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Infinitesimally tight Lagrangian submanifolds in adjoint orbits: A classification of real forms

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ABSTRACT

In this paper, we study the geometry of real flag manifolds within complex flag manifolds, focusing on their Lagrangian properties. We prove that the natural immersion of real flag manifolds into their corresponding complex flag manifolds can be characterized as infinitesimally tight Lagrangian submanifolds with respect to the Kirillov-Kostant-Souriau (KKS) symplectic form. This property of tightness provides a significant geometric constraint, indicating that the submanifolds are locally minimal and cannot be deformed infinitesimally to reduce their volume further in the ambient space. We further provide a comprehensive classification of these immersions, detailing the conditions under which such submanifolds exist across various symmetric pairs. This classification elucidates the relationship between the structure of the real flags and the associated complex flags, contributing to a deeper understanding of the interplay between symplectic geometry and representation theory.

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RESUMEN

En este artículo, estudiamos la geometría de variedades bandera reales dentro de variedades bandera complejas, con foco en sus propiedades Lagrangianas. Demostramos que la inmersión natural de variedades bandera reales en sus correspondientes variedades bandera complejas puede caracterizarse como subvariedades Lagrangianas infinitesimalmente estrechas con respecto a la forma simpléctica de Kirillov-Kostant-Souriau (KKS). Esta propiedad de estrechez provee una restricción geométrica significativa, indicando que las subvariedades son localmente mínimas y no pueden deformarse infinitesimalmente para reducir aún más su volumen en el espacio ambiente. Además entregamos una clasificación completa de estas inmersiones, detallando las condiciones bajo las cuales tales subvariedades existen entre varios pares simétricos. Esta clasificación aclara la relación entre la estructura de las banderas reales y las banderas complejas asociadas, contribuyendo a un entendimiento más profundo de la interacción entre la geometría simpléctica y la teoría de representaciones.

Keywords and Phrases: Flag manifolds, homogeneous space, Lagrangian submanifolds, infinitesimally tight

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1 Introduction

Lagrangian submanifolds in symplectic homogeneous spaces have been extensively studied, with significant contributions to their classification in various contexts. For instance, compact symplectic homogeneous manifolds have been classified in [24]. In this paper, we focus on the coadjoint orbits of semisimple Lie groups, exploring the applications of semisimple Lie theory to symplectic geometry, specifically in identifying Lagrangian submanifolds within adjoint orbits. Our motivation stems from the homological mirror symmetry conjecture and, in particular, from concepts in Fukaya–Seidel categories, where objects and morphisms are generated by Lagrangian vanishing cycles and their thimbles, exhibiting specific behaviors within symplectic fibrations (see [10] and [12]).

The primary objective of this paper is to investigate the locally, globally, and infinitesimally tight Lagrangian submanifolds on adjoint orbits, a concept first introduced by Y.-G. Oh in 1991 (see [17]). Oh defined tightness for closed Lagrangian submanifolds in compact Hermitian symmetric spaces as follows:

Definition 1.1. *Let (M,ω,J) be a Hermitian symmetric space of compact type and $\mathcal{L}$ a closed embedded Lagrangian submanifold of M. Then $\mathcal{L}$ is said to be* **globally tight** *(resp.* **locally tight***) if it satisfies*

$$\#\left(\mathcal{L}\cap g\cdot\mathcal{L}\right)=\mathrm{SB}\left(\mathcal{L},\mathbb{Z}_2\right)$$

for any isometry $g\in G$ (resp. sufficiently close to the identity) such that $\mathcal{L}$ intersects $g\cdot\mathcal{L}$ transversely. Here, $\mathrm{SB}\left(\mathcal{L},\mathbb{Z}_2\right)$ denotes the sum of the $\mathbb{Z}_2$-Betti numbers of $\mathcal{L}$.

In the same work, Oh demonstrated that the standard $\mathbb{RP}^n$ inside $\mathbb{CP}^n$ is tight and minimizes volume among all its Hamiltonian deformations (see [17]), linking tightness to Hamiltonian volume minimization (see [18]). This concept is further connected to the Arnold–Givental conjecture, which posits that the number of intersection points between a Lagrangian $\mathcal{L}$ and its image under a Hamiltonian flow $\phi(\mathcal{L})$ is bounded below by the sum of its $\mathbb{Z}_2$-Betti numbers:

$$\#\left(\mathcal{L}\cap\phi(\mathcal{L})\right)\geq\sum b_k(\mathcal{L};\mathbb{Z}_2).$$

The study of tight Lagrangian submanifolds is therefore of significant interest in symplectic geometry. Oh also posed the open problem:

Problem 1.2. *Classify all possible tight Lagrangian submanifolds in other Hermitian symmetric spaces.*

By [17], Oh proposed the following conjecture:

Conjecture 1.3. *Are the real forms in these spaces the only possible tight Lagrangian submanifolds?*

While Oh's conjecture suggests that real forms may be the only possible tight Lagrangian submanifolds in Hermitian symmetric spaces, our study is restricted to the case of flag manifolds. In particular, we examine the natural immersion of real flag manifolds into their corresponding complex flag manifolds and demonstrate that they can be characterized as infinitesimally tight Lagrangian submanifolds with respect to the Kirillov-Kostant-Souriau (KKS) symplectic form. This characterization provides a significant geometric constraint, indicating that these submanifolds are locally minimal and cannot be deformed infinitesimally to further reduce their volume in the ambient space.

Furthermore, we provide a comprehensive classification of these immersions, detailing the conditions under which such submanifolds exist across various symmetric pairs. This classification elucidates the relationship between the structure of real flags and their associated complex flag manifolds, contributing to a deeper understanding of the interplay between symplectic geometry and representation theory.

In a similar vein, Iriyeh and Sakai classified tight Lagrangian submanifolds in $S^2 \times S^2$ (see [15]), showing that if $\mathcal{L}$ is a closed, embedded, tight Lagrangian surface in $S^2 \times S^2$, then $\mathcal{L}$ must be one of the following:

- $\mathcal{L} = \{(x, -x) \in S^2 \times S^2 : \ x \in S^2\}$ (global tight submanifold).
- $\mathcal{L} = S^1(a) \times S^1(b) \subset S^2 \times S^2$, where $S^1(a)$ is a round circle of radius $0 < a \leq 1$ (locally tight submanifold).

This classification forms a special case of tight submanifolds in products of flag manifolds, which were studied in [13]. There, the authors demonstrated that a product of flag manifolds $\mathbb{F}_{\Theta_1} \times \mathbb{F}_{\Theta_2}$ admits a Lagrangian orbit under the diagonal action (or shifted diagonal action) if and only if $\Theta_2 = \Theta_1^*$, where $\Theta_2 = \sigma\Theta_1$ with σ being the symmetry of the Dynkin diagram, given by $\sigma = -w_0$, and w_0 being the longest element of the Weyl group $\mathcal{W}$. Such a Lagrangian orbit is described by the graph of

$$-\operatorname{id} : \operatorname{Ad}(U)(iH) \to \operatorname{Ad}(U)(i\sigma(H)),$$

or by the graph of $-\operatorname{Ad}(m)$, where $m \in U$ for the shifted diagonal action.

A significant contribution of [13] was the introduction of the concept of infinitesimally tight submanifolds. The authors proved that Lagrangian orbits resulting from the diagonal (or shifted diagonal) action are infinitesimally tight. This notion is formally defined as follows:

Definition 1.4. *Let $\mathcal{L}$ be a submanifold of $M = G/H$. An element $X \in \mathfrak{g} = \mathrm{Lie}(G)$ is called* **transversal** *to $\mathcal{L}$ if it satisfies the following two conditions:*

(1) *For any $x \in \mathcal{L}$, if $\widetilde{X}(x) \in T_x\mathcal{L}$, then $\widetilde{X}(x) = 0$.*

(2) *The set*

$$f_{\mathcal{L}}(X) = \left\{x \in \mathcal{L} : \ 0 = \widetilde{X}(x) \in T_x\mathcal{L}\right\}$$

is finite.

That is, $\widetilde{X}$ is tangent to $\mathcal{L}$ only at finitely many points where it vanishes.

A Lagrangian submanifold $\mathcal{L}$ in $M = G/H$ is called **infinitesimally tight** if

$$\#\left(f_{\mathcal{L}}(X)\right) = \mathrm{SB}\left(\mathcal{L}, \mathbb{Z}_2\right)$$

for any $X \in \mathfrak{g}$ such that $\widetilde{X}$ is transversal to $\mathcal{L}$. Moreover, [13] presents the following theorem:

Theorem 1.5. *Let $M = G/H$ be a homogeneous space with a G-invariant symplectic form ω. Then a Lagrangian submanifold $\mathcal{L} \subset M$ is infinitesimally tight if and only if it is locally tight.*

As discussed in [6] and [13], isotropic submanifolds can be characterized through the moment map of a Hamiltonian action. In particular, Gorodski and Podestà [6] classified compact tight Lagrangian submanifolds in irreducible compact homogeneous Kähler manifolds that have the $\mathbb{Z}_2$-homology of a sphere. This classification is closely related to our study, as it provides structural constraints on the existence of tight Lagrangian submanifolds within compact homogeneous spaces. Our work builds upon these ideas by characterizing the complex flag manifolds that admit real flag manifolds as Lagrangian submanifolds.

To establish this characterization, we equip the complex flag manifolds with the Kirillov-Kostant-Souriau (KKS) symplectic form and consider the compact orbits of the real forms of the associated complex Lie group. This approach aligns with recent developments related to the Ph.D. thesis of Báez, where the author studied Lagrangian submanifolds of adjoint semisimple orbits. The results from this thesis are directly related to the findings presented in this paper, further reinforcing the connection between Lagrangian submanifolds and the geometry of adjoint orbits in semisimple Lie theory.

Regarding the work of Gorodski and Podestà [6], although our conclusions share similarities, the methodologies differ significantly. While their approach focuses on homogeneous Kähler manifolds with topological constraints on homology, our classification provides a systematic study of complex flag manifolds and their real forms that possess compact Lagrangian orbits. This classification is explicitly detailed in Table 1 at the end of Subsection 3.1, with a case-by-case proof given in Appendix A.

Specifically, in Section 4, we prove that real flag manifolds can be seen as infinitesimally tight submanifolds of the corresponding complex flag manifolds. This result establishes a direct link between the structure of flag manifolds, symplectic geometry, and representation theory, offering a broader perspective on the classification of Lagrangian orbits within homogeneous symplectic spaces.

2 Flag manifolds

Flag manifolds play a central role in the study of Lie groups and their geometric structures. However, their treatment varies significantly depending on whether they are considered within the framework of complex semisimple Lie groups or real semisimple Lie groups. This distinction is crucial, as notation and conventions often diverge in the literature, with most works focusing exclusively on either the real or the complex setting. To provide a unified perspective, this section introduces both real and complex flag manifolds, along with fundamental concepts such as Weyl chambers and Weyl groups. The goal is to establish a consistent notation and clarify potential ambiguities, ensuring that the reader can navigate seamlessly through subsequent discussions.

There exist several equivalent definitions of flag manifolds, and they are sometimes referred to as *generalized flag manifolds*. This terminology appears in various sources, with one of the most well-known references being Alekseevsky's work (see [1]), where these spaces are studied from a broader geometric perspective. A fundamental definition, which serves as a starting point for our discussion, is the following:

Definition 2.1. *Let $\mathfrak{g}$ be a semisimple non-compact Lie algebra, and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. The flag manifold $\mathbb{F}_H$ is the homogeneous space*

$$\mathbb{F}_H = G/P_H,$$

where P_H is a parabolic subgroup of G, determined by an element $H \in \mathfrak{g}$, which can be chosen within the closure of a positive Weyl chamber of $\mathfrak{g}$.

The construction of the parabolic subgroup P_H depends on whether $\mathfrak{g}$ is a real or complex Lie algebra. In what follows, we shall present these constructions using fundamental tools from semisimple Lie theory. Although different approaches provide valuable insights, in this work, we adopt the perspective that complex flag manifolds are most naturally understood as adjoint orbits of compact semi-simple Lie groups. This viewpoint not only highlights their intrinsic geometric structure but also establishes a direct connection with symplectic geometry and representation theory, which will be further explored in the subsequent discussion.

To avoid confusion, let us denote the following:

- The notation $\mathfrak{g}^{\mathbb{C}}$ will be used to explicitly indicate that $\mathfrak{g}$ is considered as a complex Lie algebra, and similarly, $G^{\mathbb{C}}$ will denote a complex Lie group when necessary. When this notation is omitted, $\mathfrak{g}$ and G should be understood in a general sense or as real structures, depending on the context.
- The notation $\mathfrak{g}_{\mathbb{C}}$ denotes the complexification of the Lie algebra $\mathfrak{g}$, which in this case is a real Lie algebra.

For a more detailed study of these flag manifolds, we recommend referring to [1–3,19]. Additionally, for further geometric insights, see [4, 8].

2.1 Complex flag manifolds

Let $\mathfrak{g}^{\mathbb{C}}$ be a semisimple complex Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We define the following:

- $\Pi_{\mathbb{C}}$ is a root system, where for each $\alpha \in \Pi_{\mathbb{C}}$, there exists an element $H_\alpha \in \mathfrak{h}^{\mathbb{C}}$ such that
$$\alpha(H) = \langle H_\alpha, H\rangle, \quad \forall H \in \mathfrak{h}^{\mathbb{C}},$$
where $\langle\cdot,\cdot\rangle$ denotes the Cartan–Killing form of $\mathfrak{g}^{\mathbb{C}}$.
- $\Sigma_{\mathbb{C}}$ is a simple root system, such that $\Pi_{\mathbb{C}}^{+}$ denotes the set of positive roots in $\Pi_{\mathbb{C}}$, and $\{H_\alpha : \alpha \in \Sigma_{\mathbb{C}}\}$ forms a basis of $\mathfrak{h}^{\mathbb{C}}$.
- $\mathfrak{a}^{+}$ is the corresponding positive Weyl chamber, given by
$$\mathfrak{a}^{+} = \{H \in \mathfrak{h}^{\mathbb{C}} : \alpha(H) > 0, \quad \forall \alpha \in \Sigma_{\mathbb{C}}\}.$$

Thus, we have the root space decomposition:
$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha\in\Pi_{\mathbb{C}}} \mathfrak{g}^{\mathbb{C}}_{\alpha},$$
where each root space is given by
$$\mathfrak{g}^{\mathbb{C}}_{\alpha} = \{X \in \mathfrak{g}^{\mathbb{C}} : [H, X] = \alpha(H)\cdot X, \quad \forall H \in \mathfrak{h}^{\mathbb{C}}\}.$$

The *Borel subalgebra* $\mathfrak{b}$, which is the maximal solvable subalgebra, is defined as

$$\mathfrak{b} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \Pi_{\mathbb{C}}^{+}} \mathfrak{g}_{\alpha}^{\mathbb{C}}.$$

A subalgebra $\mathfrak{p}$ of $\mathfrak{g}^{\mathbb{C}}$ is called *parabolic* if it contains a Borel subalgebra. The parabolic subalgebra associated with an element H is defined as

$$\mathfrak{p}_H = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha(H) \geq 0} \mathfrak{g}_{\alpha}^{\mathbb{C}}. \tag{2.1}$$

Remark 2.2. *In some sources, the parabolic subalgebra defined in Equation* (2.1) *is denoted by* $\mathfrak{p}_{\Theta_H}$*, where* $\Theta_H = \{\alpha \in \Sigma_{\mathbb{C}} : \alpha(H) = 0\}$.

Let $G^{\mathbb{C}}$ be a connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. The *parabolic subgroup* P_H is the normalizer of $\mathfrak{p}_H$ in $G^{\mathbb{C}}$, given by

$$P_H = \{g \in G^{\mathbb{C}} : \mathrm{Ad}(g) \cdot \mathfrak{p}_H = \mathfrak{p}_H\}.$$

The **complex flag manifold** associated with H is then defined as the quotient space:

$$\mathbb{F}_H = G^{\mathbb{C}}/P_H.$$

Furthermore, we will see that the complex flag manifold can be seen as an adjoint orbit of a compact Lie group. For instance, choosing a Weyl basis given by H_α for $\alpha \in \Sigma_{\mathbb{C}}$ and $X_\alpha \in \mathfrak{g}_{\alpha}^{\mathbb{C}}$ for $\alpha \in \Pi_{\mathbb{C}}$, we have:

- $[X_\alpha, X_{-\alpha}] = H_\alpha$,
- $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$ with $m_{\alpha,\beta} \in \mathbb{R}$, where $m_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root and $m_{\alpha,\beta} = -m_{-\alpha,-\beta}$.

Defining $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = i(X_\alpha + X_{-\alpha})$, we obtain the compact real form:

$$\mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{iH_\alpha, A_\alpha, S_\alpha : \alpha \in \Pi_{\mathbb{C}}^{+}\}.$$

Let $U = \exp \mathfrak{u}$ be a compact real form of $G^{\mathbb{C}}$, and define

$$U_H = P_H \cap U.$$

The adjoint action of U is transitive on $\mathbb{F}_H$ with isotropy subgroup U_H at H, yielding

$$\mathbb{F}_H \simeq U/U_H \simeq \mathrm{Ad}(U)\cdot H.$$

Additionally, denoting $b_H = 1\cdot U_H$ as the origin of $\mathbb{F}_H$, its tangent space at b_H is given by

$$T_{b_H}\mathbb{F}_H = \mathrm{span}_{\mathbb{R}}\{A_\alpha, S_\alpha : \alpha(H) > 0\} = \sum_{\alpha(H)>0} \mathfrak{u}_\alpha,$$

where $\mathfrak{u}_\alpha = \left(\mathfrak{g}^{\mathbb{C}}_\alpha \oplus \mathfrak{g}^{\mathbb{C}}_{-\alpha}\right) \cap \mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{A_\alpha, S_\alpha\}$.

Remark 2.3. *Given a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$, a real Lie algebra $\mathfrak{g}_0$ is called a* real form *of $\mathfrak{g}^{\mathbb{C}}$ if its complexification satisfies $\mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{g}^{\mathbb{C}}$. A real form of $\mathfrak{g}^{\mathbb{C}}$ can be either compact or non-compact. Additionally, all compact semisimple Lie algebras are real.*

2.2 Real flag manifolds

Let $\mathfrak{g}$ be a semisimple, non-compact real Lie algebra. To construct real flag manifolds, we introduce the following fundamental elements of real semisimple Lie theory:

- Let θ be a *Cartan involution*, that is, an involutive automorphism such that the associated bilinear form

  $$B_\theta(X,Y) = -\langle X, \theta Y\rangle, \quad X, Y \in \mathfrak{g}$$

  defines an inner product on $\mathfrak{g}$, where $\langle \cdot,\cdot\rangle$ denotes the Cartan–Killing form of $\mathfrak{g}$. The Cartan involution induces a *Cartan decomposition*

  $$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s},$$

  where

  $$\mathfrak{k} = \{X \in \mathfrak{g} : \theta X = X\}, \quad \text{and} \quad \mathfrak{s} = \{Y \in \mathfrak{g} : \theta Y = -Y\}.$$

  The subspaces $\mathfrak{k}$ and $\mathfrak{s}$ are orthogonal with respect to both B_θ and the Cartan–Killing form. Notably, $\mathfrak{k}$ is often referred to as the compact component of the Cartan decomposition, although it is not necessarily compact. Furthermore, we define the maps $\kappa : \mathfrak{g} \to \mathfrak{k}$ and $\sigma : \mathfrak{g} \to \mathfrak{s}$, given by

  $$\kappa(X) = \frac{X + \theta X}{2}, \quad \text{and} \quad \sigma(X) = \frac{X - \theta X}{2},$$

  which correspond to the parallel projections onto $\mathfrak{k}$ and $\mathfrak{s}$, respectively.

- Let $\mathfrak{a} \subset \mathfrak{s}$ be a *maximal Abelian subalgebra*. Then, there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that contains $\mathfrak{a}$. Given a pair $(\theta, \mathfrak{a})$, we denote by $\Pi_{\mathbb{R}}$ the set of roots associated with $(\theta, \mathfrak{a})$, where each root is a linear functional $\alpha : \mathfrak{a} \to \mathbb{R}$ satisfying

$$B_\theta(H_\alpha, H) = \alpha(H), \quad \forall H \in \mathfrak{a}.$$

These roots can be interpreted as restrictions of the roots of $\mathfrak{h}^{\mathbb{C}}$, the Cartan subalgebra of the complexification of $\mathfrak{g}$, denoted as $\mathfrak{g}_{\mathbb{C}}$.

- The *Weyl group* associated with $\mathfrak{a}$ is the finitely generated group of reflections across the hyperplanes defined by $\alpha = 0$ in $\mathfrak{a}$, for α in the root system of $\mathfrak{a}$. The generators of the Weyl group corresponding to these reflections are called *simple reflections*.

- The *Weyl chambers* associated with $(\theta, \mathfrak{a})$ are the connected components of

$$\{H \in \mathfrak{a} : \alpha(H) \neq 0, \quad \forall \alpha \in \Pi_{\mathbb{R}}\}.$$

Selecting one of these chambers as the *positive Weyl chamber* $\mathfrak{a}^+$, we define the set of positive roots as

$$\Pi_{\mathbb{R}}^+ = \{\alpha \in \Pi_{\mathbb{R}} : \alpha|_{\mathfrak{a}^+} > 0\}.$$

Consequently, we define

$$\mathfrak{n} = \sum_{\alpha \in \Pi_{\mathbb{R}}^+} \mathfrak{g}_\alpha, \quad \text{and} \quad \mathfrak{n}^- = \sum_{\alpha \in \Pi_{\mathbb{R}}^+} \mathfrak{g}_{-\alpha},$$

where $\theta \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$ and $\theta \mathfrak{n} = \mathfrak{n}^-$. Furthermore, there exists a simple root system $\Sigma_{\mathbb{R}}$ associated with $\mathfrak{a}^+$, such that $\{H_\alpha \in \mathfrak{a} : \alpha \in \Sigma_{\mathbb{R}}\}$ forms a basis of $\mathfrak{a}$.

Moreover, we obtain the B_θ-orthogonal decomposition

$$\mathfrak{s} = \mathfrak{a} \oplus \sigma(\mathfrak{n}).$$

The triplet $(\theta, \mathfrak{a}, \mathfrak{a}^+)$ is called an *admissible triple* of $\mathfrak{g}$, and it gives rise to the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

known as the *Iwasawa decomposition*. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. If K, A, and N are the connected subgroups generated by $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$, respectively, then G is diffeomorphic to $K \times A \times N$. This leads to the *global Iwasawa decomposition*:

$$G = KAN.$$

For $H \in \mathrm{cl}(\mathfrak{a}^+)$, we define

$$\mathfrak{n}_H^+ = \sum_{\alpha(H)>0} \mathfrak{g}_\alpha, \quad \text{and} \quad \mathfrak{n}_H^- = \sum_{\alpha(H)<0} \mathfrak{g}_\alpha.$$

Remark 2.4. *If $H \in \mathfrak{a}^+$, i.e., H is a regular element, then $\mathfrak{n} = \mathfrak{n}_H^+$ and $\mathfrak{n}^- = \mathfrak{n}_H^-$. In some literature, $\mathfrak{n}_H^+$ is denoted by $\mathfrak{n}_\Theta^+$, where $\Theta = \{\alpha \in \Sigma_{\mathbb{R}} : \alpha(H) = 0\}$.*

Given an admissible triple $(\theta, \mathfrak{a}, \mathfrak{a}^+)$, *the parabolic subalgebra associated with* $H \in \mathrm{cl}(\mathfrak{a}^+)$ is

$$\mathfrak{p}_H = \mathfrak{k}_H \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. The parabolic subgroup associated with H is defined as the normalizer of $\mathfrak{p}_H$ in G. By the global Iwasawa decomposition of G, we obtain:

$$K_H = \{k \in K : \mathrm{Ad}(k)|_{\mathfrak{a}_H} = \mathrm{id}_{\mathfrak{a}_H}\}$$

where $\mathfrak{a}_H = \mathfrak{a} \ominus \mathfrak{a}(H)$ and $\mathfrak{a}(H)$ be a subalgebra generated by $\{H_\alpha : \alpha(H) \neq 0\}$. Then, the parabolic subgroup P_H is given by:

$$P_H = K_H \cdot A \cdot N.$$

Consequently, we have the quotient structure:

$$G/P_H = \frac{K \cdot A \cdot N}{K_H \cdot A \cdot N} \simeq K/K_H,$$

and it follows that:

$$K/K_H \simeq \mathrm{Ad}(K) \cdot H$$

which represents the K-adjoint orbit passing through H, commonly known as the **real flag manifold**.

Remark 2.5. *Given $\widetilde{H} \in \mathfrak{s}$, we have that $\mathrm{Ad}(K) \cdot \widetilde{H} \cap \mathrm{cl}(\mathfrak{a}^+) \neq \varnothing$. Since the action of K is transitive, we can choose an element $H \in \mathrm{cl}(\mathfrak{a}^+)$ which determines the same manifold.*

Remark 2.6. *We denote by $\mathbb{F}_H$ the flag manifold passing through $H \in \mathrm{cl}(\mathfrak{a}^+)$ when there is no ambiguity regarding the compact group acting on it. Otherwise, we will specify it as an adjoint orbit. To maintain clarity, we will represent flag manifolds in terms of the adjoint action (as the orbit of U in the complex case and K in the real case).*

3 Lagrangian immersion of real flags on complex flag

In this section, we investigate the conditions under which a given real flag manifold can be realized as a Lagrangian submanifold within a complex flag manifold. Specifically, given an adjoint orbit $\mathrm{Ad}(K)\cdot H$ corresponding to a real flag manifold, we determine in which complex flag manifolds it can be immersed as a Lagrangian submanifold. Importantly, this classification depends on the choice of H, which we analyze using Satake diagrams, as well as the structural properties of K. Contrary to a universal embedding, our approach highlights the interplay between the choice of H and the ambient complex flag manifold.

As discussed in [3], given a compact semisimple Lie group U with Lie algebra $\mathfrak{u}$, the adjoint orbits of U in $\mathfrak{u}$ correspond to the flag manifolds of its complexified Lie group $U_{\mathbb{C}}$, whose Lie algebra is $\mathfrak{u}_{\mathbb{C}}$. These adjoint orbits naturally inherit a symplectic structure, providing a geometric foundation for our analysis.

The *Kostant–Kirillov–Souriau (KKS) symplectic form* on an adjoint orbit $\mathrm{Ad}(U)\cdot H$ is given by

$$\omega_x\left(\widetilde{X}(x),\widetilde{Y}(x)\right)=\langle x,[X,Y]\rangle_{\mathfrak{u}},\quad X,Y\in\mathfrak{u},\tag{3.1}$$

where $\langle\cdot,\cdot\rangle_{\mathfrak{u}}$ denotes the Cartan–Killing form on $\mathfrak{u}$, and $\widetilde{X}=\mathrm{ad}(X)$ represents the Hamiltonian vector field associated with the Hamiltonian function $H_X(x)=\langle x,X\rangle_{\mathfrak{u}}$. As a consequence, the moment map μ of the U-adjoint action is simply the identity map, which is inherently equivariant.

To identify specific *isotropic submanifolds* within $\mathrm{Ad}(U)\cdot H$, we rely on the following key result:

Proposition 3.1. *Let (M,ω) be a connected symplectic manifold equipped with a Hamiltonian action of a Lie group G, given by $G\times M\to M$, along with an equivariant moment map μ. Let $L\subset G$ be a Lie subgroup.*

Then, the orbit $L\cdot x$ is isotropic if and only if $\mu(x)$ belongs to the annihilator $(\mathfrak{l}')^0$ of the derived algebra $\mathfrak{l}'$ of $\mathfrak{l}$.

This proposition was established in [13] and [14] using distinct methodologies.

3.1 Lagrangian immersion of real flags

Let U be a compact semisimple Lie group with Lie algebra $\mathfrak{u}$, and let $\mathfrak{k}\subset\mathfrak{u}$ be a Lie subalgebra. The pair $(\mathfrak{u},\mathfrak{k})$ is called a **symmetric pair** if

$$[\mathfrak{k},\mathfrak{k}^{\perp}]\subset\mathfrak{k}^{\perp},\quad\text{and}\quad[\mathfrak{k}^{\perp},\mathfrak{k}^{\perp}]\subset\mathfrak{k},$$

where $\perp$ denotes the orthogonal complement with respect to the Cartan–Killing form on $\mathfrak{u}$.

For any symmetric pair $(\mathfrak{u},\mathfrak{k})$, if we define $K = \langle \exp \mathfrak{k} \rangle$, then the quotient space U/K forms a symmetric space. The *dual symmetric pair* is given by $(\mathfrak{g},\mathfrak{k})$, where $\mathfrak{g}$ is a non-compact semisimple Lie algebra that serves as the real form of $\mathfrak{u}_{\mathbb{C}}$ and admits a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}, \quad \text{where} \quad \mathfrak{s} = i\mathfrak{k}^{\perp} \subset \mathfrak{u}_{\mathbb{C}}.$$

By construction, the orbits of the K-isotropy representation on $\mathfrak{s}$ (or equivalently on $\mathfrak{k}^{\perp}$) correspond to the flag manifolds of $\mathfrak{g}$.

Given $H \in \mathfrak{k}^{\perp}$, the Lagrangian immersion of real flag manifolds into their corresponding complex flag manifolds is constructed as follows: Let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal abelian subalgebra. Then, there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{h}$ and $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Consequently, for $H \in \mathfrak{a}$, we obtain

$$K/K_H = \mathrm{Ad}(K)\cdot H \hookrightarrow \mathrm{Ad}(U)\cdot iH = U/U_H = \mathbb{F}_H. \tag{3.2}$$

Thus, the flag manifolds of $\mathfrak{g}$ are determined by the adjoint action of K on H and are immersed in the flag manifolds of $\mathfrak{g}_{\mathbb{C}}$ (complexification), which are determined by the adjoint action of U on iH. Moreover, since $\mathfrak{u}$ is compact, the connected component of the identity of $\mathfrak{k}'^{\perp}$ corresponds to the orthogonal complement of $\mathfrak{k}'$ with respect to the invariant scalar product on $\mathfrak{u}$. Consequently, we arrive at the following proposition:

Proposition 3.2. *Given a symmetric pair $(\mathfrak{u},\mathfrak{k})$ and an element $H \in \mathfrak{a} \subset i\mathfrak{k}^{\perp}$, the real flag manifold $\mathrm{Ad}(K)\cdot H$ is a Lagrangian submanifold of $\mathbb{F}_H$ with respect to the Kirillov-Kostant-Souriau (KKS) symplectic form.*

Proof. Since $\mathfrak{k}' \subset \mathfrak{k}$, then $\mathfrak{k}^{\perp} \subset (\mathfrak{k}')^{\perp}$ and $\mathrm{Ad}(K)\cdot H \subset \mathfrak{k}^{\perp} = i\mathfrak{s}$, then $\mathrm{Ad}(K)\cdot H \cap (\mathfrak{k}')^{\perp} \neq \varnothing$ and by Proposition 3.1, the adjoint K-orbit (real flag) is an isotropic submanifold.

Furthermore, if $b_H = 1\cdot K$, we have that

$$\dim\left(T_{b_H}\,\mathrm{Ad}(K)\cdot H\right) = \dim\left(\sum_{\alpha(H)<0} \mathfrak{g}_{\alpha}\right) = \#\left\{\alpha \in \Pi_{\mathbb{C}} : \ \alpha(H) < 0\right\},$$

and as the root spaces of $\mathfrak{g}_{\mathbb{C}}$ are 1-dimensional complex spaces (*i.e.*, 2-dimensional real spaces), then

$$2\dim_{\mathbb{R}}\left(\mathrm{Ad}(K)\cdot H\right) = \dim_{\mathbb{R}}(\mathbb{F}_H).$$

Hence $\mathrm{Ad}(K)\cdot H$ is a Lagrangian submanifold of $\mathbb{F}_H$. $\square$

Remark 3.3. *Intuitively, one can observe that each complex root space effectively doubles its dimension when considered as a real vector space. However, this identification is purely at the level of vector spaces and does not yet take into account the underlying Lie algebraic or geometric structure. In [16, 21, 23], the authors provide a detailed exposition of this vector space approach, emphasizing how the real and complex structures relate in the context of flag manifolds.*

Our focus now shifts to identifying the complex flag manifolds of $\mathfrak{g}_{\mathbb{C}}$ (complexification of $\mathfrak{g}$ real non-compact semi-simple) that admit a real flag manifold, generated by the action of $K = \langle \exp \mathfrak{k} \rangle$ for the symmetric pair $(\mathfrak{u}, \mathfrak{k})$, as a Lagrangian submanifold. Consider a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{s}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{h}$. Let $\Pi_{\mathbb{C}}$ be the set of roots of $\mathfrak{h}_{\mathbb{C}}$, where the roots of $\mathfrak{a}$ correspond to their restrictions on $\mathfrak{h}_{\mathbb{C}}$.

If θ is a Cartan involution associated with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, then there exists an involutive extension of θ to $\mathfrak{g}_{\mathbb{C}}$, which we also denote by θ. As shown in [21], the restriction of $\Pi_{\mathbb{C}}$ to $\mathfrak{a}$ is given by

$$P = \frac{1}{2}\left(1 - \theta^*\right), \quad \text{where} \quad \theta^* \alpha = \alpha \circ \theta.$$

Define $\Pi_{\mathrm{im}} \subset \Pi_{\mathbb{C}}$ as the set of *imaginary roots*, where $\alpha \in \Pi_{\mathrm{im}}$ if and only if $P(\alpha) = 0$. Letting $\Pi_{\mathrm{co}} = \Pi_{\mathbb{C}} \setminus \Pi_{\mathrm{im}}$, the set of restricted roots is given by $P(\Pi_{\mathrm{co}})$.

Considering an appropriate ordering (such as the lexicographic order on $\mathfrak{a}^*$), let Σ_{im} denote the system of imaginary simple roots, and let Σ_{co} be its complement. The projection of Σ_{co} onto $\mathfrak{a}^*$ forms a *system of restricted roots* Σ, with $\mathfrak{a}^+$ denoting the positive Weyl chamber of $\mathfrak{g}$ determined by Σ.

For $H \in \mathrm{cl}\left(\mathfrak{a}^+\right)$, define

$$\Theta_H = \{\beta \in \Sigma : \beta(H) = 0\} \subset \Sigma.$$

Next, define $\widetilde{\Theta}_H \subset \Sigma_{\mathbb{C}}$ by

$$\widetilde{\Theta}_H = P^{-1}\left(\Theta_H\right) \cup \Sigma_{\mathrm{im}}, \tag{3.3}$$

which is determined by the **Satake diagram** of $\mathfrak{g}$ (see [16, 21]).

Remark 3.4. *In general, we select $H \in \mathrm{cl}\left(\mathfrak{a}^+\right)$ because for any other choice $H' \in \mathfrak{a}$ such that the orbits remain the same, there exists a Weyl conjugation σ satisfying $\sigma \cdot H = H'$. Consequently, the associated sets of admissible roots remain unchanged, i.e., $\widetilde{\Theta}_H = \widetilde{\Theta}_{H'}$.*

Proposition 3.5. $\widetilde{\Theta}_H = \{\alpha \in \Sigma_{\mathbb{C}} : \alpha(H) = 0\}$.

Proof. If $H \in \mathfrak{a}$, then for all $\alpha \in \Sigma_{\mathbb{C}}$

$$\theta^* \alpha(H) = \alpha \circ \theta(H) = -\alpha(H), \tag{3.4}$$

because $\theta|_{\mathfrak{s}} = -\,\mathrm{id}$. Also, if $\alpha \in \Sigma_{\mathrm{im}}$, then $\theta^*\alpha = \alpha$, and by (3.4) we have that $\alpha(H) = 0$, therefore it is enough to see for roots in Σ_{co}. If $\alpha \in P^{-1}(\Theta_H)$, then $(\alpha - \theta^*\alpha)(H) = 0$ implies that $\alpha(H) = \theta^*\alpha(H)$, and by (3.4) we have that $\alpha(H) = 0$. Thus $\widetilde{\Theta}_H \subseteq \{\alpha \in \Sigma_{\mathbb{C}} :\ \alpha(H) = 0\}$. Conversely, if $\alpha \in \Sigma_{\mathrm{co}}$ such that $\alpha(H) = 0$, then $\theta^*\alpha(H) = -\alpha(H) = 0$, thus $P(\alpha)(H) = 0$ and implies that $P(\alpha) \in \Theta_H$, *i.e.* $\alpha \in P^{-1}(\Theta_H)$. □

Therefore,

Theorem 3.6. *Given a symmetric pair $(\mathfrak{u}, \mathfrak{k})$, the complex flags of $\mathfrak{u}_{\mathbb{C}}$ of type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$ admit, as Lagrangian submanifold, the real flag of $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}^{\perp}$ of type $\Theta \subset \Sigma$ if and only if*

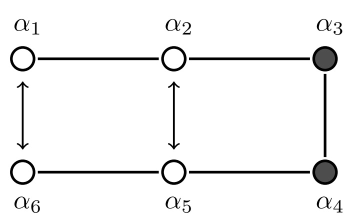
$$\widetilde{\Theta} = P^{-1}(\Theta) \cup \Sigma_{\mathrm{im}}.$$

That is, $\widetilde{\Theta}$ is determined by the Satake diagram of $\mathfrak{g}$.

In particular, we can conclude

Corollary 3.7. *A maximal flag $\mathbb{F}$ of $\mathfrak{g}_{\mathbb{C}}$ admits a real flag $\mathrm{Ad}(K) \cdot H$ as Lagrangian submanifold if and only if $\Sigma_{\mathrm{im}} = \varnothing$ and $\varnothing = \Theta_H$.*

Example 3.8. *Let $\mathfrak{u} = \mathfrak{su}(7)$, $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(5))$ and $\mathfrak{g} = \mathfrak{su}(2,5)$ that determine the symmetric pair $(\mathfrak{u}, \mathfrak{k})$ and its respective dual symmetric pair $(\mathfrak{g}, \mathfrak{k})$. The Satake diagram of $\mathfrak{su}(2,5)$ is*



By Theorem 3.6, the flags of type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$ that admit as Lagrangian submanifold a real flag of type $\Theta \subset \Sigma = \{\beta_1 = P(\alpha_1) = P(\alpha_6), \beta_2 = P(\alpha_2) = P(\alpha_5)\}$ are

- *If $\Theta_0 = \varnothing$, then $\widetilde{\Theta}_0 = \Sigma_{\mathrm{im}} = \{\alpha_3, \alpha_4\}$.*
- *If $\Theta_1 = \{\beta_1\}$, then $\widetilde{\Theta}_1 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}$.*
- *If $\Theta_2 = \{\beta_2\}$, then $\widetilde{\Theta}_2 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.*

Analogously, this is equivalent to that given in the Table 1, for $n = 7$:

- $\widetilde{\Theta}_0 = \Sigma_{\mathbb{C}} \setminus \{\alpha_1, \alpha_2, \alpha_{n-2}, \alpha_{n-1}\}$,
- $\widetilde{\Theta}_1 = \Sigma_{\mathbb{C}} \setminus \{\alpha_2, \alpha_{n-2}\}$,
- $\widetilde{\Theta}_2 = \Sigma_{\mathbb{C}} \setminus \{\alpha_1, \alpha_{n-1}\}$.

Hence, using the Satake diagrams we can determine which are the complex flags of type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$, for which there exists Θ such that Theorem 3.6 is satisfied.

Corollary 3.9. *The complex flags of type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$ admits as Lagrangian submanifold a real flag given by the K-adjoint orbit if and only if $\widetilde{\Theta}$ appears in Table 1.*

Remark 3.10. *Corollary 3.9 states that given a complex flag manifold $\mathbb{F}_H$ associated with the semisimple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, we can determine which real flag manifolds of $\mathfrak{g}_0$ are Lagrangian submanifolds of $\mathbb{F}_H$ by analyzing the Satake diagram of $\mathfrak{g}_0$. Here, $\mathfrak{g}_0$ denotes a real form of $\mathfrak{g}^{\mathbb{C}}$.*

The proof of this result is given in the following subsection. For that we will use a convenient notation of partitioning an integer, that is, we define $\flat(n)$ for $n \in \mathbb{N}$, as the set of ordered l-tuples of integers $(n_1, \ldots, n_l)$ such that $0 < n_1 < \cdots < n_l \leq n$, for example:

$$\flat(3) = \{(1), (2), (3), (1,2), (1,3), (2,3), (1,2,3)\}.$$

Using this notation, we build the Table 1. The case-by-case analysis used to construct Table 1 is detailed in Appendix A.

4 Infinitesimally tight

In this section, we establish the main result of this paper. Specifically, we demonstrate that the Lagrangian submanifolds listed in Table 1 are infinitesimally tight. To achieve this, we compute the sum of the $\mathbb{Z}_2$-Betti numbers of the real flag manifolds and identify the transversal elements. To lay the groundwork for our proof, we first provide the necessary definitions to understand Schubert cells, which play a fundamental role in computing the homology of real flag manifolds. This exposition is based on [5] and [9].

Let $\mathfrak{g}$ be a semisimple non-compact real Lie algebra, and let $\mathcal{W}$ be the Weyl group associated with a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}$, with Σ denoting the corresponding system of simple roots.

Table 1: Complex flags that admit a Lagrangian immersion of the real flag determined by the action of $K = \exp \mathfrak{k}$.

	$\mathfrak{g}_{\mathbb{C}}$	$\mathfrak{g}$	$\mathfrak{k}$	Flags type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$
A	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{so}(n)$	All possibilities
	$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(n)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2n_1},\dots,\alpha_{2n_j} : (n_1,\dots,n_j) \in \flat(n-1)\right\}$
	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(k,n-k)$, $k < n$	$\mathfrak{s}\left(\mathfrak{u}(k) \oplus \mathfrak{u}(n-k)\right)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j},\alpha_{n-n_j},\dots,\alpha_{n-n_1} : (n_1,\dots,n_j) \in \flat(k)\right\}$
	$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{su}(n,n)$	$\mathfrak{s}\left(\mathfrak{u}(n) \oplus \mathfrak{u}(n)\right)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j},\alpha_n,\alpha_{2n-n_j},\dots,\alpha_{2n-n_1} : (n_1,\dots,n_j) \in \flat(n-1)\right\}$, or $\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j},\alpha_{2n-n_j},\dots,\alpha_{2n-n_1} : (n_1,\dots,n_j) \in \flat(n-1)\right\}$
B	$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(n,n+1)$	$\mathfrak{so}(n) \oplus \mathfrak{so}(n+1)$	All possibilities
		$\mathfrak{so}(k,2n+1-k)$, $k \leq n$	$\mathfrak{so}(k) \oplus \mathfrak{so}(2n+1-k)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j} : (n_1,\dots,n_j) \in \flat(k)\right\}$
C	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{u}(n)$	All possibilities
		$\mathfrak{sp}(k,n-k)$, $k \leq n-k$	$\mathfrak{sp}(k) \times \mathfrak{sp}(n-k)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2n_1},\dots,\alpha_{2n_j} : (n_1,\dots,n_j) \in \flat(k)\right\}$
D	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(n,n)$	$\mathfrak{so}(n) \oplus \mathfrak{so}(n)$	All possibilities
		$\mathfrak{so}(k,2n-k)$, $k < n-1$	$\mathfrak{so}(k) \oplus \mathfrak{so}(2n-k)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j} : (n_1,\dots,n_j) \in \flat(k)\right\}$
		$\mathfrak{so}(n-1,n+1)$,	$\mathfrak{so}(n-1) \oplus \mathfrak{so}(n+1)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j},\alpha_{n-1},\alpha_n : (n_1,\dots,n_j) \in \flat(n-2)\right\}$, or $\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j} : (n_1,\dots,n_j) \in \flat(n-2)\right\}$
		$\mathfrak{so}^*(2n)$, n even	$\mathfrak{u}(n)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2n_1},\dots,\alpha_{2n_j} : (n_1,\dots,n_j) \in \flat(k)\right\}$
		$\mathfrak{so}^*(2n)$, n odd	$\mathfrak{u}(n)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2n_1},\dots,\alpha_{2n_j},\alpha_{n-1},\alpha_n : (n_1,\dots,n_j) \in \flat\left(\frac{n-3}{2}\right)\right\}$, or $\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2n_1},\dots,\alpha_{2n_j} : (n_1,\dots,n_j) \in \flat\left(\frac{n-3}{2}\right)\right\}$
E	E_6	E_6^6	$\mathfrak{sp}(4)$	All possibilities
		E_6^2	$\mathfrak{su}(2) \oplus \mathfrak{su}(6)$	See appendix A
		E_6^{-14}	$\mathfrak{so}(10) \oplus \mathbb{R}$	See appendix A
		E_6^{-26}	F_4	See appendix A
	E_7	E_7^7	$\mathfrak{su}(8)$	All possibilities
		E_7^{-5}	$\mathfrak{su}(2) \oplus \mathfrak{so}(12)$	See appendix A
		E_7^{-25}	$E_6 \oplus \mathbb{R}$	See appendix A
	E_8	E_8^8	$\mathfrak{so}(16)$	All possibilities
		E_8^{-24}	$\mathfrak{su}(2) \oplus E_7$	See appendix A
F	F_4	F_4^4	$\mathfrak{su}(2) \oplus \mathfrak{sp}(3)$	All possibilities
		F_4^{-20}	$\mathfrak{so}(9)$	See appendix A
G	G_2	G_2^2	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	All possibilities

- For $\Theta \subseteq \Sigma$, the subgroup $\mathcal{W}_\Theta$ of $\mathcal{W}$ is generated by the roots in Θ. This subgroup acts transitively on the cosets of $\mathcal{W}$.

- Given $\Theta \subseteq \Sigma$, the *Bruhat decomposition* of the real flag manifold $\mathbb{F}_\Theta = G/P_\Theta$ expresses it as the disjoint union of N-orbits, where N is determined by the Iwasawa decomposition of $P_\Theta = K_\Theta AN$. That is,

$$\mathbb{F}_\Theta = \coprod_{w \in \mathcal{W}/\mathcal{W}_\Theta} N \cdot wb_\Theta,$$

  where the equivalence relation $N \cdot w_1 b_\Theta = N \cdot w_2 b_\Theta$ if $w_1 \cdot \mathcal{W}_\Theta = w_2 \cdot \mathcal{W}_\Theta$ holds.

- Each N-orbit passing through $w \in \mathcal{W}$ is diffeomorphic to a Euclidean space, and the orbit $N \cdot wb_\Theta$ is referred to as a *Bruhat cell.*

- Every Bruhat cell is open and dense in $\mathbb{F}_\Theta$.

- The *Schubert cell* associated with $w \in \mathcal{W}/\mathcal{W}_\Theta$ is denoted by S_w^Θ and defined as

$$S_w^\Theta = \mathrm{cl}(N \cdot wb_\Theta), \quad w \in \mathcal{W}/\mathcal{W}_\Theta.$$

Using the Schubert cells S_w^Θ, the authors of [20] introduced a boundary map ∂, which was employed to compute the homology of the real flag manifold $\mathbb{F}_\Theta$. In particular, for any $H \in \mathrm{cl}(\mathfrak{a}^+)$, there exists a subset $\Theta_H \subset \Sigma$ such that the $\mathbb{Z}_2$-homology of $\mathbb{F}_{\Theta_H} = \mathrm{Ad}(K) \cdot H$ is freely generated by the *Schubert cells* $\mathcal{S}_w^{\Theta_H}$, where $w \in \mathcal{W}/\mathcal{W}_{\Theta_H}$.

Therefore,

$$\mathrm{SB}\left(\mathrm{Ad}(K) \cdot H, \mathbb{Z}_2\right) = \#\left(\mathcal{W}/\mathcal{W}_{\Theta_H}\right), \tag{4.1}$$

That is, the cardinality of the quotient $\mathcal{W}/\mathcal{W}_\Theta$.

Since $\mathrm{Ad}(K) \cdot H \subseteq \mathfrak{s} = i\mathfrak{k}^\perp$, for $x \in \mathrm{Ad}(K) \cdot H$ we have:

$$T_x\left(\mathrm{Ad}(K) \cdot H\right) = \{\mathrm{ad}(A)(x) : A \in \mathfrak{k}\}.$$

Then,

- If $X \in \mathfrak{k}^\perp$, then $\widetilde{X} = \mathrm{ad}(X)$ is a Hamiltonian field of the function $H_X = \langle X, x\rangle$. Thus the singularities of X are the singularities of H_X, and their number is finite, if and only if X is regular.

  Therefore, the transversal elements are the regular elements X, and they satisfy

$$\#\left(f_{\mathrm{Ad}(K)\cdot H}(X)\right) = \#\left(\mathcal{W}/\mathcal{W}_{\Theta_H}\right).$$

- If $Y \in \mathfrak{k}$, then $\widetilde{Y}$ is tangent, thus it cannot be transversal.

- If $Z = X + Y$ for $X \in \mathfrak{k}^{\perp}$ and $Y \in \mathfrak{k}$, then $\widetilde{Z}(x) \notin T_x \operatorname{Ad}(K) \cdot H$ if $\widetilde{X}(x) \neq 0$, so for Z to have singularity in x we need that $\widetilde{X}(x) = \widetilde{Y}(x) = 0$ in a finite quantity. But this only happens for X regular, such that $[X, Y] = 0$. Thus:

$$\#\left(f_{\operatorname{Ad}(K)\cdot H}(Z)\right) = \#\left(\mathcal{W}/\mathcal{W}_{\Theta_H}\right).$$

Consequently,

Theorem 4.1. *The real flags are infinitesimally tight submanifolds of their corresponding complex flag manifolds, as listed in Table 1.*

As a result of Theorem 1.5, we have:

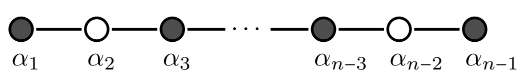
Corollary 4.2. *The real flags are locally tight submanifolds of their corresponding complex flag manifolds.*

A Appendix

In this appendix, we analyze each Satake diagram case by case to identify all complex flag manifolds that permit the Lagrangian immersion of the corresponding real flag, as determined by the possible symmetric pairs. This analysis culminates in the construction of Table 1, where for the classical cases AI, CI, G_2, F_4I, E_6I, E_7I, and E_8I, all possible sets $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$ are admissible.

Type AII

In this case, we have $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{H})$, with $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2n, \mathbb{C})$. The Satake diagram is represented as:



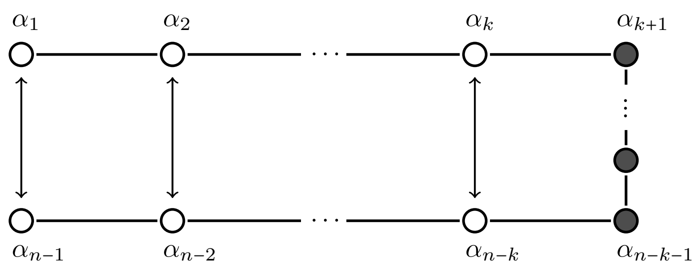
Here, $\Sigma_{\text{im}} = \{\alpha_{2j-1} : 1 \leq j \leq n\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) : 1 \leq j \leq n-1\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : (s_1, \ldots, s_l) \in \flat(n-1)\}. \tag{A.1}$$

Type *AIII*

For $\mathfrak{g} = \mathfrak{su}(k, n-k)$

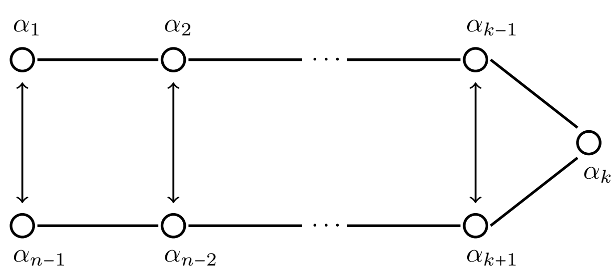
- If $k < n-k$, the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_j : k < j < n-k\}$ and $\Sigma = \{\beta_j = P(\alpha_j) = P(\alpha_{n-j}) : 1 \leq j \leq k\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_{n-s_l}, \ldots, \alpha_{n-s_1} : (s_1, \ldots, s_l) \in \flat(k)\}. \tag{A.2}$$

- If $k = n-k$, the Satake diagram is



As $\Sigma_{\text{im}} = \varnothing$ and $\Sigma = \{\beta_j = P(\alpha_j) = P(\alpha_{n-j}),\ \beta_k = P(\alpha_k) : 1 \leq j \leq k-1\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

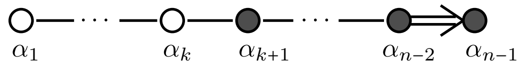
$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_{n-s_l}, \ldots, \alpha_{n-s_1} :\ (s_1, \ldots, s_l) \in \flat(k-1)\}, \tag{A.3}$$

or

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_k, \alpha_{n-s_l}, \ldots, \alpha_{n-s_1} : (s_1, \ldots, s_l) \in \flat(k-1)\}. \tag{A.4}$$

Type B

For $\mathfrak{g} = \mathfrak{so}(k, 2n+1-k)$, then the Satake diagram is



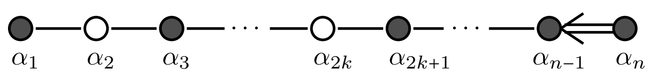
As $\Sigma_{\mathrm{im}} = \{\alpha_j : k < j \le n\}$ and $\Sigma = \{\beta_j = P(\alpha_j) : 1 \le j \le k\}$. If $k = n$ then $\mathfrak{g}$ is normal, and the sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l} : (s_1, \ldots, s_l) \in \flat(k)\}. \tag{A.5}$$

Type CII

For $\mathfrak{g} = \mathfrak{sp}(k, n-k)$.

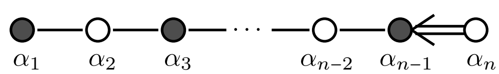
- If $k < n-k$, the Satake diagram is



  As $\Sigma_{\mathrm{im}} = \{\alpha_{2j-1}, \alpha_q : 1 \le j \le k,\ q > 2k\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) : 1 \le j \le k\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : (s_1, \ldots, s_l) \in \flat(k)\}. \tag{A.6}$$

- If $n = 2m$ and $k = m$, the Satake diagram is



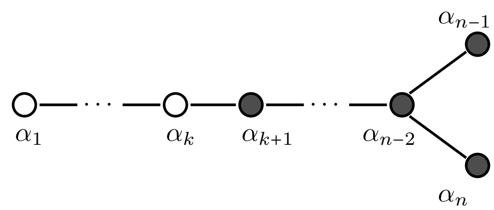
  As $\Sigma_{\mathrm{im}} = \{\alpha_{2j-1} :\ 1 \le j \le m\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) :\ 1 \le j \le m\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : (s_1, \ldots, s_l) \in \flat(k)\}. \tag{A.7}$$

Type DI

For $\mathfrak{g} = \mathfrak{so}(k, 2n-k)$.

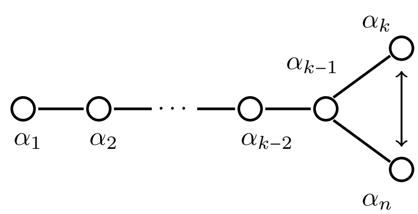
- If $k = n$ then $\mathfrak{g}$ is a normal form.
- If $k < n-1$ then the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_j : j > k\}$ and $\Sigma = \{\beta_j = P(\alpha_j) : 1 \leq j \leq k\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \dots, \alpha_{s_l} : (s_1, \dots, s_l) \in \flat(k)\}. \tag{A.8}$$

- If $k = n-1$ then the Satake diagram is



As $\Sigma_{\text{im}} = \varnothing$ and $\Sigma = \{\beta_j = P(\alpha_j),\ \beta_k = P(\alpha_k) = P(\alpha_n) :\ 1 \leq j < k\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \dots, \alpha_{s_l} : (s_1, \dots, s_l) \in \flat(k-1)\}, \tag{A.9}$$

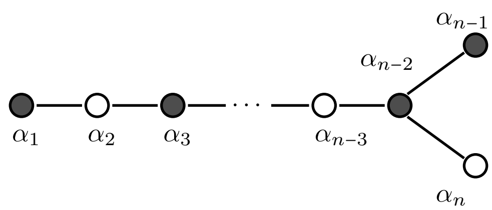
or

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \dots, \alpha_{s_l}, \alpha_k, \alpha_n : (s_1, \dots, s_l) \in \flat(k-1)\}. \tag{A.10}$$

Type DII

For $\mathfrak{g} = \mathfrak{so}^*(2n)$.

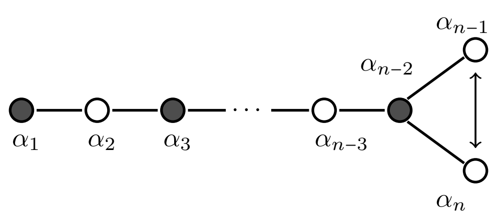
- If n is even, the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_j : j \text{ is odd}\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) : 1 \leq j \leq n\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{2s_1}, \dots, \alpha_{2s_l} : (s_1, \dots, s_l) \in \flat(k)\}. \tag{A.11}$$

- If n is odd, the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_j : j \text{ is odd and } j < n\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}), \beta_k = P(\alpha_{n-1}) = P(\alpha_n) : 1 \leq j \leq k,\ k = (n-1)/2\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2s_1}, \dots, \alpha_{2s_l} : (s_1, \dots, s_l) \in \flat\left(\frac{n-3}{2}\right)\right\}, \tag{A.12}$$

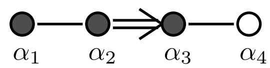
or

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2s_1}, \dots, \alpha_{2s_l}, \alpha_{n-1}, \alpha_n : (s_1, \dots, s_l) \in \flat\left(\frac{n-3}{2}\right)\right\}. \tag{A.13}$$

Exceptional cases

Type $F4II$

For $\mathfrak{g} = F_4^{-20}$, then the Satake diagram is

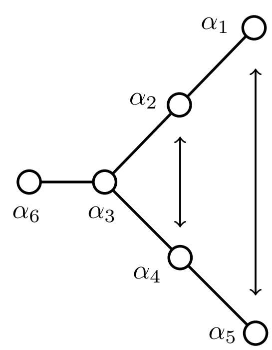


Therefore the only non-trivial possibility of $\widetilde{\Theta}$ that satisfy the Theorem 3.6 is

$$\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3\} = \Sigma_{\mathrm{im}}. \tag{A.14}$$

Type $E6II$

For $\mathfrak{g} = E_6^2$, then the Satake diagram is

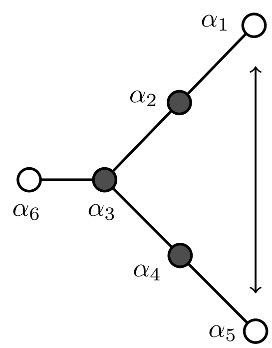


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy Theorem 3.6 are:

- $\widetilde{\Theta} = \varnothing$,
- $\widetilde{\Theta} = \{\alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_3\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_4\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_5\}$,
- $\widetilde{\Theta} = \{\alpha_3, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_5, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_5\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

Type $E6III$

For $\mathfrak{g} = E_6^{-14}$, then the Satake diagram is

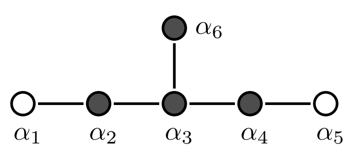


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are:

- $\widetilde{\Theta} = \{\alpha_2, \alpha_3\alpha_4\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\ \}$.

Type $E6IV$

For $\mathfrak{g} = E_6^{-26}$, then the Satake diagram is

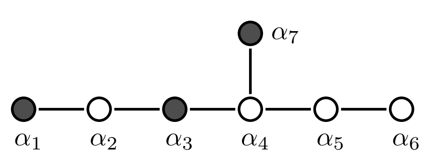


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are:

- $\widetilde{\Theta} = \{\alpha_2, \alpha_3\alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\ \}$.

Type $E7II$

For $\mathfrak{g} = E_7^{-5}$, then the Satake diagram is

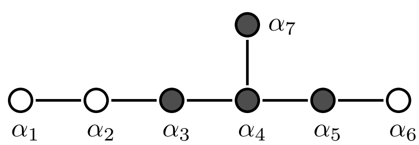


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are:

- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$.
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7\}$.

Type *E7III*

For $\mathfrak{g} = E_7^{-25}$, then the Satake diagram is

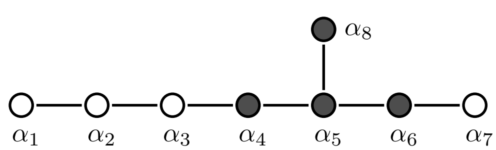


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are:

- $\widetilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$.

Type *E8II*

For $\mathfrak{g} = E_8^{-24}$, then the Satake diagram is



Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are:

- $\widetilde{\Theta} = \{\alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$.

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