= W^+\left(\nabla\log\left|e^z(\mathcal{L}^+(F) - 1)\right|,\;\cdot,\;\cdot,\;\cdot\right). \end{aligned} \tag{2.29}$$

Derdzinski's conformal change formula, equation (19) of [15], is

$$\tilde{\delta}\,\widetilde{W}^+ = \delta W^+ - \frac{1}{2}W^+\left(\nabla\log\left(e^z C\right),\;\cdot,\;\cdot,\;\cdot\right) \tag{2.30}$$

so changing the metric back with conformal factor $e^z C$, (2.29) and (2.30) give

$$\delta W^+ = W^+\left(\nabla\log\left|e^{\frac{3}{2}z}(\mathcal{L}^+(F) - 1)\sqrt{C}\right|,\;\cdot,\;\cdot,\;\cdot\right). \tag{2.31}$$

The argument for δW^- is entirely the same, after conformally changing so $\tilde{\omega}^-$ not $\tilde{\omega}$ is closed. □

Proposition 2.10 (The Bach Tensor)**.** *The Bach tensor of* (2.1) *is*

$$\begin{aligned} Bach = \frac{16}{3C^2}F\big(\mathcal{L}^-(\mathcal{L}^+(F)) - 1\big)\cdot\Big(-2(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\Big) \\ + \frac{8}{3C^2}\mathcal{B}(F,F)\cdot\Big(-(\sigma^0)^2 - (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\Big). \end{aligned} \tag{2.32}$$

Proof. In the Kähler case we decompose the Bach tensor into its J-invariant and J-anti-invariant parts $Bach^+$, $Bach^-$ respectively. It is known that $Bach^+ = \frac{1}{3}(\nabla^2 s)^+_0 + \frac{1}{6}s\,\mathring{\mathrm{Ric}}$ and $Bach^- = -\frac{1}{6}(\nabla^2 s)^-$; see Eq. (39) of [15], Eq. (20) of [1] or Lemma 6 of [10]. We have

$$\begin{aligned}\nabla^2 s &= \frac{4F}{C}s_{zz}\sigma^0\otimes\sigma^0 + s_z\nabla dz\\ &= \left(\frac{4F}{C}s_{zz} - \frac{2F^2}{C^2}s_z\left(F^{-1}C\right)_z\right)(\sigma^0)^2 + \frac{2}{C^2}s_z\left(FC\right)_z(\sigma^1)^2 + \frac{2F}{C^2}s_zC_z\left((\sigma^2)^2+(\sigma^3)^2\right).\end{aligned} \tag{2.33}$$

In the Kähler case where $C = e^{-z}$ and $s = -8e^z(\mathcal{L}^+(F)-1)$, we compute

$$\begin{aligned}(\nabla^2 s)^- &= -32e^{2z}F\left(\mathcal{L}^-(\mathcal{L}^+)(F)-1\right)\left((\sigma^0)^2-(\sigma^1)^2\right)\\ (\nabla^2 s)^+ &= -16e^{2z}\left(2F(\mathcal{L}^-(\mathcal{L}^+(F)-1) - F_z\mathcal{L}^+(F_z+F)-1\right)\left((\sigma^0)^2+(\sigma^1)^2\right)\\ &\quad + 16e^{2z}F\left(\mathcal{L}^+(F_z+F)-1\right)\left((\sigma^0)^2+(\sigma^1)^2+(\sigma^2)^2+(\sigma^3)^2\right)\\ \triangle s &= 4e^{-2z}\left(FC\right)\left(FCs_z\right)_z.\end{aligned} \tag{2.34}$$

Then $(\nabla s)^+_0 = (\nabla^2 s)^+ - \frac{1}{4}(\triangle s)g$ and using the expression for $\mathring{\mathrm{Ric}}$ of (2.23),

$$\begin{aligned}Bach^+ &= \frac{16e^{2z}}{3}\left(\frac{1}{2}\mathcal{B}(F,F) + F\cdot\left(\mathcal{L}^-(\mathcal{L}^+(F))-1\right)\right)\cdot\Big(-(\sigma^0)^2-(\sigma^1)^2+(\sigma^2)^2+(\sigma^3)^2\Big)\\ Bach^- &= \frac{16e^{2z}}{3}\cdot F\cdot\left(\mathcal{L}^-(\mathcal{L}^+(F))-1\right)\cdot\left((\sigma^0)^2-(\sigma^1)^2\right)\end{aligned} \tag{2.35}$$

Conformally changing from $C = e^{-z}$ to arbitrary C, we obtain (2.32). □

Compare also with Proposition 14 of [1].

Compare equation (2.32) with (3.3) of [34]; after substituting $C = 1$, $F = f^2$ and $dz = 2fdt$ the expression here and the expression there are identical.

2.3 Topology: "nuts", "bolts", and asymptotics

Here we discuss global aspects of $U(2)$-invariant metrics. Ostensibly the metric (2.1) is well defined on $\mathbb{R}\times\mathbb{S}^3$ but topology changes occur if F or C attain 0 somewhere. If F reaches zero, the metric most naturally lives on a quotient $I\times(\mathbb{S}^3/\Gamma)/\sim$ where Γ is some discrete subgroup of $SU(2)$, and $\sim$ identifies some 3-sphere to a 2-sphere, via the Hopf map. Where F or C is infinite, there is a (possibly incomplete) manifold end.

2.3.1 Bolts, Nuts

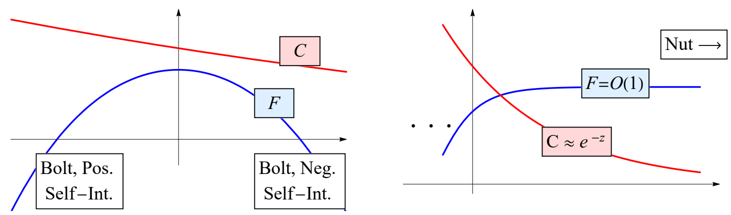

Figure 1: *A compact manifold with a bolt of positive and of negative self-intersection. A nut at* $z = +\infty$.

The first kind of topology change occurs when the Hopf fiber collapses but the conformal factor remains non-zero, meaning F but not C reaches zero. When $F(z_0) = 0$, the locus $z = z_0$ is not a 3-sphere but a 2-sphere, colloquially known as a "bolt" [21] (see also [17, 29, 32]).

As this is well known, we describe it only briefly. Recalling the coordinates of Section 2.1, transversals to the bolt are 2-dimensional submanifolds locally given by $\theta = const$, $\varphi = const$, and the metric is smooth at the bolt provided it is smooth on such transversals. The inherited metric on the transversal is $\hat{g}_2 = \frac{1}{4F}dz^2 + \frac{F}{4}d\psi^2$ with $\psi \in [-2\pi, 2\pi)$, which we write $\hat{g}_2 = dr^2 + (\sqrt{F}d(\frac{1}{2}\psi))^2$ by solving $dr = \frac{1}{\sqrt{4F}}dz$ with $r = 0$ at $z = z_0$. If $\sqrt{F} = kr + O(r^2)$, where $k \neq 0$, then $(\sqrt{F}d(\frac{1}{2}\psi))^2 \approx r^2(d(\frac{k}{2}\psi))^2$ so the metric $\hat{g}_2$ will be conical at $r = 0$ with cone angle $2\pi|k|$ (so smooth if and only if $k = \pm 1$). If $k \in \mathbb{Z} \setminus \{0\}$ however, we can obtain a smooth metric on the quotient $I \times \mathbb{S}^3/\Gamma$ where Γ is a cyclic subgroup of order $|k|$ of the Hopf action. From $\sqrt{F} = kr + O(r^2)$ we have $k = \frac{d\sqrt{F}}{dr}$, and because $\frac{d}{dr} = 2\sqrt{F}\frac{d}{dz}$, $k = \frac{dF}{dz}$. We summarize this in the following Proposition.

Proposition 2.11 (The "bolting condition")**.** *Assume* $z = z_0$ *is a zero of* $F(z)$ *but not* $C(z)$*. If*

$$\left.\frac{dF}{dz}\right|_{z=z_0} = k \tag{2.36}$$

where $k \neq 0$ *then we may identify the locus* $\{z = z_0\}$ *with a 2-sphere (a "bolt"). Assuming* $k \in \mathbb{Z} \setminus \{0\}$*, then taking the* $|k|$*-to-1 quotient of the* $\mathbb{S}^3$ *factor, the metric is smooth near* $\{z = z_0\}$ *and the "bolt" is a 2-sphere of self intersection number* k*.*

It is possible that two bolts occur, one at z_0 and one at z_1 where $z_0 < z_1$, as in Figure 1. We certainly must have $\frac{dF}{dz} \geq 0$ at z_0 and $\frac{dF}{dz} \leq 0$ at z_1, so the bolts, assuming they are both smooth after resolution, must have self-intersection numbers k and $-k$ where $k \in \mathbb{Z} \setminus \{0\}$. With either complex structure J^+ or J^-, this is the "odd" Hirzebruch surface Σ_{2k-1}; see [33].

A nut, by contrast, occurs when the $\mathbb{S}^3$ factor contracts to a point; the nearby topology is that of a ball in $\mathbb{R}^4$. This occurs when C becomes zero but F remains finite. When ω is Kähler and

$C = C_0 e^{-z}$, a nut may occur at $z = +\infty$; this is depicted in Figure 1. When ω^- is Kähler and $C = C_0 e^{-z}$ a nut may occur at $z = -\infty$.

Proposition 2.12 (The Nut condition at $z = \infty$)**.** *Assume $C = O(e^{-z})$ and $F = 1 + O(e^{-z})$ as $z \to \infty$. Adding a point at $z = \infty$, this point is a finite distance away and has a neighborhood with bounded curvature and the topology of a ball.*

2.3.2 ALE, ALF, and cusp-like ends

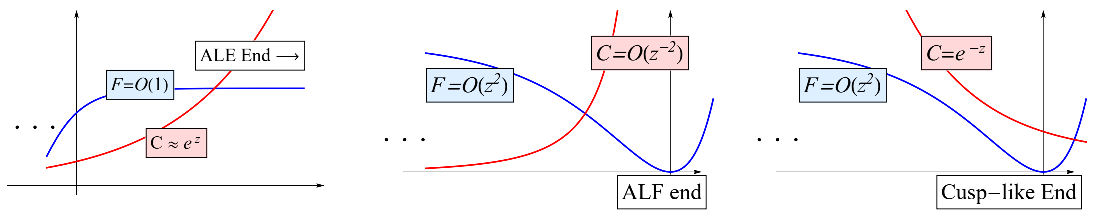

Figure 2: *ALE, ALF, and cusp-like ends in the $U(2)$ ansatz.*

If g is Kähler with respect to J^- so $C = C_0 e^{z}$, an ALE end can occur as $z \to \infty$, as depicted in Figure 2. If instead g is Kähler with respect to J^+ then replacing z by $-z$, Figure 2 is flipped and an ALE end occurs as $z \to -\infty$.

Proposition 2.13. *Assume g is Kähler with respect to J^+, so $C = e^{-z}$. If $F = 1 + O(z^{-2})$ as $z \to -\infty$, the metric is ALE with better-than-quadratically decaying curvature.*

Proof. Letting r be the distance function that solves $dr = \frac{1}{2}\sqrt{C/F}dz = \frac{1}{2}e^{-\frac{1}{2}z}(1 + O(z^{-2}))dz$, by assumption we have $r = e^{-\frac{1}{2}z} + O(z^{-1})$. Then $C = e^{-z} = r^2 + O(r^{-4})$, so the metric is $g \approx dr^2 + (r^2 + O(r^{-4}))d\sigma_{\mathbb{S}^3}$ as $r \to \infty$, so it is ALE. To check curvature decay, Proposition 2.6 gives

$$\rho = -2C^{-1}\left(\frac{1}{2}F_{zz} - \frac{3}{2}F_z + F - 1\right)\omega + 2C^{-1}\left(\frac{1}{2}F_{zz} - \frac{1}{2}F_z - F + 1\right)\omega^- \tag{2.37}$$

so asymptotically $\rho \approx e^z O(z^{-2})\omega + e^z O(z^{-2})\omega^- = o(r^{-2})$. The expressions for $|W^+|$, $|W^-|$ in (2.27) give the same decay rates. Thus the Riemann tensor decays like $|\,\mathrm{Rm}\,| = o(r^{-2})$. ☐

The ALF end has cubic volume growth, cubic curvature decay, and $\mathbb{R}^3$ tangent cone at infinity. See for example [13, 16, 18, 26]. By a "cusp-like" end, we mean an end that locally resembles a Riemannian product of a tractrix of revolution (sometimes called a pseudosphere) with a sphere. Toward infinity the scalar and Weyl curvatures decrease rapidly, whereas the Ricci curvature approaches a constant bilinear form of signature $(-,-,+,+)$. These two kinds of ends are conformal to each other: we have $C = \frac{e^z}{(1-e^z)^2}$ in the ALF case and $C = e^{-z}$ or $C = e^z$ in the cusp-like case. In both cases F has a second-order zero at $z = 0$. See Figure 2.

Proposition 2.14. *Assume $F = z^2 + O(z^3)$ near $z = 0$.*

If C remains finite then the manifold forms a complete, cusp-like end near $z = 0$. Asymptotically the Hopf fiber shrinks to zero and the metric has the local geometry of the product of a psuedosphere times a sphere.

If $C = O(z^{-2})$ then the the metric forms an ALF end near $z = 0$.

Proof. The distance function r satisfies $dr = \frac{1}{2}\sqrt{\frac{C}{F}}dz$ so in the cusp-like case, where C remains finite, then $\sqrt{F} = O(z)$ gives $r \approx \frac{1}{2}\log|z|$ near 0 and indeed the distance to 0 is infinite so the metric is complete. From $\omega \wedge \omega = -C^2 dz \wedge \eta^1 \wedge \eta^2 \wedge \eta^3$, we see the volume is finite. Checking the tensors $W^\pm$, from $F = z^2 + O(z^3)$ we find that $\mathcal{L}^\pm(F) - 1 = O(z)$ and so $|W^\pm| \searrow 0$ as $z \to 0$. In the Kähler case ρ is a multiple of ω added to a multiple of ω^-. The multiple on ω is also $O(z)$, but the multiple on ω^-, by (2.17), approaches $4C^{-1}$. This justifies the assertion that, in the Kähler case, the local geometry approaches a $+1$ times a -1 curvature surface. In the non-Kähler case, the usual conformal change formulas for Ricci curvature shows this remains true.

Next we verify that when $C = z^{-2} + O(1)$ near $z = 0$, the metric has an ALF end. Then $dr = \frac{1}{2}\sqrt{\frac{C}{F}}dz = \left(\frac{1}{2}z^{-2} + O(1)\right)dz$ so $r = z^{-1} + O(z)$ near $z = 0$. To compute volume, we use $C^{\frac{3}{2}} = O(r^3)$ and $F^{\frac{1}{2}} = O(z) = O(r^{-1})$, so we have

$$dVol = -C^{\frac{3}{2}}F^{\frac{1}{2}}dr \wedge d\sigma_{\mathbb{S}^3} \approx r^2 dr \wedge d\sigma_{\mathbb{S}^3}. \tag{2.38}$$

Integrating (2.38) and noting that r is a distance function, indeed we observe cubic volume growth. Next we check curvature decay. From (2.27) we have $\mathcal{L}^\pm(F) - 1 = O(1)$ so that $|W^+| \approx \frac{32}{3}C^{-2} = O(z^2) = O(r^{-2})$ and similarly for $|W^-|$. Inserting F, C into the Ricci form ρ from (2.19), we see Ricci curvature decays quadratically. □

We close by noting that ALE ends are conformal to nuts and vice-versa—by changing between $C = e^{-z}$ and $C = e^{z}$—and similarly ALF ends and cusp-like ends are conformal to each other.

3 Special Metrics

We use the computations of Section 2.2 to determine what conditions are needed to make a $U(2)$-invariant metric special or canonical.

3.1 Scalar Curvature

From (2.21) of Proposition 2.7, specifying scalar curvature is equivalent to

$$sC^{\frac{3}{2}} + 4C^{\frac{1}{2}}\left(\frac{\partial^2 F}{\partial z^2} + \frac{1}{2}F - 2\right) + 24\frac{\partial}{\partial z}\left(F\frac{\partial}{\partial z}C^{\frac{1}{2}}\right) = 0, \tag{3.1}$$

for given $s = s(z)$. This underdetermined equation is linear in F. Imposing the Kähler condition $C = C_0e^{\pm z}$ creates a critically determined linear equation.

3.2 Extremal Kähler metrics

A Kähler metric is extremal if the functional $g \mapsto \int s^2 dVol$ is stable under those perturbations of g that preserve the Kähler class. From [9] the Euler-Lagrange equations are that the gradient ∇s is a holomorphic vector field, but there are several ways to assess whether (1.3) is extremal. In our context we are less concerned with global functionals such as $\int s^2$. We use the local condition that a Kähler metric is extremal if and only if $J\nabla s$ is Killing.

Proposition 3.1 (The extremal condition)**.** *The metric* (2.1) *with complex structure* J^+ *is extremal Kähler if and only if* $C = C_0e^{-z}$ *and* $\mathcal{L}^-(\mathcal{L}^+(F)) = 1$*, which is*

$$F = 1 + \frac{1}{2}C_1e^{-2z} + C_2e^{-z} + C_3e^z + \frac{1}{2}C_4e^{2z}. \tag{3.2}$$

Its scalar curvature is $s = -\frac{24}{C_0}(C_1e^{-z} + C_2)$*.*

Likewise, the metric with complex structure J^- *is extremal Kähler if and only if* $C = C_0e^z$ *and again* $\mathcal{L}^-(\mathcal{L}^+(F)) = 1$*. Its scalar curvature is* $s = -\frac{24}{C_0}(C_3 + C_4e^z)$*.*

Proof. From (2.1) and (2.5), we have $\nabla z = 4\frac{F}{C}\frac{\partial}{\partial z} = \frac{4}{C}J\frac{\partial}{\partial \psi}$. Because the coordinate field $\frac{\partial}{\partial \psi}$ is itself a Killing field and because $s = s(z)$ is a function of z alone, the extremal condition is $\nabla s = -4\alpha J\frac{\partial}{\partial \psi} = -\alpha e^z\nabla z = \nabla\left(\alpha e^{-z}\right)$ where α is a constant. Therefore $s = \alpha e^{-z} + \beta$ where β is another constant. Using $s = -8C_0^{-1}e^z(\mathcal{L}^+(F) - 1)$, from (2.18) we obtain

$$-8C_0^{-1}e^z\left(\frac{1}{2}\frac{\partial^2 F}{\partial z^2} - \frac{3}{2}\frac{\partial F}{\partial z} + F - 1\right) = \alpha e^{-z} + \beta. \tag{3.3}$$

After setting $C_1 = -\frac{1}{24}\alpha C_0$ and $C_2 = -\frac{1}{24}\beta C_0$ we obtain (3.2).

For J^- in place of J^+, reverse the sign on z in all computations. $\square$

3.3 Einstein metrics

By (2.22), $\mathring{\mathrm{Ric}} = 0$ if and only if

$$\frac{\partial^2}{\partial z^2} C^{-\frac{1}{2}} = \frac{1}{4} C^{-\frac{1}{2}} \quad \text{and} \quad C^{\frac{1}{2}} \frac{\partial}{\partial z}\left(F \frac{\partial}{\partial z} C^{-\frac{1}{2}} \right) = \left(\frac{1}{2}\frac{\partial^2 F}{\partial z^2} - \frac{3}{4} F + 1 \right). \tag{3.4}$$

This is critically determined and partly decoupled. It is 4^{th} order in total so we will have a 4-parameter solution space. The general solution is

$$\begin{aligned} &F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2} C_4 e^{2z}, \quad C = \frac{e^{-z}}{(C_5 + C_6 e^{-z})^2}, \\ &\text{where} \quad C_1 C_5 - C_2 C_6 = 0, \quad \text{and} \quad C_3 C_5 - C_4 C_6 = 0. \end{aligned} \tag{3.5}$$

With six constants and two algebraic relations we have the expected four-parameter family of solutions. Compare with Proposition 2.4. The algebraic relations on the C_i are equivalent to the pairs (C_1, C_2), (C_3, C_4), and (C_5, C_6) being proportional to each other. These imply also that $C_1 C_4 - C_2 C_3 = 0$, so we recover the fact that Einstein metrics are Bach-flat; see (3.14) below. By Lemma 2.2 the metric is Kähler when $C_6 = 0$ (for J^+) or $C_5 = 0$ (for J^-).

To be Ricci-flat, C and F require, in addition to (3.4), that $s = 0$. This third relation appears to make the overall system overdetermined, but it does not, for the reason that s is a first integral for the system (3.4) so only contributes an algebraic relation. From (3.1),

$$s = -24(C_2 C_5^2 - 2C_5 C_6 + C_3 C_6^2). \tag{3.6}$$

Proposition 3.2 (The Einstein conditions)**.** *The metric* (1.3) *is Einstein if and only if*

$$\begin{aligned} &F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2} C_4 e^{2z}, \quad C = \frac{e^{-z}}{(C_5 + C_6 e^{-z})^2}, \\ &C_1 C_5 - C_2 C_6 = 0, \quad \textit{and} \quad C_3 C_5 - C_4 C_6 = 0. \end{aligned} \tag{3.7}$$

Its scalar curvature is the constant $s = -24(C_2 C_5^2 - 2C_5 C_6 + C_3 C_6^2)$.

Up to homothety, there is a 2-dimensional family of Einstein metrics. Up to homothety, there is a 1-dimensional family of Ricci-flat metrics, a 1-dimensional family of KE metrics with respect to J^+*, and a 1-dimensional family of KE metrics with respect to* J^-*. Up to homothety and biholomorphism, there are exactly five Ricci-flat Kähler metrics, three of which are complete.*

Proof. We have proven everything except the final assertion, that exactly five metrics of the form (1.3) are Ricci-flat Kähler, up to homothety. We prove this regardless of the complex structure, whether one of the structures considered here or not. A $U(2)$-invariant metric is Einstein if and only if it has the form (3.7). By Derdzinski's theorem [15], if a scalar-flat metric is Kähler—

regardless of the complex structure—then it is half-conformally flat. In particular $C_1 = C_2 = 0$ or $C_3 = C_4 = 0$.

So assume $C_3 = C_4 = 0$; the case $C_1 = C_2 = 0$ is identical under the isomorphism $z \mapsto -z$. We have four remaining variables C_1, C_2, C_5, C_6 and two relations: $C_1C_5 - C_2C_6 = 0$ from (3.5) and $C_2C_5^2 - 2C_5C_6 = 0$ from (3.6). If in addition to $C_3 = C_4 = 0$ we have both $C_1 = C_2 = 0$ then either $C_5 = 0$ or else $C_6 = 0$ and in either case we have the flat metric: $F = 1$ and $C = C_0e^{\pm z}$.

Suppose $C_1 = 0$ but $C_2 \neq 0$; then the two relations force $C_5 = C_6 = 0$, an impossibility. Suppose $C_1 \neq 0$ but $C_2 = 0$; then the relations force $C_6 = 0$ so

$$F = 1 + \frac{1}{2}C_1e^{-2z}, \quad C = \frac{1}{C_5^2}e^{-z} \tag{3.8}$$

which is Kähler with respect to J^+. Up to homothety, there are exactly two such metrics: the first is given by $F = 1 - e^{-2z}$, $C = e^{-z}$, which is the Eguchi-Hanson metric, and the second is given by

$$F = 1 + e^{-2z}, \quad C = e^{-z} \tag{3.9}$$

which is incomplete and has a curvature singularity at $z = -\infty$.

Lastly it is possible that neither C_1 nor C_2 are zero. The two relations now give $\frac{C_6}{C_5} = \frac{C_1}{C_2}$ and $\frac{C_6}{C_5} = \frac{C_2}{2}$, so $C_1 = \frac{1}{2}C_2^2$. Therefore the metric is

$$F = 1 + \frac{1}{4}C_2^2e^{-2z} + C_2e^{-z} = \left(1 + \frac{1}{2}C_2e^{-z}\right)^2, \quad C = \frac{C_5^2e^{-z}}{\left(1 + \frac{1}{2}C_2e^{-z}\right)^2}. \tag{3.10}$$

Under the isomorphism $z \mapsto -z$ this is the Kähler metric of Proposition 2.4 which is Kähler with respect to the complex structure I^-; therefore the metric (3.10) is Kähler with respect to the complex structure I^+. As in Proposition 2.5 there are two such metrics: one where $C_2 < 0$ (which is the Taub-NUT metric) and one where $C_2 > 0$ (which has a curvature singularity). □

3.4 Half-conformally flat, half-harmonic, and Bach-flat metrics

Proposition 3.3. *The metric* (1.3) *has* $W^\pm = 0$ *if and only if* $\mathcal{L}^\pm(F) - 1 = 0$, *meaning*

$$F = 1 + C_3e^z + \frac{1}{2}C_4e^{2z} \quad \text{or} \quad F = 1 + \frac{1}{2}C_1e^{-2z} + C_2e^{-z}, \tag{3.11}$$

respectively. Up to homothety, each case constitutes a 1-parameter family of such metrics, each a subspace of the 2-parameter family of Bach-flat metrics.

In the case g *is Kähler with respect to* J^+ *so* $C = C_0e^{-z}$, *then* $W^+ = 0$ *implies* $s = 0$, *and* $W^- = 0$ *implies* $s = -\frac{24}{C_0}(C_1e^{-z} + C_2)$.

The half-harmonic condition $\delta W^+ = 0$ (or $\delta W^- = 0$) is underdetermined, and requires an additional condition to be critically determined. Three possibilities are $s = const$, the Kähler condition, and both $\delta W^\pm = 0$.

Proposition 3.4. *The metric* (1.3) *has* $\delta W^+ = 0$ *if and only if a constant* k_1 *exists so* $e^{\frac{3}{2}z}\left(\mathcal{L}^+(F) - 1\right)C = k_1$*, and* $\delta W^- = 0$ *if and only if* $e^{-\frac{3}{2}z}\left(\mathcal{L}^-(F) - 1\right)C = k_2$ *for some* $k_2 \in \mathbb{R}$.

Assume (2.1) *is Kähler with respect to* J^+*, meaning* $C = C_0 e^{-z}$*. Then*

a) $\delta W^+ = 0$ *if and only if* $F = 1 + C_2 e^{-z} + C_3 e^z + \frac{1}{2} C_4 e^{2z}$*. In particular* $s = \frac{-24C_2}{C_0}$ *is constant.*

b) $\delta W^- = 0$ *if and only if* $F = 1 + \frac{1}{2}C_1 e^{-2z} + C_2 e^{-z} + \frac{1}{2}C_4 e^{2z}$*. In particular the metric is extremal and* $s = -24\frac{1}{C_0}(C_1 e^{-z} + C_2)$.

Proof. For $\delta W^+ = 0$ this follows from Proposition 2.9 with $C = C_0 e^{-z}$, $e^{\frac{3}{2}z}(\mathcal{L}^+(F) - 1)\sqrt{C} = k_1$ and finding the general solution. In the Kähler case, *a)* and *b)* follow from Proposition 3.1. □

In the $U(2)$-invariant case, $\delta W = 0$ is equivalent to the Einstein condition.

Proposition 3.5 (Harmonic curvature). *The metric* (2.1) *has* $\delta W = 0$ *if and only if* g *is Einstein.*

Proof. Because $\delta W^+ \in T^*M \otimes \bigwedge^+$ and $\delta W^- \in T^*M \otimes \bigwedge^-$, we have $\delta W = 0$ if and only if δW^+ and δW^- are both zero. Then by Lemma 2.9 constants k_1, k_2 exist so

$$e^{\frac{3}{2}z}(\mathcal{L}^+(F) - 1)\sqrt{C} = k_1 \quad \text{and} \quad e^{-\frac{3}{2}z}(\mathcal{L}^-(F) - 1)\sqrt{C} = k_2. \tag{3.12}$$

Eliminating C, we obtain $k_2 e^{\frac{3}{2}z}(\mathcal{L}^+(F) - 1) = k_1 e^{-\frac{3}{2}z}(\mathcal{L}^-(F) - 1)$ which has solution

$$F = 1 + k_1\left(\frac{1}{2}C_1 e^{-2z} + C_2 e^{-z}\right) + k_2\left(C_1 e^z + \frac{1}{2}C_2 e^{2z}\right). \tag{3.13}$$

Using either equation in (3.12), $C = \frac{C_0 e^{-z}}{(C_2 + C_1 e^{-z})^2}$. By Proposition 3.2, the metric is Einstein. □

Next we consider the case of Bach-flat metrics. By Proposition 2.10, F solves the fourth order linear equation $\mathcal{L}^-(\mathcal{L}^+(F)) - 1 = 0$ and the third order non-linear equation $\mathcal{B}(F, F) = 0$. This seems to be overdetermined, but due to (3.15) the two equations are not independent.

Lemma 3.6. *If* F *solves* $\mathcal{L}^+(\mathcal{L}^-(F)) - 1$ *then* $\mathcal{B}(F, F) = const$*. If* F *solves* $\mathcal{B}(F, F) = 0$*, then* $\mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$*. Lastly* $\mathcal{B}(F, F) = \mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$ *if and only if*

$$F = 1 + \frac{1}{2}C_1 e^{-2z} + C_2 e^{-z} + C_3 e^z + \frac{1}{2}C_4 e^{2z} \quad \text{and} \quad C_1 C_4 - C_2 C_3 = 0. \tag{3.14}$$

Proof. A tedious but completely elementary computation shows

$$\frac{\partial}{\partial z}\mathcal{B}(F,F) = 2\left(\mathcal{L}^+(\mathcal{L}^-(F)) - 1\right)\frac{\partial F}{\partial z}. \tag{3.15}$$

Therefore $\mathcal{B}(F,F)$ is indeed constant on solutions of $\mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$. Next, $\mathcal{B}(F,F) = 0$ implies either $F = const$ or $\mathcal{L}^+(\mathcal{L}^-(F)) = 1$. By direct computation the only constant that satisfies $\mathcal{B}(F,F) = 0$ is $F = 1$, which indeed solves $\mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$. We conclude that $\mathcal{B}(F,F) = 0$ implies $\mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$.

The general solution of $\mathcal{L}^+(\mathcal{L}^-(F)) = 1$ is $F = 1 + \frac{1}{2}C_1e^{-2z} + C_2e^{-z} + C_3e^z + \frac{1}{2}C_4e^{2z}$, and in this case direct computation shows that $\mathcal{B}(F,F) = 3(C_2C_3 - C_1C_4)$. Therefore the general solution of $\mathcal{L}^+(\mathcal{L}^-(F)) = 1$, $\mathcal{B}(F,F) = 0$ is the three parameter family of (3.14). □

Proposition 3.7. *The metric* (2.1) *is Bach-flat if and only if*

$$F = 1 + \frac{1}{2}C_1e^{-2z} + C_2e^{-z} + C_3e^z + \frac{1}{2}C_4e^{2z} \quad \textit{and} \quad C_1C_4 - C_2C_3 = 0. \tag{3.16}$$

In particular g is Bach-flat if and only if it is conformally Einstein. Up to conformal factors and translation in z, the Bach-flat metrics constitute a 2-parameter family of metrics.

Proof. The metric g is Bach-flat if and only if $\mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$ and $\mathcal{B}(F,F) = 0$. From Lemma 3.6, this holds if and only if $F = 1 + \frac{1}{2}C_1e^{-2z} + C_2e^{-z} + C_3e^z + \frac{1}{2}C_4e^{2z}$ and $C_1C_4 - C_2C_3 = 0$, giving a 3-parameter family of solutions. Factoring out by translation in z, this is a 2-parameter family, as claimed. To see that any Bach-flat metric is conformal to an Einstein metric, simply let C be a conformal factor from Proposition 3.2. □

3.5 B^t-flat metrics

The B^t-flat metrics [25] extremize the functional $B^t(g) = \int |W|^2 + t\int s^2$, where we take $B^\infty = \int s^2$. The Euler-Lagrange equations of this functional [25] are

$$-4Bach + t\,\mathcal{C} = 0 \tag{3.17}$$

where $\mathcal{C} = 2\left(\nabla^2 s - (\triangle s)g - s\,\mathring{\mathrm{Ric}}\right)$. The Bach tensor is always trace-free and $Tr(\mathcal{C}) = -6\triangle s$, so tracing the B^t-flat condition (3.17) gives $\triangle s = 0$. Then we can rewrite the B^t-flat condition as the two equations $2Bach + t(s\,\mathring{\mathrm{Ric}} - \nabla^2 s) = 0$ and $\triangle s = 0$. We can express these as an ODE system.

Lemma 3.8 (The unreduced B^t-flat equations)**.** *In the metric* (2.1) *the B^t-flat equations $\triangle s = 0$, $2Bach + t(s\,\mathring{\mathrm{Ric}} - \nabla^2 s) = 0$ are equivalent to*

$$\frac{\partial}{\partial z}\left(CF\frac{\partial s}{\partial z}\right) = 0, \quad \mathcal{F}_1(F,C) = 0, \quad \mathcal{F}_2(F,C) = 0, \quad \mathcal{T}(F,C) = 0 \tag{3.18}$$

where $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{T}$ are the operators

$$\begin{aligned}
\mathcal{F}_1(F,C) &= 24\frac{\partial}{\partial z}\left(F\frac{\partial}{\partial z}C^{\frac{1}{2}}\right)+4C^{\frac{1}{2}}\left(\frac{\partial^2 F}{\partial z^2}+\frac{1}{2}F-2\right)+s\,C^{\frac{3}{2}}\\
\mathcal{F}_2(F,C) &= \frac{8}{3}\left(\mathcal{L}^+(\mathcal{L}^-(F))-1\right)+t\,s\,C^{\frac{3}{2}}\left(\frac{\partial^2}{\partial z^2}C^{-\frac{1}{2}}-\frac{1}{4}C^{-\frac{1}{2}}\right)+\frac{t}{2}\frac{C}{F}\frac{\partial F}{\partial z}\frac{\partial s}{\partial z}+t\frac{\partial C}{\partial z}\frac{\partial s}{\partial z}\\
\mathcal{T}(F,C) &= 16\,\mathcal{B}(F,F)-18tF\frac{\partial C}{\partial z}\frac{\partial s}{\partial z}-6tC\frac{\partial F}{\partial z}\frac{\partial s}{\partial z}\\
&\quad-\frac{3}{4}tsC^{-1}\left(C^2(-16+4F+Cs)+12F\left(\frac{\partial C}{\partial z}\right)^2+8C\frac{\partial C}{\partial z}\frac{\partial F}{\partial z}\right)
\end{aligned}\tag{3.19}$$

and $\mathcal{B}$ is the operator from (1.7).

Proof. In coordinates, $\triangle = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(\sqrt{\det g}\,g^{ij}\frac{\partial}{\partial x^j}\right)$. Using (Z,τ,x,y)-coordinates of (2.15) we have $\det g = \frac{1}{16\cosh^2(x)}C^2$ and $g^{11}=4FC$. Because $s=s(Z)$ is a function of Z alone, then $0=\triangle s$ is

$$0=\frac{4\cosh^2(x)}{C}\frac{\partial}{\partial Z}\left(\frac{C}{4\cosh^2(x)}4FC\frac{\partial s}{\partial Z}\right)=\frac{4}{C}\frac{\partial}{\partial Z}\left(FC^2\frac{\partial s}{\partial Z}\right).\tag{3.20}$$

The coordinate change from z to Z of (2.15) gives $C\frac{\partial}{\partial Z}=\frac{\partial}{\partial z}$, so we obtain the first equation of (3.18). The second equation $\mathcal{F}_1(F,C)=0$ is precisely the scalar curvature equation (3.1). With $\triangle s=0$ the Hessian $\nabla^2 s$ is trace-free; then a straightforward computation gives

$$\nabla^2 s=-2C^{-4}\frac{\partial s}{\partial z}\frac{\partial(FC^3)}{\partial z}(\sigma^0)^2+2C^{-2}\frac{\partial s}{\partial z}\frac{\partial(FC)}{\partial z}(\sigma^1)^2+2FC^{-2}\frac{\partial s}{\partial z}\frac{\partial C}{\partial z}\left((\sigma^2)^2+(\sigma^3)^2\right).\tag{3.21}$$

Now for the third and fourth equations we use (2.22), (2.32), and (3.21). We expect precisely two additional relations, due to the fact that each of the tensors $Bach$, $\mathring{\mathrm{Ric}}$, and $\nabla^2 s$ have four non-zero components, but also the two algebraic relations of being trace-free, and having identical $(3,3)$ and $(4,4)$ entries. We take one relation from $2(Bach_{00}+Bach_{22})+t(s\,\mathring{\mathrm{Ric}}_{00}+s\,\mathring{\mathrm{Ric}}_{22}-s_{,00}-s_{,22})=0$. Using (1.9), (1.11), and (3.21), this is

$$\frac{8}{3}\left(\mathcal{L}^-(\mathcal{L}^+(F))-1\right)+tsC^{\frac{3}{2}}\left(\frac{\partial^2\frac{1}{\sqrt{C}}}{\partial z^2}-\frac{1}{4}\frac{1}{\sqrt{C}}\right)+\frac{t}{2}\frac{C}{F}\frac{\partial F}{\partial z}\frac{\partial s}{\partial z}+t\frac{\partial C}{\partial z}\frac{\partial s}{\partial z}=0\tag{3.22}$$

which is $\mathcal{F}_2(C,F)=0$. We take another relation from $2Bach_{00}+t(s\,\mathring{\mathrm{Ric}}_{00}-s_{,00})=0$, which is

$$\begin{aligned}
0 &= 16\mathcal{B}(F,F)-18tF\frac{\partial C}{\partial z}\frac{\partial s}{\partial z}-6tC\frac{\partial F}{\partial z}\frac{\partial s}{\partial z}\\
&\quad-\frac{3}{4}tsC^{-1}\left(C^2(-16+4F+sC)+12F\left(\frac{\partial C}{\partial z}\right)^2+8C\frac{\partial C}{\partial z}\frac{\partial F}{\partial z}\right)\\
&\quad+\frac{3}{4}tsC^{\frac{1}{2}}\left(4C^{\frac{1}{2}}\left(\frac{\partial^2 F}{\partial z^2}+\frac{1}{2}F-2\right)+24\frac{\partial}{\partial z}\left(F\frac{\partial C^{\frac{1}{2}}}{\partial z}\right)+sC^{\frac{3}{2}}\right).
\end{aligned}\tag{3.23}$$

Using (3.1) to eliminate the last term, this is $\mathcal{F}_1(F,C)=0$. $\square$

The equations (3.18) give four equations for the three unknowns s, F, C, so the system appears to be overdetermined. But the equations of (3.18) are not independent.

Lemma 3.9. *We have the following relation:*

$$\frac{\partial \mathcal{T}}{\partial z} = \frac{-3t}{2\sqrt{C}}\frac{\partial(sC)}{\partial z}\mathcal{F}_1 + 12\frac{\partial F}{\partial z}\mathcal{F}_2 - 6t\frac{\partial \log(C^3F)}{\partial z}\frac{\partial}{\partial z}\left(CF\frac{\partial s}{\partial z}\right). \tag{3.24}$$

In particular $\mathcal{T}(F,C)$ is constant along solutions of the system $\mathcal{F}_1(F,C) = \mathcal{F}_2(F,C) = \triangle s = 0$.

Proof. This follows from a tedious but completely elementary computation. □

Lemma 3.10. *At all points where $C \neq 0$ and $F \neq 0$, the 8^{th} order system*

$$\frac{\partial}{\partial z}\left(CF\frac{\partial s}{\partial z}\right) = 0, \quad \mathcal{F}_1(F,C) = 0, \quad \mathcal{F}_2(F,C) = 0 \tag{3.25}$$

is critically determined, $\mathcal{T}$ is a constant of the motion, and (3.25) combined with the restraint $\mathcal{T}(F,C) = 0$ admits a 7-parameter family of solutions.

Up to homothety, in the $U(2)$-invariant setting the B^t-flat metrics form a 5-parameter family, and the CSC B^t-flat metrics form a 4-parameter family.

Proof. To ascertain whether the system (3.25) is critically determined, we examine the coefficients on the derivatives of s, F, and C. These coefficients of the form FC, CF^{-1}, $C^{\frac{1}{2}}$, $C^{-\frac{1}{2}}$, $FC^{-\frac{3}{2}}$ and so on. Provided F and C remain bounded away from 0 and $+\infty$, we have a non-singular principal symbol. We conclude that the system (3.25), which has three unknowns and three equations, remains critically determined when F and C remain positive.

We count the degrees of freedom in the solution space. The equations $\frac{\partial}{\partial z}\left(CF\frac{\partial s}{\partial z}\right) = 0$, $\mathcal{F}_1 = 0$, and $\mathcal{F}_2 = 0$ are fourth order in F, second order in C, and second order in s, which makes eight derivatives in total, requiring eight initial conditions. Then we restrict to $\mathcal{T} = 0$. From Lemma 3.9, $\mathcal{T}$ is constant along solutions so is completely determined by the system's initial conditions. $\mathcal{T}(F,C)$ is third order in F, second order in C, and first order in s, so $\mathcal{T} = 0$ is a single algebraic relationship among the initial conditions, and reduces the solution space from eight dimensions to seven. Up to homothety the solution space is therefore 5-dimensional. Finally, requiring $s = const$ is the same as imposing an initial condition of $s_z = 0$, so the CSC B^t-flat solution space is 4-dimensional up to homothety. □

Theorem 3.11. *The ZSC B^t-flat metrics, $t \neq \infty$, are the ZSC Bach-flat metrics.*

Assume g is B^t-flat, conformally extremal, and $t \neq 0, \infty$. Then it is CSC if and only if it is ZSC or Einstein.

If $t \neq 0, \frac{1}{3}, \infty$ there exist CSC B^t-flat solutions that are not conformally extremal.

Proof. The CSC B^t-flat equations are (3.18) with initial condition $s_z = 0$. As discussed above, this is a system with 6 degrees of freedom (4 up to homothety). First we examine the ZSC case, where $s = 0$. In this case $\mathcal{T} = 16\mathcal{B}$, so $\mathcal{B}(F, F) = 0$ and so the metric is Bach-flat. Thus F lies in the 3-parameter family given by Lemma 3.6. Fixing F, $\mathcal{F}_1 = 0$ gives a 2-parameter family of solutions for C and we obtain the expected 5-parameter solution space of ZSC Bach-flat metrics (which has 3 parameters up to homothety).

Next assume the metric is CSC B^t-flat, $s \neq 0$, and g conformally extremal. By Proposition 3.1, $F = \frac{1}{2}C_1 e^{-2z} + C_2 e^{-z} + C_3 e^z + \frac{1}{2}C_4 e^{2z}$. Plugging in this, along with $\frac{\partial s}{\partial z} = 0$ into $\mathcal{F}_2 = 0$, we obtain

$$\left(\frac{\partial^2}{\partial z^2} C^{-\frac{1}{2}} - \frac{1}{4} C^{-\frac{1}{2}}\right) = 0. \tag{3.26}$$

Therefore $C = \frac{e^{-z}}{(C_5 + C_6 e^{-z})^2}$. Plugging this into $\mathcal{F}_1 = 0$ provides

$$0 = C_5(C_1C_5 - C_2C_6)e^{-z} + \left(-\frac{s}{24} + C_2C_5^2 - 2C_5C_6 - C_3C_6^2\right) + C_6(C_4C_6 - C_3C_5)e^z. \tag{3.27}$$

We have the seven unknown constants $C_1, C_2, C_3, C_4, C_5, C_6$, and s, and (3.27) contributes three relations so we have a 4-parameter solution space. We consider the possibilities. First, the expression for C makes it impossible that C_5 and C_6 are both zero. If $C_5 \neq 0, C_6 = 0$ then $C = C_5^{-2} e^{-z}$ so the metric is Kähler with respect to J^+, and (3.27) forces $C_1 = 0$, $C_2 = \frac{s}{24C_5^2}$. Then $0 = \mathcal{T}$ is

$$0 = -\frac{1}{2} e^{2z} s \left(3st - 4e^{2z}(1 - 3t)C_3C_5^2\right), \tag{3.28}$$

and because $t \neq 0$, this forces $s = 0$, contradicting the assumption $s \neq 0$. (Similarly assuming $C_5 = 0, C_6 \neq 0$ also gives $s = 0$, again contradicting $s \neq 0$.)

Therefore both $C_5, C_6 \neq 0$. Then (3.27) forces $C_1C_5 - C_2C_6 = 0$, $C_4C_6 - C_3C_5 = 0$, and by Proposition 3.2 the metric is Einstein. We conclude that if a CSC B^t-flat metric is conformally extremal, it is ZSC or Einstein.

Finally we prove that some CSC B^t-flat metrics are not conformally extremal. The family of Einstein solutions is 4-dimensional, and therefore, by what we just proved, the family of CSC B^t-flat that are conformally extremal is also 4-dimensional. But the space of CSC B^t-flat metrics is 6-dimensional. We conclude that some CSC B^t-flat metrics fail to be conformally extremal. □

4 AmbiKähler Pairs

AmbiKähler pairs are from [2]. An *ambiKähler structure* on a manifold is a pair of Kähler manifolds (M^n, J_1, g_1) and (M^n, J_2, g_2) where the complex stuctures J_1 and J_2 produce opposite orientations and the Kähler metrics g_1 and g_2 are conformal. Either member of the pair can be called the *ambiKähler transform* of the other. From Lemma 2.2, every $U(2)$-invariant metric on a 4-manifold has an ambiKähler structure using $J^{\pm}$, conformally related by letting C be e^{+z} or e^{-z}.

Consequently the classic $U(2)$-invariant Kähler metrics all have ambiKähler transforms. Most of these ambiKähler transforms produce nothing interesting. The ambiKähler transform of the Burns metric is the Fubini-study metric, for example, and the transforms of the other LeBrun instanton metrics are extremal Kähler metrics on weighted projective spaces—these are Bochner-flat metrics found by Bryant in [8, Section 2.2], although their conformal relationship with the LeBrun instantons was not discussed there. The transform of an odd Hirzebruch surface is precisely itself. The transforms of the Taub-NUT-Λ and Eguchi-Hanson-Λ metrics have curvature singularities.

The Taub-NUT and Taub-bolt cases, however, are more interesting. The Taub-NUT is hyperKähler with its family of complex structures being I^- and its left-translates. By Propositions 2.4 and 2.5

$$F = (1 - e^{-z})^2, \quad C = \frac{C_0 e^{-z}}{(1 - e^{-z})^2} \tag{4.1}$$

with coordinate range $z \in (0, \infty]$. The nut is located at $z = \infty$, and the ALF end is at $z = 0$; see Section 2.3 and Figure 3. Separate from the hyperKähler structure an ambiKähler structure exists, given by complex structures J^- and J^+ and conformal factors $C = C_0 e^z$, $C = C_0 e^{-z}$. Thus the conformal orbit of the Taub-NUT meets three complete canonical metrics: itself which is hyperKähler, a 2-ended ZSC Kähler metric, and a 1-ended extremal Kähler metric. We call the latter two the *modified Taub-NUT metrics of the first and second kinds.*

The modified Taub-NUT of the first kind has complex structure J^- and conformal factor $C = C_0 e^z$, which gives it the same orientation as the original Taub-NUT. This metric is two-ended: the nut at $z = -\infty$ becomes an ALE end, and the ALF end at $z = 0$ becomes a cusp-like end. This complete, 2-ended metric is scalar flat by Proposition 1.1. Letting J^+ be the complex structure with conformal factor $C = C_0 e^{-z}$ produces the modified Taub-NUT of the second kind. This metric is one-ended: it still has a nut at $z = \infty$, but the conformal change turns the ALF end into a cusp-like end. By Theorem 3.1 it is extremal Kähler. It has scalar curvature $s = 48(1 - e^{-z})$, which is positive and approaches 0 asymptotically along the cusp.

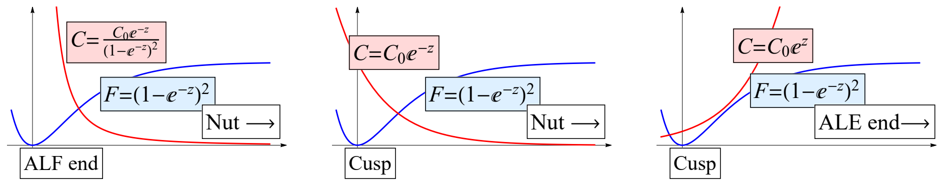

Figure 3: *The Taub-NUT and modified Taub-NUTs of the first and second kinds.*

The modified Taub-NUT of the first kind on $\mathbb{C}^2 \setminus \{(0,0)\}$ is the ZSC Kähler metric of [19] for $n = 2$, and the modified Taub-NUT of the second kind is a complete Bochner-flat metric from [8, Section 2.2] (see also [39]) and is explored in [20].

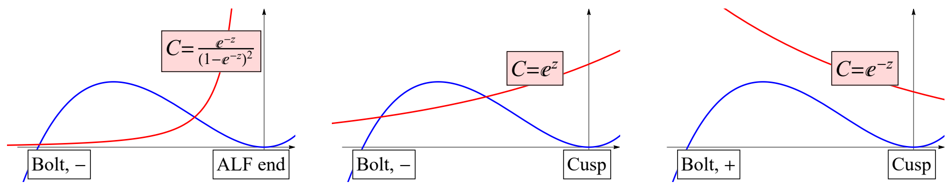

Figure 4: *The Taub-bolt, and the modified Taub-bolts of the first and second kinds.*

The classic Taub-bolt is Ricci-flat but not Kähler (and certainly not hyperKähler) with respect to any complex stucture[2]. The Taub-bolt metric is

$$C = \frac{C_0 e^{-z}}{(1-e^{-z})^2}, \quad F = 1 - \frac{1}{8}e^{-2z} + \frac{1}{4}e^{-z} - \frac{9}{4}e^{z} + \frac{9}{8}e^{2z} \tag{4.2}$$

on $z \in [-\log(3), 0)$. This metric is complete, Ricci-flat, Bach-flat, but not half-conformally flat: both W^+ and W^- are non-zero by Proposition 3.3; see [35, 36]. It has an ALF end at $z = 0$ and a bolt of self-intersection -1 at $z = -\log(3)$. The underlying manifold is the total space of $\mathcal{O}(-1)$. It is conformally Kähler with respect to either J^- or J^+, creating an ambiKähler pair—the *modified Taub-bolt metrics of the first and second kinds*, respectively. Changing between J^- and J^+ reverses the orientation, so changes the self-intersection number of the bolt from -1 to $+1$.

With the complex structure J^- and conformal factor $C = C_0 e^z$ we obtain an extremal Kähler metric we call the *modified Taub-bolt of the first kind*. This metric continues to have a bolt of self-intersection -1 at $z = -\log(3)$, but the ALF end at $z = 0$ has been transformed into a cusp-like end. The scalar curvature is $s = 54C_0^{-1}(1 - e^z)$, which is positive and approaches 0 along the cusp. Its underlying complex manifold is the total space of $\mathcal{O}(-1)$. Its ambiKähler transform has complex structure J^+ and conformal factor $C = C_0 e^{-z}$; we call this extremal Kähler metric the *modified Taub-bolt of the second kind*. The orientation has been reversed and the bolt has self-intersection $+1$ at $z = -\log(3)$. The ALF end at $z = 0$ has again been transformed into a cusp-like end. The scalar curvature is $s = 6C_0^{-1}(-1 + e^{-z})$, which again is positive and approaches

[2] If it were Kähler with respect to *any* complex structure, whether a complex structure considered here or not, Derdzinski's theorem would imply it is half-conformally flat which it is not.

zero asymptotically along the cusp. Its underlying complex manifold is the total space of $\mathcal{O}(+1)$, which is $\mathbb{C}P^2 \setminus \{pt\}$.

Like the Taub-NUT, the Taub-bolt's conformal orbit meets three canonical metrics: itself, which is Ricci flat, and two extremal Kähler metrics. See also [6] which explores the Taub-bolt among other topics (electronically released almost simultaneously with this paper). Neither of the modified Taub-bolts is Bochner-flat or half-conformally flat.

Notable is the presence of a rational curve of positive self intersection in the modified Taub-bolt of the second kind. This is the only example of a complete extremal Kähler metric with a curve of positive self-intersection, that is known to the authors. By contrast there are many examples with curves of zero or negative self intersection. These include the Burns, Eguchi-Hanson, and LeBrun metrics which are all Kähler metrics on $\mathcal{O}(k)$ with $k < 0$ [29]; the Chen-Teo metrics [11, 12] and conformally related Kähler metrics [6] which are on surfaces with rational curves of non-positive self-intersection; and the extremal Kähler "asymptotically equivariantly $\mathbb{R}^2 \times \mathbb{S}^2$" [40, 41] metrics which all have rational curves of non-positive self-intersection.

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