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Rational approximation of the finite sum of some sequences

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ABSTRACT

In this paper, we give some rational approximations of $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$ by the multiple-correction method and present the bounds of its error.

RESUMEN

En este artículo, entregamos algunas aproximaciones racionales de $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$ por el método de corrección múltiple y presentamos las cotas de su error.

Keywords and Phrases: Rational approximation, continued fraction, inequalities, multiple-correction method.

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1 Introduction

It is well known that we often need to deal with the problem of approximating the factorial function $n!$, and its extension to real numbers called the gamma function, defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad \mathrm{Re}(x) > 0,$$

and the logarithmic derivative of $\Gamma(x)$ called the psi-gamma function, denoted by

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

For $x > 0$, the derivative $\psi'(x)$ is called the tri-gamma functions, while the derivatives $\psi^{(k)}(x)$, $k = 1, 2, 3, \dots$ are called the poly-gamma functions.

In recent years, some authors paid attention to giving increasingly better approximations for the gamma function using continued fractions. For detailed information, please refer to the papers [1,2, 9,11,12] and references cited therein. In fact, it is quite well-known in the theory the algorithm for transforming every formal power series into an associated continued fraction, see [6]. In particular, there are certain methods of transforming the power series $\sum_{n=0}^{\infty} c_n x^{-n-1}$ into continued fractions, see [10, Section III].

For any integer i and $x > 0$, we have

$$\psi^{(i)}(x+1) - \psi^{(i)}(x) = (-1)^i \frac{i!}{x^{i+1}},$$

and when $i = 0$, it yields

$$\psi(x+1) - \psi(x) = \frac{1}{x}.$$

By adding equalities of the form

$$\psi(j+1) - \psi(j) = \frac{1}{j}$$

from $j = n^2 + 1$ to $j = n^2 + n$, we get

$$\psi(n^2+n+1) - \psi(n^2+1) = \sum_{j=1}^{n} \frac{1}{n^2+j} = S(n) \tag{1.1}$$

Graham, Knuth and Patashnik [5] proposed the problem of obtaining the asymptotic value of the finite sum

$$S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j} = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n} \tag{1.2}$$

with a given absolute error.

In this paper, we handle the problem with the aid of the multiple-correction method [3,4,13]. We will give some rational approximations of $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$ by the multiple-correction method, and prove some inequalities for the upper and lower bounds. Throughout the paper, the notation $P(x;k)$ means a polynomial of degree k in x, which may be different at each occurrence.

2 Some lemmas

The following lemma gives a method for measuring the rate of convergence, for its proof see Mortici [7,8].

Lemma 2.1. *If the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to zero and there exists the limit*

$$\lim_{n\to+\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty], \tag{2.1}$$

with $s > 1$, then

$$\lim_{n\to+\infty} n^{s-1}x_n = \frac{l}{s-1}. \tag{2.2}$$

We also need the following intermediary result.

Lemma 2.2. *For every positive integer k, we define*

$$f_k(x) = \ln x + \frac{s_1}{x + t_1 + \frac{s_2}{x+t_2+\cdots+\frac{s_k}{x+t_k}}},$$

where $s_1 = -\frac{1}{2}$, $t_1 = -\frac{1}{6}$; $s_2 = \frac{1}{36}$, $t_2 = -\frac{13}{30}$; $s_3 = \frac{9}{25}$, $t_3 = -\frac{17}{630}$; $s_4 = \frac{6241}{15876}$, $t_4 = -\frac{417941}{786366}$; ...
Then for $x > 1$, we have

$$f_2(x+1) - f_2(x) < \frac{1}{x} < f_3(x+1) - f_3(x). \tag{2.3}$$

Proof. We will apply the multiple-correction method [3, 4, 13] to study the two-sided inequality (2.3) as follows.

(Step 1) The initial-correction. Since $(\ln x)' = \frac{1}{x}$, so we choose $f_0(x) = \ln x$ and develop $F_0(x) := f_0(x+1) - f_0(x) - \frac{1}{x}$ into power series expansion in $\frac{1}{x}$, we have

$$F_0(x) = f_0(x+1) - f_0(x) - \frac{1}{x} = -\frac{1}{2}\frac{1}{x^2} + \frac{1}{3}\frac{1}{x^3} + O\left(\frac{1}{x^4}\right). \tag{2.4}$$

(Step 2) The first-correction. Let $f_1(x) = \ln x + \frac{s_1}{x+t_1}$ and develop $F_1(x) := f_1(x+1) - f_1(x) - \frac{1}{x}$ into power series expansion in $\frac{1}{x}$, we have

$$F_1(x) = \left(-\frac{1}{2} - s_1\right)\frac{1}{x^2} + \left(\frac{1}{3} + s_1 + 2s_1t_1\right)\frac{1}{x^3} + O\left(\frac{1}{x^4}\right). \tag{2.5}$$

Then let the coefficients of $\frac{1}{x^2}$ and $\frac{1}{x^3}$ in (2.5) equal zero, we have $s_1 = -\frac{1}{2}$, $t_1 = -\frac{1}{6}$ and

$$F_1(x) = \frac{1}{24}\frac{1}{x^4} + O\left(\frac{1}{x^5}\right). \tag{2.6}$$

(Step 3) The second-correction. Let $f_2(x) = \ln x + \frac{s_1}{x+t_1+\frac{s_2}{x+t_2}}$ and develop $F_2(x) := f_2(x+1) - f_2(x) - \frac{1}{x}$ into power series expansion in $\frac{1}{x}$, it can be derived that

$$F_2(x) = \left(\frac{1}{24} - \frac{3s_2}{2}\right)\frac{1}{x^4} + \left(-\frac{11}{270} + \frac{7s_2}{3} + 2s_2t_2\right)\frac{1}{x^5} + O\left(\frac{1}{x^6}\right). \tag{2.7}$$

Then let the coefficients of $\frac{1}{x^4}$ and $\frac{1}{x^5}$ in (2.7) equal zero, we have $s_2 = \frac{1}{36}$, $t_2 = -\frac{13}{30}$ and

$$F_2(x) = f_2(x+1) - f_2(x) - \frac{1}{x} = -\frac{1}{40}\frac{1}{x^6} + O\left(\frac{1}{x^7}\right). \tag{2.8}$$

Furthermore, we obtain

$$F_2'(x) = \frac{P(x)}{3x^2(1+x)(1-6x+10x^2)^2(5+14x+10x^2)^2},$$

where $P(x) = 75 - 480x - 508x^2 + 3680x^3 + 4500x^4$.

As all coefficients of $P(x+1) = 7267 + 27544x + 37532x^2 + 21680x^3 + 4500x^4$ are positive, which implies that $F_2(x)$ is strictly increasing. Since $F_2(\infty) = 0$, it can be found that $F_2(x) < 0$ on $x > 1$. This finishes the proof of the left-hand inequality in (2.3).

(Step 4) The third-correction. Similarly, let $f_3(x) = \ln x + \frac{s_1}{x+t_1+\frac{s_2}{x+t_2+\frac{s_3}{x+t_3}}}$ and develop $F_3(x) := f_3(x+1) - f_3(x) - \frac{1}{x}$ into power series expansion in $\frac{1}{x}$, we have

$$F_3(x) = \left(-\frac{1}{40} + \frac{5s_3}{72}\right)\frac{1}{x^6} + \frac{802 - 2275s_3 - 1750s_3t_3}{21000}\frac{1}{x^7} + O\left(\frac{1}{x^8}\right). \tag{2.9}$$

Then let the coefficients of $\frac{1}{x^6}$ and $\frac{1}{x^7}$ in (2.9) equal zero, we have $s_3 = \frac{9}{25}$, $t_3 = -\frac{17}{630}$ and

$$F_3(x) = f_3(x+1) - f_3(x) - \frac{1}{x} = \frac{6241}{453600}\frac{1}{x^8} + O\left(\frac{1}{x^9}\right). \tag{2.10}$$

Furthermore, we obtain

$$F_3'(x) = \frac{Q(x)}{3x^2(1+x)(-79+600x-790x^2+1260x^3)^2(991+2800x+2990x^2+1260x^3)^2},$$

where $Q(x) = 18387502563 - 175398675600\,x - 226510750180\,x^2 - 500966546560x^3 - 1400497343100x^4 - 1903983580800x^5 - 832289774400x^6$.

As all coefficients of $Q(x+1) = -5021259168077 - 22246965738440x - 41656576872460x^2 - 41788587214960x^3 - 23404761863100x^4 - 6897722227200x^5 - 832289774400x^6$ are negative, which implies that $F_3(x)$ is strictly decreasing. Since $F_3(\infty) = 0$, it can be found that $F_3(x) > 0$ on $x > 1$. This finishes the proof of the right-hand inequality in (2.3).

The proof of Lemma 2.2 is completed. □

3 Main results

By adding inequalities (2.3) of the form

$$f_2(x+1) - f_2(x) < \frac{1}{x} < f_3(x+1) - f_3(x)$$

from $x = n^2+1$ to $x = n^2+n$, we get

$$f_2(n^2+n+1) - f_2(n^2+1) < \sum_{j=1}^{n} \frac{1}{n^2+j} < f_3(n^2+n+1) - f_3(n^2+1). \tag{3.1}$$

This two-sided inequalities give the estimate of $\sum_{j=1}^{n} \frac{1}{n^2+j}$. So we have

Theorem 3.1. *For positive integer $n > 1$,*

$$\ln\left(1+\frac{n}{n^2+1}\right) + \frac{P(n;5)}{3P_1(n;4)P_2(n;4)} < \sum_{j=1}^{n} \frac{1}{n^2+j} < \ln\left(1+\frac{n}{n^2+1}\right) + \frac{5P(n;9)}{3P_1(n;6)P_2(n;6)}, \tag{3.2}$$

where

$$\begin{aligned}
P(n;5) &= 44n + 85n^2 + 170n^3 + 150n^4 + 150n^5,\\
P_1(n;4) &= 5 + 14n^2 + 10n^4,\\
P_2(n;4) &= 5 + 14n + 24n^2 + 20n^3 + 10n^4,\\
P(n;9) &= 387838n + 655457n^2 + 1744984n^3 + 1983990n^4 + 2717310n^5\\
&\quad + 2199960n^6 + 1942920n^7 + 952560n^8 + 476280n^9,\\
P_1(n;6) &= 991 + 2800n^2 + 2990n^4 + 1260n^6,\\
P_2(n;6) &= 991 + 2800n + 5790n^2 + 7240n^3 + 6770n^4 + 3780n^5 + 1260n^6.
\end{aligned}$$

Proof. The double inequality (3.1) can be equivalently written as (3.2). □

Theorem 3.1 gives an asymptotic formula of the sum $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$, but we want to obtain the rational approximation. It ensures the following approximation formula as $n \to \infty$, $\ln\left(1+\frac{n}{n^2+1}\right) \sim \frac{n}{n^2+1}$, but the rate of convergence is not satisfied. Now we estimate the function $\ln\left(1+\frac{n}{n^2+1}\right)$ as following.

Theorem 3.2. *For positive integer $n > 1$, we have*

$$\frac{n^2+\frac{133}{109}n-\frac{769}{6540}}{n^3+\frac{375}{218}n^2+\frac{768}{545}n+\frac{2401}{2180}} < \ln\left(1+\frac{n}{n^2+1}\right) < \frac{n-\frac{1}{22}}{n^2+\frac{5}{11}n+\frac{59}{66}}. \tag{3.3}$$

Proof. Developing the function $\ln\left(1+\frac{n}{n^2+1}\right) - \frac{s_2n^2+s_1n+s_0}{n^3+t_2n^2+t_1n+t_0}$ into power series expansion in $\frac{1}{n}$, we have

$$\begin{aligned}
&\ln\left(1+\frac{n}{n^2+1}\right) - \frac{s_2n^2+s_1n+s_0}{n^3+t_2n^2+t_1n+t_0} \qquad (3.4)\\
&= (1-s_2)\frac{1}{n} + \left(-\frac{1}{2}-s_1+s_2t_2\right)\frac{1}{n^2} + \left(-\frac{2}{3}-s_0+s_2t_1+s_1t_2-s_2t_2^2\right)\frac{1}{n^3}\\
&+ \left(\frac{3}{4}+s_2t_0+s_1t_1+s_0t_2-2s_2t_1t_2-s_1t_2^2+s_2t_2^3\right)\frac{1}{n^4}\\
&+ \left(\frac{1}{5}+s_1t_0+s_0t_1-s_2t_1^2-2s_2t_0t_2-2s_1t_1t_2-s_0t_2^2+3s_2t_1t_2^2+s_1t_2^3-s_2t_2^4\right)\frac{1}{n^5}\\
&+ \left(-\frac{2}{3}+s_0t_0-2s_2t_0t_1-s_1t_1^2-2s_1t_0t_2-2s_0t_1t_2+3s_2t_1^2t_2+3s_2t_0t_2^2+3s_1t_1t_2^2\right.\\
&\left.+s_0t_2^3-4s_2t_1t_2^3-s_1t_2^4+s_2t_2^5\right)\frac{1}{n^6} + O\left(\frac{1}{n^7}\right).
\end{aligned}$$

According to Lemma 2.1, to get the highest rate of convergence, we have $s_2 = 1$, $s_1 = \frac{133}{109}$, $s_0 = -\frac{769}{6540}$, $t_2 = \frac{375}{218}$, $t_1 = \frac{768}{545}$, $t_0 = \frac{2401}{2180}$ and

$$\ln\left(1+\frac{n}{n^2+1}\right) - \frac{s_2n^2+s_1n+s_0}{n^3+t_2n^2+t_1n+t_0} = \frac{31721}{305200}\frac{1}{n^7} + O\left(\frac{1}{n^8}\right).$$

Furthermore, we denote $G_1(x) = \ln\left(1+\frac{x}{x^2+1}\right) - \frac{x^2+\frac{133}{109}x-\frac{769}{6540}}{x^3+\frac{375}{218}x^2+\frac{768}{545}x+\frac{2401}{2180}}$, then we can get

$$G_1'(x) = -\frac{1409315+4813232x+3457589x^2}{(1+x^2)(1+x+x^2)(2401+3072x+3750x^2+2180x^3)^2} < 0,$$

which implies that $G_1(x)$ is strictly decreasing. Since $G_1(\infty) = 0$, it can be found that $G_1(n) > 0$ for every positive integer n. Then we have

$$\frac{n^2+\frac{133}{109}n-\frac{769}{6540}}{n^3+\frac{375}{218}n^2+\frac{768}{545}n+\frac{2401}{2180}} < \ln\left(1+\frac{n}{n^2+1}\right). \tag{3.5}$$

This finishes the proof of the left-hand inequality in (3.3).

Similarly, developing the function $\ln\left(1+\frac{n}{n^2+1}\right)-\frac{u_1n+u_0}{n^2+v_1n+v_0}$ into power series expansion in $\frac{1}{n}$, we have

$$\begin{aligned}
&\ln\left(1+\frac{n}{n^2+1}\right)-\frac{u_1n+u_0}{n^2+v_1n+v_0} \qquad (3.6)\\
&=(1-u_1)\frac{1}{n}+\left(-\frac{1}{2}-u_0+u_1v_1\right)\frac{1}{n^2}+\left(-\frac{2}{3}+u_1v_0+u_0v_1-u_1v_1^2\right)\frac{1}{n^3}\\
&+\left(\frac{3}{4}+u_0v_0-2u_1v_0v_1-u_0v_1^2+u_1v_1^3\right)\frac{1}{n^4}+O\left(\frac{1}{n^5}\right).
\end{aligned}$$

According to Lemma 2.1, to get the highest rate of convergence, we have $u_1=1$, $u_0=-\frac{1}{22}$, $v_1=\frac{5}{11}$, $v_0=\frac{59}{66}$ and

$$\ln\left(1+\frac{n}{n^2+1}\right)-\frac{u_1n+u_0}{n^2+v_1n+v_0}=-\frac{109}{1980}\frac{1}{n^5}+O\left(\frac{1}{n^6}\right).$$

Furthermore, we denote $G_2(x)=\ln\left(1+\frac{x}{x^2+1}\right)-\frac{x-\frac{1}{22}}{x^2+\frac{5}{11}x+\frac{59}{66}}$, then we can get

$$G_2'(x)=\frac{-503-840x+1199x^2}{(1+x^2)(1+x+x^2)(59+30x+66x^2)^2}>0$$

when $x>1$, which implies that $G_2(x)$ is strictly increasing. Since $G_2(\infty)=0$, it can be found that $G_2(n)>0$ for positive integer $n>1$. Then we have

$$\ln\left(1+\frac{n}{n^2+1}\right)<\frac{n-\frac{1}{22}}{n^2+\frac{5}{11}n+\frac{59}{66}}. \qquad (3.7)$$

This finishes the proof of the right-hand inequality in (3.3).

The proof of Theorem 3.2 is completed. □

Combining (3.2) and (3.3), we have

Theorem 3.3. *As $n\to\infty$,*

$$\frac{P(n;10)}{3P(n;3)P_1(n;4)P_2(n;4)}<\sum_{j=1}^{n}\frac{1}{n^2+j}<\frac{P(n;13)}{3P(n;2)P_1(n;6)P_2(n;6)}, \qquad (3.8)$$

where

$$\begin{aligned}
P(n;10)&=-19225+251314n+915243n^2+2580666n^3+4566456n^4+6735890n^5\\
&\quad+7304720n^6+6514900n^7+4331300n^8+2106000n^9+654000n^{10},\\
P(n;3)&=2401+3072n+3750n^2+2180n^3,
\end{aligned}$$

$$\begin{aligned} P(n;13) = & -8838729 + 283891048n + 724331705n^2 + 2291454430n^3 + 3803306340n^4 \\ & + 6508603530n^5 + 7775628660n^6 + 9153584460n^7 + 8099239500n^8 + 6891737400n^9 \\ & + 4319179200n^{10} + 2549232000n^{11} + 928746000n^{12} + 314344800n^{13}, \\ P(n;2) = & \ 59 + 30n + 66n^2. \end{aligned}$$

So we can get the rational approximation $\frac{P(n;10)}{3P(n;3)P_1(n;4)P_2(n;4)}$ of the finite sum $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$, and the error can be bounded as following,

Theorem 3.4. *As $n \to \infty$, we have*

$$\sum_{j=1}^{n} \frac{1}{n^2+j} \sim T(n) = \frac{P(n;10)}{3P(n;3)P_1(n;4)P_2(n;4)}. \tag{3.9}$$

Furthermore, we can give the bounds of the error estimation,

$$0 < \sum_{j=1}^{n} \frac{1}{n^2+j} - T(n) < \frac{109}{1980} \frac{1}{n^5}. \tag{3.10}$$

Proof. Set $D = \frac{109}{1980}$, from (3.8) we can get

$$\begin{aligned} & \frac{P(n;13)}{3P(n;2)P_1(n;6)P_2(n;6)} - T(n) - \frac{D}{n^5} \\ & = -\frac{P(n;24)}{1980n^5 P(n;2)P_1(n;3)P_1(n;4)P_2(n;4)P_1(n;6)P_2(n;6)} < 0, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} P(n;24) = & \ 379103668732775 + 2810435887808320n + 14242250073272280n^2 \\ & + 52307052296627116n^3 + 157936445498291068n^4 + 399973820542120296n^5 \\ & + 882209143385828432n^6 + 1711892774844546448n^7 + 2970795182632943800n^8 \\ & + 4635720249539129840n^9 + 6558910458343361680n^{10} + 8434105620517736160n^{11} \\ & + 9897520754047548080n^{12} + 10594749646379864160n^{13} + 10355798883536793600n^{14} \\ & + 9208131536164270400n^{15} + 7433462344335679600n^{16} + 5402752686291200000n^{17} \\ & + 3514488757828417600n^{18} + 2012863116859364800n^{19} + 1001770606450320000n^{20} \\ & + 417999105909504000n^{21} + 141577633391040000n^{22} + 34754556120480000n^{23} \\ & + 5414684436000000n^{24}. \end{aligned}$$

Proof of Theorem 3.4 is completed. □

Remark 3.5. *As $n \to \infty$, we also can get the rational approximation*

$$W(n) = \frac{P(n;13)}{3P(n;2)P_1(n;6)P_2(n;6)} \tag{3.12}$$

of the finite sum $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$.

Remark 3.6. *Using the Maclaurin series of the left and right hand of (3.2), we obtain*

$$\frac{29}{440}\frac{1}{n^{11}} + \frac{1}{30}\frac{1}{n^{12}} + O\left(\frac{1}{n^{13}}\right) \le \sum_{j=1}^{n} \frac{1}{n^2+j} - U(n) \le \frac{1}{11}\frac{1}{n^{11}} - \frac{1}{24}\frac{1}{n^{12}} + O\left(\frac{1}{n^{13}}\right). \tag{3.13}$$

So we have another approximation, as $n \to \infty$,

$$\sum_{j=1}^{n} \frac{1}{n^2+j} \sim U(n) = \frac{1}{n} - \frac{1}{2}\frac{1}{n^2} - \frac{1}{6}\frac{1}{n^3} + \frac{1}{4}\frac{1}{n^4} - \frac{2}{15}\frac{1}{n^5} + \frac{1}{12}\frac{1}{n^6} - \frac{1}{42}\frac{1}{n^7} - \frac{1}{24}\frac{1}{n^8} + \frac{7}{90}\frac{1}{n^9} - \frac{1}{10}\frac{1}{n^{10}}. \tag{3.14}$$

Furthermore, we denote $H_1(x) = \ln\left(1 + \frac{x}{x^2+1}\right) + \frac{P(x;5)}{3P_1(x;4)P_2(x;4)} - U(x) - \frac{29}{440}\frac{1}{x^{11}}$, then we can get $H_1'(x) = \frac{P(x;19)}{120x^{12}(1+x^2)(1+x+x^2)P_1^2(x;4)P_2^2(x;4)}$, where

$$\begin{aligned} P(x;19) = {}& 54375 + 283875x + 1223550x^2 + 3541475x^3 + 8928955x^4 + 18003620x^5 \\ & + 32386512x^6 + 48945976x^7 + 66608504x^8 + 76840064x^9 + 79734920x^{10} \\ & + 68524380x^{11} + 52231532x^{12} + 29887232x^{13} + 14214864x^{14} + 1988640x^{15} \\ & - 1179920x^{16} - 2468400x^{17} - 927200x^{18} - 480000x^{19}. \end{aligned}$$

As all coefficients of

$$\begin{aligned} P(x+3;19) = {}& -1095798626414130 - 6922138869735924x - 20458381656316617x^2 \\ & - 37730683241040109x^3 - 48798043215225557x^4 - 47107553905950172x^5 \\ & - 35247917132102064x^6 - 20940823139217776x^7 - 10032400214888248x^8 \\ & - 3912613116855772x^9 - 1247976394963924x^{10} - 325701204911892x^{11} \\ & - 69291596265604x^{12} - 11915674458880x^{13} - 1632596145936x^{14} - 174202919520x^{15} \\ & - 13962062720x^{16} - 791257200x^{17} - 28287200x^{18} - 480000x^{19} \end{aligned}$$

are negative, which implies that $H_1(x)$ is strictly decreasing on $x > 3$. Since $H_1(\infty) = 0$, it can be found that $H_1(n) > 0$ for positive integer $n > 3$. Then we have

$$\frac{29}{440}\frac{1}{n^{11}} \le \sum_{j=1}^{n} \frac{1}{n^2+j} - U(n). \tag{3.15}$$

Similarly, we denote $H_2(x) = \ln\left(1+\frac{x}{x^2+1}\right) + \frac{5P(x;9)}{3P_1(x;6)P_2(x;6)} - U(x) - \frac{1}{11}\frac{1}{x^{11}}$, *then we can get* $H_2'(x) = \frac{P(x;27)}{30x^{12}(1+x^2)(1+x+x^2)P_1^2(x;6)P_2^2(x;6)}$, *where*

$$\begin{aligned} P(x;27) = {} & 28934492716830 + 163504701528000x + 781783155292011x^2 \\ & + 2561640891519341x^3 + 7366886663076127x^4 + 17465244022945601x^5 \\ & + 37293047508784116x^6 + 69715428169427545x^7 + 119236982847280685x^8 \\ & + 183471922929904370x^9 + 260745743812768040x^{10} + 338060035189670685x^{11} \\ & + 406969201616917085x^{12} + 450014549032420100x^{13} + 463005366631670400x^{14} \\ & + 438405464461473000x^{15} + 385877522700724000x^{16} + 311756448527065800x^{17} \\ & + 233075982007921000x^{18} + 158623848613552500x^{19} + 98916577490962500x^{20} \\ & + 55177732215522000x^{21} + 27657182228634000x^{22} + 11962175918742000x^{23} \\ & + 4459721484330000x^{24} + 1316647483200000x^{25} + 296695768320000x^{26} \\ & + 37807106400000x^{27}. \end{aligned}$$

As all coefficients of $P(x;27)$ *are positive, which implies that* $H_2(x)$ *is strictly increasing. Since* $H_2(\infty) = 0$, *it can be found that* $H_2(n) < 0$ *for every positive integer* n. *Then we have*

$$\sum_{j=1}^{n} \frac{1}{n^2+j} - U(n) \le \frac{1}{11}\frac{1}{n^{11}}. \tag{3.16}$$

So we can give the upper and lower bounds as follow, for positive integer $n > 3$,

$$\frac{29}{440}\frac{1}{n^{11}} \le \sum_{j=1}^{n} \frac{1}{n^2+j} - U(n) \le \frac{1}{11}\frac{1}{n^{11}}. \tag{3.17}$$

4 Some new estimates and double side inequalities

In order to prove the announced inequalities, we use the direct consequence of Theorem 8 of Alzer [2] who proved that the double-sided inequalities for the function of arbitrary accuracies

$$\ln x - \frac{1}{2x} - \sum_{i=1}^{2n-1} \frac{B_{2i}}{2ix^{2i}} < \psi(x) < \ln x - \frac{1}{2x} - \sum_{i=1}^{2n} \frac{B_{2i}}{2ix^{2i}}, \quad (x > 0,\, n \in N), \tag{4.1}$$

where B_j, $j \ge 0$ denote the Bernoulli numbers which may be generated by

$$\frac{z}{e^z - 1} = \sum_{j=1}^{\infty} B_j \frac{z^j}{j!}.$$

In particular, for $i = 2$, we deduce that:

$$\ln x - \frac{1}{2x} - Q_6(x) < \psi(x) < \ln x - \frac{1}{2x} - Q_8(x), \tag{4.2}$$

where $Q_6(x) = \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6}$, $Q_8(x) = \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6} - \frac{1}{240x^8}$. Combining (1.1) and (4.2), we get

$$\begin{aligned}\ln \frac{n^2+n+1}{n^2+1} + \frac{P_1(n;25)}{5040(n^2+1)^8(n^2+n+1)^6} &< S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j} \\ = \psi(n^2+n+1) - \psi(n^2+1) &< \ln \frac{n^2+n+1}{n^2+1} + \frac{P_2(n;25)}{5040(n^2+1)^8(n^2+n+1)^6},\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}P_1(n;25) = &-21 + 3186n + 15651n^2 + 69238n^3 + 202356n^4 + 529934n^5 \\ &+ 1122353n^6 + 2160262n^7 + 3588004n^8 + 5473222n^9 + 7408367n^{10} \\ &+ 9267866n^{11} + 10416693n^{12} + 10852108n^{13} + 10193994n^{14} + 8875980n^{15} \\ &+ 6943146n^{16} + 5020008n^{17} + 3220812n^{18} + 1898232n^{19} + 966000n^{20} \\ &+ 446880n^{21} + 167580n^{22} + 56280n^{23} + 12600n^{24} + 2520n^{25},\end{aligned}$$

and

$$\begin{aligned}P_2(n;25) = &\ 21 + 3312n + 22842n^2 + 105784n^3 + 354605n^4 + 972552n^5 \\ &+ 2229004n^6 + 4439168n^7 + 7749915n^8 + 12075104n^9 + 16850506n^{10} \\ &+ 21261744n^{11} + 24267221n^{12} + 25182808n^{13} + 23708364n^{14} + 20294352n^{15} \\ &+ 15714090n^{16} + 11002824n^{17} + 6899676n^{18} + 3862152n^{19} + 1894620n^{20} \\ &+ 808080n^{21} + 287700n^{22} + 84000n^{23} + 17640n^{24} + 2520n^{25}.\end{aligned}$$

So we can immediately obtain the new estimates of the finite sum $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$ as following,

Theorem 4.1. *As $n \to \infty$, we have*

$$\sum_{j=1}^{n} \frac{1}{n^2+j} \sim V(n) = \ln \frac{n^2+n+1}{n^2+1} + \frac{2520n^{25}}{5040(n^2+1)^8(n^2+n+1)^6}. \tag{4.4}$$

Remark 4.2. *If we select a lager n in the double-sided inequalities* (4.1)*, we can get others double-sided rational estimates for the considered function S_n with arbitrary accuracies.*

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