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Zeros of cubic polynomials in zeon algebra

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ABSTRACT

It is well known that every cubic polynomial with complex coefficients has three not necessarily distinct complex zeros. In this paper, zeros of cubic polynomials over complex zeons are considered. In particular, a monic cubic polynomial with zeon coefficients may have three spectrally simple zeros, uncountably many zeros, or no zeros at all. A classification of zeros is developed based on an extension of the cubic discriminant to zeon polynomials. In indeterminate cases, sufficient conditions are provided for existence of spectrally nonsimple zeon zeros. We also show that when considering zeros of cubic polynomials over the finite-dimensional complex zeon algebra $\mathbb{C}\mathfrak{Z}_2$, there are no indeterminate cases.

RESUMEN

Es bien sabido que todo polinomio cúbico con coeficientes complejos tiene tres ceros complejos no necesariamente distintos. En este artículo consideramos los ceros de polinomios cúbicos sobre los complejos zeones. En particular, un polinomio cúbico mónico con coeficientes zeones puede tener tres ceros espectralmente simples, una cantidad no numerable de ceros, o no tener ceros. Desarrollamos una clasificación de ceros en base a una extensión del discriminante cúbico a polinomios zeones. En casos indeterminados, entregamos condiciones suficientes para la existencia de ceros zeones espectralmente no simples. También mostramos que cuando consideramos ceros de polinomios cúbicos sobre el álgebra de complejos zeones finito-dimensional $\mathbb{C}\mathfrak{Z}_2$, no hay casos indeterminados.

Keywords and Phrases: Zeons, polynomials, cubic formula, symbolic computation

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1 Introduction

The *n-particle (real) zeon algebra* is a commutative $\mathbb{R}$-algebra generated by a fixed collection $\{\zeta_{\{i\}} : 1 \leq i \leq n\}$ and the scalar identity $1 = \zeta_\varnothing$, whose generators satisfy the zeon commutation relations

$$\zeta_{\{i\}}\zeta_{\{j\}} + \zeta_{\{j\}}\zeta_{\{i\}} = \begin{cases} 2\zeta_{\{i\}}\zeta_{\{j\}} & i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

We denote this algebra by $\mathbb{Z}_n$. Like fermions the algebra has null-square generators; like bosons, the generators commute. Hence the name "zeon algebra", first suggested by Feinsilver [2].

Combinatorial properties of zeons have proven useful in problems ranging from enumerating paths and cycles in finite graphs to routing problems in communication networks. Where classical approaches to routing problems require construction of trees and the use of heuristics to prevent combinatorial explosion, the zeon algebraic approach avoids tree constructions and heuristics. Much of the essential background on algebraic and combinatorial properties and applications of zeons is summarized in the books [9] and [13]. Other works involving zeons include combinatorial identities developed by Neto [5–8] and first and second order differential equations considered by Mansour and Schork [4].

Polynomials over the n-particle complex zeon algebra, denoted by $\mathbb{C}\mathfrak{Z}_n$, were first considered in [11]. We extend the finite-dimensional zeon algebras to the infinite-dimensional complex zeon algebra $\mathbb{C}\mathfrak{Z}$ and focus on zeros of cubic polynomials over $\mathbb{C}\mathfrak{Z}$. Our study is restricted to monic polynomials of the form $\varphi(u) = u^3 + \alpha u^2 + \beta u + \gamma \in \mathbb{C}\mathfrak{Z}[u]$, which generalize naturally to non-monic cubic polynomials with invertible leading coefficients. Observing that

$$\varphi\left(u - \frac{\alpha}{3}\right) = u^3 + 3qu - 2r,$$

where $q = \frac{1}{3}\beta - \frac{1}{9}\alpha^2$ and $r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3$, our work is further simplified by focusing on solutions of the *depressed cubic equation* $u^3 + 3qu - 2r = 0$.

Traditionally, the cubic discriminant $\Delta_f = 18abc - 4a^3b + a^2b^2 - 4b^3 - 27c^2$ is used to classify the zeros of the real monic cubic function $f(x) = x^3 + ax^2 + bx + c \in \mathbb{R}[x]$. In particular, $\Delta_f = 0$ implies that the polynomial has a repeated zero, $\Delta_f < 0$ implies distinct real zeros, and $\Delta_f > 0$ indicates that the polynomial has one real zero and a conjugate pair of complex zeros.

To classify the zeon zeros of monic zeon cubic function $\varphi(u)$, we define the zeon cubic discriminant by $\Delta_\varphi = q^3 + r^2$. When Δ_φ is invertible, the zeon cubic φ has three spectrally simple zeon zeros. If q is also invertible, the zeros can be obtained from the depressed zeon cubic formula (or general extension thereof). If q is nilpotent, zeros can be obtained using the spectrally simple zeros algorithm recalled in Section 2. By contrast, when Δ_φ is not invertible, the zeon cubic φ either has no zeros or uncountably many of them. Some examples and special cases are considered in detail in Section 4.

We proceed as follows. Terminology, notational conventions, and essential results on kth roots of complex zeons are established in Subsections 1.1 and 1.4. Essential background on zeon polynomials is recalled in Section 2.

Main results appear in Sections 3 and 4, where depressed and general cubic formulas are presented and a classification of zeros based on the cubic discriminant is established. Beginning with Theorem 3.2, we show that a depressed cubic $\varphi(u) = u^3 + 3qu - 2r \in \mathbb{C}\mathfrak{Z}[u]$ with invertible q has zeon zeros given by $u = A^{1/3} - qA^{-1/3}$ for the cube roots of $A = r \pm \sqrt{q^3 + r^2}$ with either choice of sign, provided $q^3 + r^2$ has square roots. The restrictions are relaxed to allow nilpotent q in Theorem 3.5, where we find that if r is invertible, then $\varphi(u)$ has three spectrally simple zeros, while if r is nilpotent, then φ has either no zeros or uncountably many nilpotent zeros. Section 3 concludes with the establishment of a general cubic formula for zeon polynomials in Theorem 3.16.

In Section 4, our attention turns to classification via the cubic discriminant. In Theorem 4.1, we consider zeon cubic $\varphi(u) = u^3 + \alpha u^2 + \beta u + \gamma \in \mathbb{C}\mathfrak{Z}[u]$, and define the discriminant $\Delta_\varphi = q^3 + r^2$, where $q = \frac{1}{3}\beta - \frac{1}{9}\alpha^2$, and $r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3$. We show that if Δ_φ is invertible, then φ has three spectrally simple zeros. On the other hand, if Δ_φ is nilpotent, then φ either has no zeros or has uncountably many zeros. Section 4 concludes with a discussion of classification of cubic polynomials over the finite-dimensional zeon algebra $\mathbb{C}\mathfrak{Z}_2$.

Examples appearing throughout the paper have been computed using *Mathematica* with the "Zeon Essentials" package freely available online via the "Research" link at `https://www.siue.edu/~sstaple`.

1.1 Preliminaries

Throughout the paper $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ represent the natural numbers (*i.e.*, positive integers), real numbers, and complex numbers, respectively.

Let $\mathbb{C}\mathfrak{Z}$ denote the infinite-dimensional complex Abelian algebra generated by a fixed collection $\{\zeta_{\{i\}} : i \in \mathbb{N}\}$ along with the scalar $1 = \zeta_\varnothing$ subject to the zeon commutation relation (ZCR):

$$\{\zeta_{\{i\}}, \zeta_{\{j\}}\} = \zeta_{\{i\}}\zeta_{\{j\}} + \zeta_{\{j\}}\zeta_{\{i\}} = 2\delta_{ij}\zeta_{\{i\}}\zeta_{\{j\}} := 2\delta_{ij}\zeta_{\{i,j\}},$$

where we employ multi-index notation for the final equality. For each finite subset I of $\mathbb{N}$, define $\zeta_I = \prod_{\iota \in I} \zeta_\iota$. Letting the finite subsets of positive integers be denoted by $[\mathbb{N}]^{<\omega}$, the algebra $\mathbb{C}\mathfrak{Z}$ has a canonical basis of the form $\{\zeta_I : I \in [\mathbb{N}]^{<\omega}\}$. Elements of this basis are referred to as the *basis blades* of $\mathbb{C}\mathfrak{Z}$. The algebra $\mathbb{C}\mathfrak{Z}$ is called the *(complex) zeon algebra*.

While nonzero scalar multiples of generators also generate the algebra $\mathbb{C}\mathfrak{Z}$, nontrivial linear combinations of generators are *not* generators. For example, $i \neq j$ and $a, b \neq 0$ imply $(a\zeta_{\{i\}} + b\zeta_{\{j\}})^2 = 2ab\zeta_{\{i,j\}}$, which is not a generator of the algebra. Hence, the representation is unique up to generator labeling and scaling.

By the null-square property of the generators $\{\zeta_i : i \in \mathbb{N}\}$, the basis blade product satisfies

$$\zeta_I \zeta_J = \begin{cases} \zeta_{I \cup J} & I \cap J = \varnothing, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

An element $u \in \mathbb{C}\mathfrak{Z}$ has canonical expansion $u = \sum_I u_I \zeta_I$, where each I is a finite subset of $\mathbb{N}$, $u_I \in \mathbb{C}$, and only finitely many of the coefficients u_I are nonzero. Two elements u, v are equal if and only if $u_I = v_I$ for every multi-index in the canonical expansions.

We note that $\mathbb{C}\mathfrak{Z}$ is graded. For non-negative integer k, the *grade-k part* of element $u = \sum_I u_I \zeta_I$ is defined as

$$\langle u \rangle_k = \sum_{\{I : |I| = k\}} u_I \zeta_I. \tag{1.2}$$

The mapping $\langle \cdot \rangle_k : \mathbb{C}\mathfrak{Z} \to \mathbb{C}\mathfrak{Z}$ is clearly a projection onto the subspace of $\mathbb{C}\mathfrak{Z}$ spanned by $\{\zeta_I : |I| = k\}$.

Given $z \in \mathbb{C}\mathfrak{Z}$ we write $\mathfrak{C}z = \langle z \rangle_0$ for the *complex (scalar) part* of z, and $\mathfrak{D}z = z - \mathfrak{C}z$ for the *dual part* of z. Here, the term "dual" is motivated by regarding zeons as higher-dimensional dual numbers.

Remark 1.1. *The algebra $\mathbb{C}\mathfrak{Z}$ can be regarded as the algebra of polynomials in commuting null-square variables $\zeta_{\{1\}}, \zeta_{\{2\}}, \ldots$ Equivalently, $\mathbb{C}\mathfrak{Z} \cong \mathbb{C}[z_1, z_2, \ldots]/\langle {z_1}^2, {z_2}^2, \ldots \rangle$, the quotient of the algebra of complex polynomials in commuting variables z_i by the ideal generated by squares of variables. The basis blades of $\mathbb{C}\mathfrak{Z}$ correspond to basis monomials of the polynomial algebra.*

Definition 1.2. *The* minimal grade *of $u \in \mathbb{C}\mathfrak{Z}$ is defined by*

$$\natural u = \begin{cases} \min\{k \in \mathbb{N} : \langle \mathfrak{D}u \rangle_k \neq 0\} & \mathfrak{D}u \neq 0, \\ 0 & u = \mathfrak{C}u. \end{cases} \tag{1.3}$$

We emphasize that $\natural u = 0$ if and only if u is a scalar, in which case u is said to be trivial. *As it is often useful to refer to the minimal grade part of an element $u \in \mathbb{C}\mathfrak{Z}$, we further define the following notation:*

$$u_\natural := \langle u \rangle_{\natural u}.$$

Example 1.3. *Let* $u = 3 - \zeta_{\{2\}} + 5\zeta_{\{3\}} - 12\zeta_{\{1,2,3\}}$. *We are looking for the minimal grade and the minimal grade part of* u. *Appealing to* (1.2), *we see that* u *has nonzero grade-*k *parts for* $k \in \{0,1,3\}$. *In particular,*

$$\begin{aligned}\langle u\rangle_0 &= 3,\\ \langle u\rangle_1 &= -\zeta_{\{2\}} + 5\zeta_{\{3\}},\\ \langle u\rangle_3 &= -12\zeta_{\{1,2,3\}}.\end{aligned}$$

Hence, by Definition 1.3, the minimal grade of u *is* $\natural u = 1$ *and the minimal grade part of* u *is* $u_\natural = \langle u\rangle_1 = -\zeta_{\{2\}} + 5\zeta_{\{3\}}$.

Finally, we note that the nilpotent elements of $\mathbb{C}\mathfrak{Z}$ form a maximal ideal, which we denote by

$$\mathbb{C}\mathfrak{Z}^\circ = \{u \in \mathbb{C}\mathfrak{Z} : \mathfrak{C}u = 0\}.$$

The invertible elements form a multiplicative abelian group denoted by

$$\mathbb{C}\mathfrak{Z}^\times = \mathbb{C}\mathfrak{Z} \setminus \mathbb{C}\mathfrak{Z}^\circ = \{u \in \mathbb{C}\mathfrak{Z} : \mathfrak{C}u \neq 0\}.$$

1.2 Finite-dimensional complex zeon algebras

Letting $[n]$ denote the n-set $\{1,\ldots,n\}$, the complex zeon algebra generated by $\{\zeta_{\{i\}} : i \in [n]\}$ along with the unit scalar 1 is denoted by $\mathbb{C}\mathfrak{Z}_n$. As a vector space over $\mathbb{C}$, $\mathbb{C}\mathfrak{Z}_n$ has dimension 2^n.

Given any zeon $u \in \mathbb{C}\mathfrak{Z}$, we define the *maximum index* of u to be the least positive integer n such that

$$u \in \mathbb{C}\mathfrak{Z}_n \subset \mathbb{C}\mathfrak{Z}_{n+1} \subset \mathbb{C}\mathfrak{Z}_{n+2} \subset \cdots .$$

Equivalently, we have the following definition.

Definition 1.4. *The* maximum index *of* $u \in \mathbb{C}\mathfrak{Z}$ *is the unique positive integer* n *such that* $u \in \mathbb{C}\mathfrak{Z}_n$ *and* $u \notin \mathbb{C}\mathfrak{Z}_{n-1}$.

For example, if $u = 1 + 3\zeta_{\{1,4\}} - 2\zeta_{\{1,3,5\}}$, the maximum index of u is $n = 5$.

1.3 Multiplicative properties of zeons

The elements of $\mathbb{C}\mathfrak{Z}$ form a multiplicative semigroup, and it is not difficult to establish convenient formulas for expanding products of zeons. Moreover, $u \in \mathbb{C}\mathfrak{Z}$ is invertible if and only if $\mathfrak{C}u \neq 0$. The following result is recalled from [1] for reference.

Proposition 1.5. *Let $u \in \mathbb{C}\mathfrak{Z}$, and let κ denote the index of nilpotency*[1] *of $\mathfrak{D}u$. It follows that u is uniquely invertible if and only if $\mathfrak{C}u \neq 0$, and the inverse is given by*

$$u^{-1} = \frac{1}{\mathfrak{C}u} \sum_{j=0}^{\kappa-1} (-1)^j (\mathfrak{C}u)^{-j} (\mathfrak{D}u)^j. \tag{1.4}$$

One way to see Proposition 1.5 is to first recall that if the geometric series $\sum_{j=0}^{\infty} x^j$ converges, its limit is $\frac{1}{1-x}$. Again letting $a = \mathfrak{C}u \neq 0$ and writing $u = a + \mathfrak{D}u$, we see that

$$u^{-1} = (a + \mathfrak{D}u)^{-1} = a^{-1} \frac{1}{1 - (-a\mathfrak{D}u)} = a^{-1} \sum_{j=0}^{\kappa-1} (-1)^j a^{-j} (\mathfrak{D}u)^j,$$

where nilpotency of $\mathfrak{D}u$ reduces the infinite series to a finite sum, eliminating any concern about lack of convergence.

1.3.1 Products and partitions

For convenience, we recall without proof the multinomial theorem. Let $\{x_1, \ldots, x_m\}$ be a collection of commuting variables. For any positive integer m and any nonnegative integer n, one has

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{\substack{k_1 + \cdots + k_m = n \\ k_1, k_2, \ldots, k_m \geq 0}} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{\ell=1}^{m} {x_\ell}^{k_\ell}, \tag{1.5}$$

where

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}$$

is a multinomial coefficient. We further take $x^0 = 1$ even when $x = 0$.

When $n = 2$, (1.5) reduces to the more commonly seen binomial theorem. The importance of the multinomial theorem when considering powers of zeons becomes evident when one realizes that the nonnegative integers $k_1, \ldots, k_m$ are restricted to values 0 or 1 when $x_1, \ldots, x_m$ are zeon basis blades.

For an immediate consequence, let $u, v \in \mathbb{C}\mathfrak{Z}$, write $u = \sum_I u_I \zeta_I$ and $v = \sum_I v_I \zeta_I$, and let the product $w = uv$ be written $w = \sum_I w_I \zeta_I$. Then for fixed multi-index I, the corresponding coefficient of ζ_I in w is given by

$$w_I = \sum_{K \subseteq I} u_K v_{I \setminus K}.$$

Extending to powers of zeons, let $u = \sum_I u_I \zeta_I \in \mathbb{C}\mathfrak{Z}$. For positive integer k, let $w = u^k$ be written

[1] In particular, κ is the least positive integer such that $(\mathfrak{D}u)^\kappa = 0$.

$w = \sum_I w_I \zeta_I$. For any fixed multi-index I, the corresponding coefficient of ζ_I in w is given by

$$w_I = \sum_{j=0}^{k} \frac{k!}{j!} {u_\varnothing}^j \sum_{\substack{\pi \in \mathcal{P}(I) \\ |\pi| = k-j}} u_\pi.$$

Here, $\mathcal{P}(I)$ denotes the collection of partitions of the multi-index I. When $\pi \in \mathcal{P}(I)$ is a partition, $|\pi|$ denotes the number of *blocks* (nonempty subsets of I) in the partition π and $u_\pi := \prod_{b \in \pi} u_b$; *i.e.*, the product of coefficients u_b in the expansion of u corresponding to blocks b in the partition π. Note that the scalar part of u is $\mathfrak{C}u = u_\varnothing$. By convention, we define ${u_\emptyset}^0 = 1$ when $u_\emptyset = 0$.

1.4 Complex zeon roots: Existence and recursive formulations

Invertible zeons have roots of all positive integer orders. Generalizing the result established in [1] for $\mathbb{Z}_n$, their existence is established recursively as follows.

Theorem 1.6. *Let $w \in \mathbb{C}\mathfrak{Z}^\times$, and let $k \in \mathbb{N}$. Then, there exists some $z \in \mathbb{C}\mathfrak{Z}^\times$ such that $z^k = w$. Further, writing $w = u + v\zeta_{\{n\}}$, where $u, v \in \mathbb{C}\mathfrak{Z}_{n-1}$, z is computed recursively by*

$$z = w^{1/k} = u^{1/k} + \frac{1}{k} u^{-(k-1)/k} v \zeta_{\{n\}}.$$

Proof. Proof is by induction on the maximum index n of w. When $n = 1$, let $w = w_\varnothing + b\zeta_{\{1\}}$, where $w_\varnothing = \mathfrak{C}w \neq 0$ and $b \in \mathbb{C}$. Applying the binomial theorem and null-square properties of zeon generators, one finds

$$\left({w_\varnothing}^{1/k} + \frac{b}{k {w_\varnothing}^{(k-1)/k}} \zeta_{\{1\}} \right)^k = w_\varnothing + k {w_\varnothing}^{(k-1)/k} \frac{b}{k {w_\varnothing}^{(k-1)/k}} \zeta_{\{1\}} = w_\varnothing + b\zeta_{\{1\}}.$$

Next, suppose the result holds for some $n - 1 \geq 1$ and let $w \in \mathbb{C}\mathfrak{Z}_n$ be written $w = u + v\zeta_{\{n\}}$, where $u, v \in \mathbb{C}\mathfrak{Z}_{n-1}$. In particular, this implies $u \in \mathbb{C}\mathfrak{Z}_n{}^\times$. Let $\alpha = u^{1/k}$, and let $z = \alpha + \frac{1}{k}\zeta_{\{n\}} \alpha^{-(k-1)} v$. Then

$$z^k = \left(\alpha + \frac{1}{k} \alpha^{-(k-1)} v \zeta_{\{n\}} \right)^k = u + k\alpha^{(k-1)} \frac{1}{k} \alpha^{-(k-1)} v \zeta_{\{n\}} = u + v\zeta_{\{n\}} = w. \qquad \square$$

Theorem 1.6 establishes the existence of kth roots of invertible zeons. The following corollary shows that for each kth root of $\mathfrak{C}w$, there exists exactly one zeon kth root of w.

Corollary 1.7. *Let $w \in \mathbb{C}\mathfrak{Z}^{\times}$, and let $k \in \mathbb{N}$. Then, w has exactly k distinct kth roots; i.e., $\sharp\{u : u^k = w\} = k$.*

Proof. Given any invertible zeon w, the nonzero scalar part $\mathfrak{C}w$ has precisely k distinct kth roots in $\mathbb{C}$. We claim that for each of these scalars λ, there is precisely one zeon z satisfying $\mathfrak{C}z = \lambda$ and $z^k = w$.

To see this, suppose $u^k = w = v^k$, where $\mathfrak{C}u = \mathfrak{C}v = \lambda$ and observe that $u - v$ is nilpotent because $u = \lambda + \mathfrak{D}u$ and $v = \lambda + \mathfrak{D}v$. Note that the product $w\delta$ of invertible w and nilpotent δ, is zero if and only if $\delta = 0$, since $0 = w^{-1}0 = \delta$. With the assumption $u^k = v^k$, we then have

$$\begin{aligned} u^k - v^k &= (u-v)(u^{k-1} + u^{k-2}v + \cdots + v^{k-1}) \\ &= (u-v)\left[(\lambda^{k-1} + \delta_1) + (\lambda^{k-1} + \delta_2) + \cdots + (\lambda^{k-1} + \delta_k)\right] \\ &= (u-v)\left[k\lambda^{k-1} + \delta\right], \end{aligned}$$

where $\delta = \delta_1 + \cdots + \delta_k$ is nilpotent because $\mathbb{C}\mathfrak{Z}^{\circ}$ is an ideal. It is clear that $k\lambda^{k-1} + \delta$ is invertible, so $(u-v)(k\lambda^{k-1} + \delta) = 0$ implies $(u - v) = 0$. □

Given invertible $u \in \mathbb{C}\mathfrak{Z}$ and positive integer k, the *principal kth root* of u is defined to be the zeon kth root of u whose scalar part is the principal kth root of $\mathfrak{C}u \in \mathbb{C}$.

1.4.1 Roots of nilpotent zeons

Generally, for positive integer $k \geq 2$, a nilpotent zeon has either no kth roots or uncountably many of them. We restrict our attention to square roots and cube roots here because these are the only roots of interest when dealing with cubic polynomials.

An element $u = \sum_{\{I \in \mathbb{N}^{\leq \omega}\}} u_I \zeta_I$ has a square root $w = \sum_J w_J \zeta_J$ if for each coefficient u_I in the expansion of u, the coefficients of w satisfy

$$\sum_{K \subset I} w_K w_{I \setminus K} = u_I. \tag{1.6}$$

For each nonempty multi index I, (1.6) is an equation in $2^{|I|} - 1$ variables. Letting n denote the smallest positive integer such that $u \in \mathbb{C}\mathfrak{Z}_n{}^{\circ}$, and observing that squares of elements in the maximal ideal $\mathbb{C}\mathfrak{Z}^{\circ}$ always have minimal grade greater than 1, it follows that there are $2^n - n - 1$ such equations to consider. The resulting underdetermined system of $2^n - n - 1$ equations in $2^n - 2$ variables then has either no solution or uncountably many solutions.

Example 1.8. *Consider the nilpotent zeon* $u = 4\zeta_{\{1,2\}} - 5\zeta_{\{1,3\}} - 10\zeta_{\{2,3\}} - 5\zeta_{\{1,2,3\}}$. *A square root* $w = \sum_I w_I\zeta_I$ *of* u *must satisfy the following system of equations:*

$$\begin{aligned} w_{\{1\}}w_{\{2\}} &= 2, \\ w_{\{1\}}w_{\{3\}} &= -\frac{5}{2}, \\ w_{\{2\}}w_{\{3\}} &= -5, \\ w_{\{3\}}w_{\{1,2\}} + w_{\{2\}}w_{\{1,3\}} + w_{\{1\}}w_{\{2,3\}} &= -\frac{5}{2}. \end{aligned}$$

One such solution is

$$w = -\zeta_{\{1\}} - 2\zeta_{\{2\}} + \frac{5}{2}\zeta_{\{3\}} + \zeta_{\{1,2\}} + \zeta_{\{1,3\}} + 3\zeta_{\{2,3\}}.$$

Similarly, a nilpotent zeon of minimal grade 3 or more having expansion $u = \sum_{\{I\in\mathbb{N}^{\leq\omega}:|I|\geq 3\}} u_I\zeta_I$ has cube root $w = \sum_J w_J\zeta_J$ if for each coefficient u_I, the coefficients of w satisfy

$$\sum_{\{K,L\subset I:K\cap L=\varnothing\}} w_K w_L w_{I\setminus(K\cup L)} = u_I.$$

This leads to an underdetermined system of $2^n - \binom{n}{2} - n - 1$ equations in $2^n - 2$ variables with either no solution or uncountably many solutions.

We turn now to a simple special case for which symbolic computation is straightforward.

1.4.2 Fundamental roots of nonzero null monomials

In this section we consider kth roots of $a\zeta_I$ for $a \in \mathbb{C}^\times$ and nonempty $I \subset \mathbb{N}$. Such elements are referred to as *nonzero null monomials*[2] of $\mathbb{C}\mathfrak{Z}$.

Remark 1.9. *Nonzero null monomials are square roots of zero. It follows that every* k*th root of a nonzero null monomial is a* $2k$*th root of zero.*

Definition 1.10. *Given a nonzero null monomial* $w = a\zeta_I$ *and a* k*-block partition* π *of* I, *a* fundamental kth root *of* w *is any nilpotent zeon of the form*

$$u_\pi = \sum_{J\in\pi} u_J\zeta_J, \tag{1.7}$$

satisfying $(u_\pi)^k = w$.

[2]In particular, $a\zeta_I$ is a zero of the monomial $\varphi(u) = u^2$ for any $I \neq \varnothing$.

For purposes of symbolic computation, two forms of roots are particularly convenient. Roots of the form (1.8) are referred to as *flat form* fundamental kth roots of u, while roots of the form (1.9) will be referred to as *spike form* fundamental kth roots of u.

Lemma 1.11. *Given a nonzero null monomial $w = a\zeta_I$, a nilpotent zeon of the form*

$$u_\pi = \sqrt[k]{\frac{a}{k!}} \sum_{J\in\pi} \zeta_J \tag{1.8}$$

satisfies ${u_\pi}^k = w$ for any k-block partition π of the multi index I and any complex kth root of $\frac{a}{k!}$. Moreover,

$$u_{\pi,M} = \sum_{J\in\pi\setminus M} \zeta_J + \frac{a}{k!}\zeta_M \tag{1.9}$$

satisfies ${u_\pi}^k = w$ for any fixed block M of the k-block partition π of the multi index I.

Proof. By direct computation via the multinomial theorem,

$$\left(\sqrt[k]{\frac{a}{k!}} \sum_{J\in\pi} \zeta_J\right)^k = \frac{a}{k!}k!\prod_{\ell=1}^{k} \zeta_{I_\ell} = a\zeta_I = \left(\sum_{J\in\pi\setminus M} \zeta_J + \frac{a}{k!}\zeta_M\right)^k. \qquad \square$$

Hence, the result.

Example 1.12. *The flat form fundamental square roots of $a\zeta_{\{1,2,3\}}$ are*

$$u_{1|23} = \pm\sqrt{\frac{a}{2}}(\zeta_{\{1\}} + \zeta_{\{2,3\}}), \qquad u_{2|13} = \pm\sqrt{\frac{a}{2}}(\zeta_{\{2\}} + \zeta_{\{1,3\}}),$$
$$u_{3|12} = \pm\sqrt{\frac{a}{2}}(\zeta_{\{3\}} + \zeta_{\{1,2\}}),$$

and the spike form fundamental square roots are

$$u_{1|23,\{2,3\}} = \left(\zeta_{\{1\}} + \frac{a}{2}\zeta_{\{2,3\}}\right), \qquad u_{2|13,\{1,3\}} = \left(\zeta_{\{2\}} + \frac{a}{2}\zeta_{\{1,3\}}\right),$$
$$u_{3|12,\{1,2\}} = \left(\zeta_{\{3\}} + \frac{a}{2}\zeta_{\{1,2\}}\right), \qquad u_{1|23,\{1\}} = \left(\frac{a}{2}\zeta_{\{1\}} + \zeta_{\{2,3\}}\right),$$
$$u_{2|13,\{2\}} = \left(\frac{a}{2}\zeta_{\{2\}} + \zeta_{\{1,3\}}\right), \qquad u_{3|12,\{3\}} = \left(\frac{a}{2}\zeta_{\{3\}} + \zeta_{\{1,2\}}\right).$$

Notation. The numbers of k-block partitions of sets containing m elements are given by Stirling numbers of the second kind, denoted $\left\{ {m \atop k} \right\}$.

Lemma 1.13. *The number of fundamental kth roots of a null monomial of grade $m \geq k$ is $k\left\{ {m \atop k} \right\}$.*

Proof. Each partition of I into k nonempty subsets $\{I_\ell : 1 \leq \ell \leq \left\{ {m \atop k} \right\}\}$ gives a principal kth root of $a\zeta_I$ since

$$(a^{1/k}\zeta_{I_\ell})^k = k!(a^{1/k})^k \prod_{\ell=1}^{k} \zeta_{I_\ell} = a\zeta_I.$$

Each $a \in \mathbb{C}^\times$ has k distinct complex kth roots, so there are k zeon kth roots of the form seen in (1.8) for each k-block partition π of I. □

2 Zeon polynomials

Let $f(z) = a_m z^m + \cdots + a_1 z + a_0$ $(a_m \neq 0)$ be a polynomial function with complex coefficients, and recall that by the Fundamental Theorem of Algebra, $f(z)$ has exactly m complex zeros. If $f(z)$ can be written in the form $f(z) = (z-r)^\ell g(z)$, where $\ell \in \mathbb{N}$ and $g(r) \neq 0$, then r is said to be a *zero of multiplicity* ℓ of $f(z)$. For convenience, $\mu_f(r)$ will denote the multiplicity of r as a zero of $f(z)$.

On the other hand, if $\varphi(u) = \alpha_m u^m + \cdots + \alpha_1 u + \alpha_0 \in \mathbb{C}\mathfrak{Z}[u]$ is a polynomial with zeon coefficients, it is not obvious how many zeros this polynomial may have in $\mathbb{C}\mathfrak{Z}$. For example, $\varphi(u) = u^2 - \zeta_{\{1\}}$ has no zeon zeros because $\zeta_{\{1\}}$ has no square root.

2.1 Spectrally simple zeros of zeon polynomials

Given a complex zeon polynomial $\varphi(u) = \alpha_m u^m + \cdots + \alpha_1 u + \alpha_0$, a *complex polynomial* $f_\varphi : \mathbb{C} \to \mathbb{C}$ is induced by

$$f_\varphi(z) = \sum_{\ell=0}^{m} (\mathfrak{C}\alpha_\ell) z^\ell.$$

It follows that

$$f_\varphi(\mathfrak{C}u) = \sum_{\ell=0}^{m} (\mathfrak{C}\alpha_\ell)(\mathfrak{C}u)^\ell = \mathfrak{C}(\varphi(u)),$$

so that $f_\varphi \circ \mathfrak{C} = \mathfrak{C} \circ \varphi$.

We restrict our attention to zeon polynomials with invertible leading coefficients because when α_m is nilpotent, the induced polynomial $f_\varphi(z)$ is of lower degree than $\varphi(u)$. Moreover, as a matter of convenience the zeros of $\varphi(u)$ are exactly the zeros of the monic polynomial ${\alpha_m}^{-1}\varphi(u)$.

The zeon extension of the Fundamental Theorem of Algebra developed in [11] shows that $\varphi(u)$ has a simple zeon zero if the complex polynomial $f_\varphi(z)$ has a simple complex zero.

Let $\varphi(u)$ be a nonconstant monic zeon polynomial. A zeon $\lambda \in \mathbb{C}\mathfrak{Z}$ is said to be a *simple* zero of φ if $\varphi(u) = (u-\lambda)g(u)$ for some zeon polynomial g satisfying $g(\lambda) \neq 0$.

The spectrum of an element u in a unital algebra is the collection of scalars λ for which $u - \lambda$ is not invertible. Hence, the spectrum of $u \in \mathbb{C}\mathfrak{Z}$ is the singleton $\{\lambda = \mathfrak{C}u\}$, motivating the next definition.

Definition 2.1. *A simple zero $\lambda_0 \in \mathbb{C}\mathfrak{Z}$ of $\varphi(u)$ is said to be a* spectrally simple *if $\mathfrak{C}\lambda_0$ is a simple zero of the complex polynomial $f_\varphi(z)$.*

2.1.1 Fundamental theorem of zeon algebra

The *Fundamental Theorem of Zeon Algebra* presented in [11] for the finite dimensional zeon algebra $\mathbb{C}\mathfrak{Z}_n$ shows that a zeon polynomial $\varphi(u) \in \mathbb{C}\mathfrak{Z}_n[u]$ has a spectrally simple zero $\lambda = \lambda_0 + \mathfrak{D}\lambda$ whenever the complex polynomial $f_\varphi(z) \in \mathbb{C}[z]$ has a simple zero $\lambda_0 \in \mathbb{C}$. The theorem also holds also for a polynomial over $\mathbb{C}\mathfrak{Z}$ by first defining the *maximum index of a zeon polynomial* φ to be the least positive integer n such that $\varphi(u) \in \mathbb{C}\mathfrak{Z}_n[u]$ and proceeding as in the finite-dimensional zeon algebra.

For reference, the theorem is recalled here without proof. We note that it also provides a method for calculating spectrally simple zeros of any zeon polynomial.

Theorem 2.2 (Fundamental Theorem of Zeon Algebra)**.** *Let $\varphi(u) \in \mathbb{C}\mathfrak{Z}[u]$ be a monic zeon polynomial of degree m and having maximum index n, and let $f_\varphi(z) \in \mathbb{C}[z]$ be induced by φ. If $\lambda_0 \in \mathbb{C}$ is a simple zero of $f_\varphi(z)$, let g be the unique complex polynomial such that $f_\varphi(\mathfrak{C}u) = (\mathfrak{C}u - \lambda_0)g(\mathfrak{C}(u))$. Then $\varphi(u)$ has a simple zero λ such that $\mathfrak{C}\lambda = \lambda_0$. Letting n denote the maximum index of $\varphi(u)$, for $1 \leq k \leq n$, the grade-k part of λ (denoted λ_k) is given by*

$$\lambda_k = -\frac{1}{g(\lambda_0)}\left\langle \varphi\left(\sum_{i=0}^{k-1}\lambda_i\right)\right\rangle_k .$$

Moreover, such a zero λ is unique.

The idea behind the proof is that when λ_0 is a simple zero of $f_\varphi(z)$, the remainder $\varphi(\lambda_0)$ of $\varphi(u)$ when divided by $u - \lambda_0$ has zero scalar part. The minimal grade part of the remainder $w = \varphi(\lambda_0)$ can then be utilized to construct a new zeon element $\lambda_0 + \lambda_{\natural w}$ having the property that $\varphi(\lambda_0 + \lambda_{\natural w})$ has higher minimal grade than $\varphi(\lambda_0)$. Grades of all remainders will be at most n (the maximum index of $\varphi(u)$), so the process terminates in a finite number of steps.

Of particular significance, Theorem 2.2 provides an algorithm by which a spectrally simple zeon zero can be calculated. Algorithm 1 returns the spectrally simple zeon zero λ of φ whose scalar part λ_0 satisfies $f_\varphi(\lambda_0) = 0$.

Algorithm 1: Compute spectrally simple zeon zero.

input : Zeon polynomial $\varphi(u)$ over $\mathbb{C}\mathfrak{Z}_n$ and a simple nonzero root λ_0 of the associated complex polynomial $\mathfrak{C}(\varphi(u))$.
output: Zeon zero λ of $\varphi(u)$ with $\mathfrak{C}\lambda = \lambda_0$.

Initialize complex polynomial $g(\mathfrak{C}u)$.

$g(\mathfrak{C}u) \leftarrow \dfrac{\mathfrak{C}(\varphi(u))}{\mathfrak{C}u - \lambda_0}$;

Note $g(\mathfrak{C}u)$ satisfies $\mathfrak{C}(\varphi(u)) = (\mathfrak{C}u - \lambda_0)g(\mathfrak{C}u)$, where $g(\lambda_0) \neq 0$.

$\xi \leftarrow \varphi(\lambda_0)_\natural / g(\lambda_0)$;
$\lambda \leftarrow \lambda_0 - \xi$;

while $0 < \natural\xi \leq n$ **do**
  $\xi \leftarrow \varphi(\lambda)_\natural / g(\lambda_0)$;
  $\lambda \leftarrow (\lambda - \xi)$;

return λ;

When $\varphi(u) \in \mathbb{C}\mathfrak{Z}[u]$ is of degree $m \geq 1$ and the zeros of $f_\varphi(z)$ are all simple, we see that $\varphi(u)$ has exactly m complex zeon zeros. For example, when $\alpha \in \mathbb{C}\mathfrak{Z}^\times$, $\varphi(u) = u^k + \alpha$ has exactly k distinct complex zeon zeros.

2.2 Spectrally nonsimple zeon zeros

Algorithm 1 is useful for computing spectrally simple zeros of $\varphi(u)$, but it is not applicable to any zero w whose scalar part $\mathfrak{C}w$ is a multiple zero of the induced complex polynomial f_φ satisfying $\mathfrak{C}(\varphi(u)) = f_\varphi(\mathfrak{C}u)$. These spectrally nonsimple zeros are considered next.

A zero $\lambda_0 \in \mathbb{C}\mathfrak{Z}$ of $\varphi(u) \in \mathbb{C}\mathfrak{Z}[u]$ is said to be *spectrally nonsimple* if $\mathfrak{C}\lambda_0$ is a multiple zero of the induced complex polynomial f_φ. We note that zeon zeros of multiplicity greater than one are included among spectrally nonsimple zeros.

It was shown in [11] that if a monic polynomial $\varphi(u) \in \mathbb{C}\mathfrak{Z}[u]$ has distinct complex zeon zeros w_1, w_2 satisfying $\mathfrak{C}w_1 = \mathfrak{C}w_2 = w_\varnothing$, then $\varphi(u)$ has uncountably many zeros of the form $w = w_\varnothing + \mathfrak{D}w$. As a consequence, if $\varphi \in \mathbb{C}\mathfrak{Z}[u]$ has a zero $z \in \mathbb{C}\mathfrak{Z}$ of multiplicity two or greater, then φ has uncountably many zeros $w \in \mathbb{C}\mathfrak{Z}$ satisfying $\mathfrak{C}w = \mathfrak{C}z$.

Lacking an algorithm for computing spectrally nonsimple zeros of zeon polynomials, our attention turns to zeon extensions of well-known special cases: quadratic and cubic polynomials.

2.2.1 The zeon quadratic formula

We close this review of zeon polynomials by recalling a basic result concerning zeros of quadratic zeon polynomials. A zeon quadratic polynomial has solutions if and only if its discriminant has a square root [3].

Theorem 2.3 (Zeon Quadratic Formula)**.** *Let $\varphi(u) = \alpha u^2 + \beta u + \gamma$ be a quadratic function with zeon coefficients from $\mathbb{C}\mathfrak{Z}$, where $\mathfrak{C}\alpha \neq 0$. Let $\Delta_\varphi = \beta^2 - 4\alpha\gamma$ denote the zeon discriminant of φ. The zeros of φ are given by*

$$\varphi^{-1}(0) = \left\{ \frac{\alpha^{-1}}{2}(w - \beta) : w^2 = \beta^2 - 4\alpha\gamma \right\}.$$

In particular,

(1) *When $\Delta_\varphi = 0$, the zeros of φ are given by $u = -\alpha^{-1}\beta/2 + \eta$ for any $\eta \in \mathbb{C}\mathfrak{Z}$ satisfying $\eta^2 = 0$.*

(2) *When $\mathfrak{C}\Delta_\varphi \neq 0$, $\varphi(u) = 0$ has two distinct solutions.*

(3) *If $\Delta_\varphi \neq 0$ is nilpotent and $\varphi(u) = 0$ has a solution, then it has uncountably many solutions.*

To see the result, begin by writing $\alpha u^2 + \beta u + \gamma = \frac{\alpha^{-1}}{4}((2\alpha u + \beta)^2 - (\beta^2 - 4\alpha\gamma))$ and expand. This reduces the problem to seeking square roots of the zeon discriminant. We are now ready to turn our attention to cubic polynomials over $\mathbb{C}\mathfrak{Z}$.

3 Cubic polynomials with zeon coefficients

Beginning with the general zeon cubic equation $z^3 + \alpha z^2 + \beta z + \gamma = 0$, where $\alpha, \beta, \gamma \in \mathbb{C}\mathfrak{Z}$ and $\alpha \neq 0$, the *depressed cubic equation* is obtained via the substitution $z = u - \alpha/3$. In particular,

$$\begin{aligned} 0 &= \left(u - \frac{\alpha}{3}\right)^3 + \alpha\left(u - \frac{\alpha}{3}\right)^2 + \beta\left(u - \frac{\alpha}{3}\right) + \gamma = u^3 + \left(\beta - \frac{\alpha}{3}\right)u + \frac{2\alpha^3}{27} - \frac{\alpha\beta}{3} + \gamma \\ &= u^3 + 3\left(\frac{\beta}{3} - \frac{\alpha^2}{9}\right)u - 2\left(\frac{-\alpha^3}{27} + \frac{\alpha\beta}{6} - \frac{\gamma}{2}\right) = u^3 + 3qu - 2r \end{aligned}$$

where $q = \frac{1}{3}\beta - \frac{1}{9}\alpha^2$ and $r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3$. It follows that depressed cubics are sufficient for our purposes.

We note that any monic cubic polynomial having a spectrally simple zero λ can be reduced via polynomial division to the product $\varphi(u) = (u - \lambda)\psi(u)$, where $\psi(\lambda) \neq 0$ is a quadratic polynomial over $\mathbb{C}\mathfrak{Z}$. The remaining zeros of $\varphi(u)$ can then be classified by the zeon quadratic formula of Theorem 2.3.

Example 3.1. *To motivate our discussion, consider the depressed zeon cubic equation* $\varphi(u) = 0$ *where*

$$\varphi(u) = u^3 + u\left(-18\zeta_{\{1,2,3\}} - 6\zeta_{\{1\}} + 9\zeta_{\{2\}} - 9\right) - 10\zeta_{\{1,2\}} - 6\zeta_{\{1,2,3\}}. \tag{3.1}$$

The induced scalar cubic polynomial is $f_\varphi(z) = z^3 - 9z$*, which has simple zeros* $\{-3, 0, 3\}$*. Consequently,* $\varphi(u)$ *has three spectrally simple zeon zeros, each of which can be found by applying Algorithm 1. Applying the algorithm with simple zero* $\lambda_0 = -3$ *of* $f_\varphi(z)$*, we obtain the first zero:*

$$u_1 = -3 + \frac{1}{18}\zeta_{\{1,2\}} - \frac{8}{3}\zeta_{\{1,2,3\}} - \zeta_{\{1\}} + \frac{3\zeta_{\{2\}}}{2}.$$

At this point, we may either repeat the algorithm with the other two zeros of $f_\varphi(z)$ *or we may perform polynomial division to write* $\varphi(u) = (u - u_1)\psi(u)$ *and apply the zeon quadratic formula to* $\psi(u)$ *to obtain the remaining zeros. In the latter method, we apply the quadratic formula to*

$$\psi(u) = u^2 + u\left(\frac{1}{18}\zeta_{\{1,2\}} - \frac{8}{3}\zeta_{\{1,2,3\}} - \zeta_{\{1\}} + \frac{3\zeta_{\{2\}}}{2} - 3\right) - \frac{10}{3}\zeta_{\{1,2\}} - 2\zeta_{\{1,2,3\}},$$

which yields the remaining zeros:

$$u_2 = -\frac{10}{9}\zeta_{\{1,2\}} - \frac{2}{3}\zeta_{\{1,2,3\}},$$
$$u_3 = 3 + \frac{19}{18}\zeta_{\{1,2\}} + \frac{10}{3}\zeta_{\{1,2,3\}} + \zeta_{\{1\}} - \frac{3}{2}\zeta_{\{2\}}.$$

We point out that the approach taken in Example 3.1 involves the application of Algorithm 1 once, followed by polynomial division and an application of the zeon quadratic formula. Alternatively, since the zeros of $\varphi(u)$ were all spectrally simple, we could have applied Algorithm 1 three times.

We further point out that when the scalar polynomial $f_\varphi(z)$ has a single zero of multiplicity three, the approach taken in Example 3.1 fails completely.

To treat such cases as well as to gain deeper insight on zeros of zeon cubics for all cases, we now consider a zeon extension of the cubic formula. The complex zeon result below is based on Cardano's approach to cubic polynomials with real coefficients, as presented in [10].

Theorem 3.2 (Depressed Zeon Cubic Formula)**.** *Let* $\varphi(u) = u^3 + 3qu - 2r \in \mathbb{C}\mathfrak{Z}[u]$*, where* $\mathfrak{C}q \neq 0$ *and square roots of* $q^3 + r^2$ *are assumed to exist. The zeon zeros of* $\varphi(u)$ *are given by* $u = A^{1/3} - qA^{-1/3}$*, for the cube roots of* $A = r \pm \sqrt{q^3 + r^2}$ *with either choice of sign.*

Proof. Note that A is invertible if and only if $\mathfrak{C}q \neq 0$, since $\mathfrak{C}A = 0$ if and only if $\mathfrak{C}r = \mp\mathfrak{C}\left(\sqrt{q^3 + r^2}\right)$. Squaring both sides yields $\mathfrak{C}q^3 = 0$. Proof is then by direct substitution, where all necessary cube roots, square roots, and inverses exist. Assuming $A = r + \sqrt{q^3 + r^2}$, it follows that

$$\begin{aligned}\varphi(A^{\frac{1}{3}} - qA^{-\frac{1}{3}}) &= (A^{\frac{1}{3}} - qA^{-\frac{1}{3}})^3 + 3q(A^{\frac{1}{3}} - qA^{-\frac{1}{3}}) - 2r \\ &= A - 3A^{\frac{2}{3}}qA^{-\frac{1}{3}} + 3A^{\frac{1}{3}}q^2A^{-\frac{2}{3}} - q^3A^{-1} + 3qA^{\frac{1}{3}} - 3q^2A^{-\frac{1}{3}} - 2r \\ &= A - 3qA^{\frac{1}{3}} + 3q^2A^{-\frac{1}{3}} - q^3A^{-1} + 3qA^{\frac{1}{3}} - 3q^2A^{-\frac{1}{3}} - 2r \\ &= A - q^3A^{-1} - 2r \\ &= (r^2 + 2r\sqrt{q^3+r^2} + q^3 + r^2 - q^3)(r + \sqrt{q^3+r^2})^{-1} - 2r \\ &= 2r(r + \sqrt{q^3+r^2})(r + \sqrt{q^3+r^2})^{-1} - 2r = 2r - 2r = 0.\end{aligned}$$

Similar calculations establish the result for $A = r - \sqrt{q^3 + r^2}$. □

Since A is assumed to be invertible in Theorem 3.2, there are three distinct zeon cube roots of A for any square root of $q^3 + r^2$.

Example 3.3 ($\mathfrak{C}q \neq 0$, $q^3 + r^2$ invertible)**.** *Consider the zeon cubic $\varphi(u) = u^3 + 3qu - 2r$ defined by*

$$\varphi(u) = u^3 + u\left(6\zeta_{\{1,2\}} - 12\zeta_{\{2,3\}} - 3\zeta_{\{3\}} - 36\right) + 6\zeta_{\{2\}} - 4\zeta_{\{3\}}.$$

We note that $\mathfrak{C}q \neq 0$, since

$$q = -12 + 2\zeta_{\{1,2\}} - 4\zeta_{\{2,3\}} - \zeta_{\{3\}}.$$

Further, $q^3 + r^2$ is invertible since r is clearly nilpotent. The zeros of $\varphi(u)$ are then found via Theorem 3.2:

$$\begin{aligned}u_1 &= -6 + \frac{1}{2}\zeta_{\{1,2\}} - \frac{215}{216}\zeta_{\{2,3\}} - \frac{5}{432}\zeta_{\{1,2,3\}} - \frac{\zeta_{\{2\}}}{12} - \frac{7}{36}\zeta_{\{3\}}, \\ u_2 &= -\frac{1}{72}\zeta_{\{2,3\}} - \frac{1}{54}\zeta_{\{1,2,3\}} + \frac{\zeta_{\{2\}}}{6} - \frac{\zeta_{\{3\}}}{9}, \\ u_3 &= 6 - \frac{1}{2}\zeta_{\{1,2\}} + \frac{109}{108}\zeta_{\{2,3\}} + \frac{13}{432}\zeta_{\{1,2,3\}} - \frac{1}{12}\zeta_{\{2\}} + \frac{11}{36}\zeta_{\{3\}}.\end{aligned}$$

When $q^3 + r^2 \in \mathbb{C}\mathfrak{Z}$ is nilpotent and has a square root, uncountably many square roots exist. In this case, the associated cubic equation has infinitely many solutions.

Example 3.4 ($\mathfrak{C}q \neq 0$, $q^3 + r^2 \in \mathbb{C}\mathfrak{Z}^\circ$)**.** *Consider the depressed zeon cubic $\varphi(u) = u^3 + 3qu - 2r$, where*

$$\begin{aligned}q &= -\zeta_{\{1,2\}} + \zeta_{\{1,3\}} + \zeta_{\{2,3\}} + 2\zeta_{\{1\}} - 2\zeta_{\{2\}} - 1, \\ r &= -3\zeta_{\{1\}} + 3\zeta_{\{2\}} + 1.\end{aligned}$$

The nilpotent element $q^3 + r^2 = 3\zeta_{\{1,2\}} + 3\zeta_{\{1,3\}} + 3\zeta_{\{2,3\}}$ *has uncountably many square roots; for example,* $\rho = \sqrt{\frac{3}{2}}\left(\zeta_{\{1\}} + \zeta_{\{2\}} + \zeta_{\{3\}}\right)$*. It follows that* $\varphi(u)$ *has uncountably many zeros of the form* $(r+\rho)^{1/3} - q(r+\rho)^{-1/3}$*. In particular,*

$$u_0 = 2 - 2\zeta_{\{1\}} + 2\zeta_{\{2\}} + \frac{10}{3}\zeta_{\{1,2\}} - \frac{2}{3}\zeta_{\{1,3\}} - \frac{2}{3}\zeta_{\{2,3\}}$$

satisfies $\varphi(u_0) = 0$*.*

Next we consider the depressed cubic $\varphi(u) = u^3 + 3qu - 2r$, where $\mathfrak{C}q = 0$. It follows that the complex polynomial induced by φ is $f_\varphi(z) = z^3 - 2\mathfrak{C}r$. If $\mathfrak{C}r = 0$, then $f_\varphi(z) = z^3$ has one zero 0 of multiplicity three. Hence, if φ has zeros, there are uncountably many and they are all nilpotent. On the other hand, if $\mathfrak{C}r \neq 0$, then $f_\varphi(z)$ has exactly three distinct complex zeros, so that φ has three spectrally simple zeros. Thus, we have derived the following theorem.

Theorem 3.5 (Depressed Cubic Zeros II)**.** *Let* $\varphi(u) = u^3 + 3qu - 2r \in \mathbb{C}\mathfrak{Z}[u]$*, where* $\mathfrak{C}q = 0$*. Then the following are true.*

(1) *If* $\mathfrak{C}r \neq 0$*, then* $\varphi(u)$ *has three spectrally simple zeros.*

(2) *If* $\mathfrak{C}r = 0$*, then* φ *has either no zeros or uncountably many nilpotent zeros.*

We illustrate Theorem 3.5 with the following example.

Example 3.6. *The case* $\mathfrak{C}q = 0$ *is illustrated by the zeon cubic polynomial*

$$\varphi(u) = u^3 + \left(-\frac{2}{3}\zeta_{\{1,2\}} + \frac{4}{3}\zeta_{\{1,3\}} + \frac{4}{3}\zeta_{\{2,3\}} - \frac{8}{3}\zeta_{\{1,2,3\}}\right)u - \frac{8}{9}\zeta_{\{1,2,3\}}.$$

In this example $r = \frac{4}{9}\zeta_{\{1,2,3\}}$*, so that* $\varphi(u)$ *either has no zeros or uncountably many. Letting* $s = \zeta_{\{1\}} + \zeta_{\{2\}} - \zeta_{\{3\}} + \zeta_{\{1,2\}} - \zeta_{\{1,3\}}$*, it is seen that* $\varphi(s) = 0$*. Moreover,* $\varphi(s + a\zeta_{\{1,2,3\}}) = 0$ *for any* $a \in \mathbb{C}$*.*

3.1 Special case: $\varphi(u) = u^3 + 3qu$

Note that if $r = 0$, the zeros of $\varphi(u)$ include $\{0, \pm\sqrt{-3q}\}$, provided the square roots exist. When q is invertible (*i.e.*, $\mathfrak{C}q \neq 0$), these are the three distinct zeros of $\varphi(u)$. When $\mathfrak{C}q = 0$, $\varphi(u) = 0$ has uncountably many solutions.

Our goal in this subsection is to describe some of the zeros of $u^3 + 3qu$ when q is nilpotent.

Definition 3.7. *Let* $q = \sum_I q_I\zeta_I \in \mathbb{C}\mathfrak{Z}$*. The* index support *of* q *is defined to be*

$$[q] = \bigcup_{\{I:q_I\neq 0\}} I. \tag{3.2}$$

The index support of a nilpotent q is used to obtain a null monomial that "annihilates" q; *i.e.*, $q\zeta_{[q]} = 0$. For this reason, $\zeta_{[q]}$ will be referred to as an *annihilator* of $q \in \mathbb{C}\mathfrak{Z}^{\circ}$. More generally, $q\zeta_{[q]} = (\mathfrak{C}q)\zeta_{[q]}$ for arbitrary $q \in \mathbb{C}\mathfrak{Z}$, so that $\zeta_{[q]}$ is an annihilator of $\mathfrak{D}q$.

Example 3.8. *Let* $q = 3 + 4\zeta_{\{2\}} - 5\zeta_{\{1,3,4\}}$. *Then* $[q] = \{1, 2, 3, 4\}$ *and*

$$q\zeta_{[q]} = (3 + 4\zeta_{\{2\}} - 5\zeta_{\{1,3,4\}})\zeta_{\{1,2,3,4\}} = 3\zeta_{\{1,2,3,4\}}.$$

While it is clear that when q is nilpotent, $q\zeta_I = 0$ for all $I \supseteq [q]$, a nilpotent q may also be annihilated by a basis blade ζ_I for one or more $I \subsetneq [q]$. Letting $\mathcal{N}_q = \{I \subseteq [q] : q\zeta_I = 0\}$, it follows that w

$$q \sum_{I\in\mathcal{N}_q} a_I\zeta_I = 0$$

for any linear combination of basis blades indexed by $\mathcal{N}_q$. The resulting subspace of $\mathbb{C}\mathfrak{Z}$ is denoted by $\mathrm{Ann}_{\mathfrak{Z}}(q)$.

It is clear that $\mathrm{Ann}_{\mathfrak{Z}}(u) \cap \mathrm{Ann}_{\mathfrak{Z}}(v) \subseteq \mathrm{Ann}_{\mathfrak{Z}}(u + v)$ because $z \in \mathrm{Ann}_{\mathfrak{Z}}(u) \cap \mathrm{Ann}_{\mathfrak{Z}}(v)$ implies $z(u + v) = zu + zv = 0$. However, the reverse inclusion need not hold, as illustrated in Example 3.9.

Example 3.9. *Let* $u = \zeta_{\{1\}} + \zeta_{\{2\}}$, $v = -\zeta_{\{2\}} \in \mathbb{C}\mathfrak{Z}^{\circ}$. *Letting* $z = \zeta_{\{1\}}$, *we see that*

$$z(u + v) = \zeta_{\{1\}}(\zeta_{\{1\}} + \zeta_{\{2\}} - \zeta_{\{2\}}) = {\zeta_{\{1\}}}^2 = 0,$$

so that $z \in \mathrm{Ann}_{\mathfrak{Z}}(u + v)$ *even though* $z \notin \mathrm{Ann}_{\mathfrak{Z}}(u)$ *and* $z \notin \mathrm{Ann}_{\mathfrak{Z}}(v)$.

With the concept of zeon annihilators in hand, we are ready to present our result on zeros of $\varphi(u) = u^3 + 3qu$ when q is nilpotent.

Theorem 3.10 (Zeros of $\varphi(u) = u^3 + 3qu$ when $\mathfrak{C}q = 0$)**.** *Let* $\varphi(u) = u^3 + 3qu \in \mathbb{C}\mathfrak{Z}[u]$, *where* $q \neq 0$ *and* $\mathfrak{C}q = 0$. *Then,*

(1) $\varphi(z) = 0$ *for any* $z \in \mathrm{Ann}_{\mathfrak{Z}}(q)$ *satisfying* $\kappa(z) \leq 3$*; and*

(2) *if* q *has square roots, then* $\varphi(z) = 0$ *for any* $z \in \{\pm\sqrt{-3q}\}$.

In particular, $\varphi(a\zeta_{[q]}) = 0$ *for* $a \in \mathbb{C}$.

Proof. First, for any $z \in \mathrm{Ann}_{\mathfrak{Z}}(q)$ satisfying $\kappa(z) \leq 3$,

$$\varphi(z) = z^3 + 3qz = 0 + 0 = 0.$$

Second, let $(-3q)^{1/2} = \{z \in \mathbb{C}\mathfrak{Z} : z^2 = -3q\}$ and recall that this set has infinite cardinality when it is nonempty. It follows that for each $z \in (-3q)^{1/2}$,

$$\varphi(z) = z(z^2 + 3q) = z(-3q + 3q) = 0.$$

Finally, $\zeta_{[q]} \in \mathrm{Ann}_{\mathfrak{Z}}(q)$ satisfies $\kappa(\zeta_{[q]}) = 2$, so $\varphi(a\zeta_{[q]}) = 0$ for all $a \in \mathbb{C}$. □

Theorem 3.10 does not characterize all zeros of the cubic $\varphi(u) = u^3 + 3qu$, as illustrated by Example 3.11.

Example 3.11. *Consider the cubic $\varphi(u) = u^3 + 3qu$, where*

$$q = \frac{1}{3}\zeta_{\{1,2,3\}} - \frac{2}{3}\zeta_{\{1,2\}} - \frac{2}{3}\zeta_{\{1,3\}} - \frac{2}{3}\zeta_{\{2,3\}}.$$

Letting $z = \zeta_{\{1\}} + \zeta_{\{2\}} + \zeta_{\{3\}}$, one finds that $z^2 = 2(\zeta_{\{1,2\}} + \zeta_{\{1,3\}} + \zeta_{\{2,3\}})$, $z^2 + 3q = \zeta_{\{1,2,3\}}$, and $z^3 = 6\zeta_{\{1,2,3\}}$, so that $\kappa(z) > 3$ and $z \notin (-3q)^{1/2}$. Further, $z \notin \mathrm{Ann}_{\mathfrak{Z}}(q)$ because

$$qz = \frac{1}{3}(\zeta_{\{1,2,3\}} - 2\zeta_{\{1,2\}} - 2\zeta_{\{1,3\}} - 2\zeta_{\{2,3\}})(\zeta_{\{1\}} + \zeta_{\{2\}} + \zeta_{\{3\}}) = -2\zeta_{\{1,2,3\}}.$$

Clearly, z fails to satisfy the sufficient conditions described in Theorem 3.10. However, $z \in \varphi^{-1}(0)$ since

$$\varphi(z) = z^3 + 3qz = 6\zeta_{\{1,2,3\}} - 6\zeta_{\{1,2,3\}} = 0.$$

Corollary 3.12. *Let $\varphi(u) = u^3 - a\zeta_I u \in \mathbb{C}\mathfrak{Z}[u]$, where $a \neq 0$ and $|I| \geq 2$. Then $\varphi(u) = 0$ has $\left\{ {|I| \atop 2} \right\}$ flat form solutions of the form*

$$u_\pi = \sqrt{\frac{a}{2}} \sum_{J \in \pi} \zeta_J,$$

where π ranges over the 2-block partitions of the multi index I.

Proof. Note that $u^3 - a\zeta_I u = u(u^2 - a\zeta_I) = 0$. Let π be a 2-block partition of I. Let K be one block of the partition. It follows that

$$u_\pi = \sqrt{\frac{a}{2}}(\zeta_K + \zeta_{I \setminus K}),$$

so that

$$\begin{aligned}\varphi(u_\pi) = u_\pi({u_\pi}^2 - a\zeta_I) &= u_\pi\left(\left(\sqrt{\frac{a}{2}}(\zeta_K + \zeta_{I\setminus K})\right)^2 - a\zeta_I\right) \\ &= u_\pi \frac{a}{2}(2a\zeta_I - a\zeta_I) = u_\pi(a\zeta_I - a\zeta_I) = 0.\end{aligned}$$

The number of two block partitions π of I is $\left\{ {|I| \atop 2} \right\}$, so the result follows from Lemma 1.13. ☐

3.2 Special case: $q = 0$

Lemma 3.13 (Depressed cubics: $q = 0$)**.** *Let $\varphi(u) = u^3 - 2r \in \mathbb{C}\mathfrak{Z}[u]$. Then the following are true.*

(1) *If $r = 0$, then $\varphi^{-1}(0) = \{\eta \in \mathbb{C}\mathfrak{Z}^\circ : \kappa(\eta) \leq 3\}$.*

(2) *If $\mathfrak{C}r \neq 0$, then φ has three spectrally simple zeros: $\varphi^{-1}(0) = (2r)^{1/3}$.*

(3) *If $r \neq 0$ and $\mathfrak{C}r = 0$, then φ has either no zeros or uncountably many zeros; in particular, $\varphi^{-1}(0) = \{\omega : \omega^3 = 2r\}$.*

Proof. Consider the zeon cubic $\varphi(u) = u^3 - 2r$.

(1) Clearly $\varphi(\eta) = \eta^3 = 0$ if and only if η is nilpotent of index 3 or less.

(2) If r is invertible, then $u^3 - 2r = 0$ if and only if u is a cube root of $2r$. There are three such zeros, one for each complex cube root of $\mathfrak{C}2r$.

(3) When r is nonzero and nilpotent, the zeros of $\varphi(u)$ are precisely the nilpotent cube roots of $2r$. As seen in Section 1.4.1, $2r$ has either no cube roots or uncountably many of them. ☐

Corollary 3.14. *Let $\varphi(u) = u^3 - a\zeta_I \in \mathbb{C}\mathfrak{Z}[u]$, where $a \neq 0$ and $|I| \geq 3$. It follows that $\varphi(u) = 0$ has $\left\{ {|I| \atop 3} \right\}$ flat form solutions of the form*

$$u_\pi = \sqrt[3]{\frac{a}{6}} \sum_{J \in \pi} \zeta_J,$$

where π ranges over the 3-block partitions of the multi index I.

Proof. Proceeding as in the proof of Corollary 3.12, let π be a 3-block partition of I. Let J, K, L be the blocks of partition π. It follows that

$$u_\pi = \sqrt[3]{\frac{a}{6}}(\zeta_J + \zeta_K + \zeta_L),$$

so that

$$\varphi(u_\pi) = \left(\sqrt[3]{\frac{a}{6}}(\zeta_J + \zeta_K + \zeta_L)\right)^3 - a\zeta_I = \frac{a}{6}6\zeta_J\zeta_K\zeta_L - a\zeta_I = a\zeta_I - a\zeta_I = 0.$$

The number of three block partitions π of I is $\left\{ {|I| \atop 3} \right\}$, so the result follows from Lemma 1.13. ☐

Example 3.15. *Consider the cubic polynomial*

$$\varphi(u) = u^3 + u^2\left(3 - \zeta_{\{1\}}\right) + u\left(-\zeta_{\{1,2\}} - 2\zeta_{\{1\}} + 3\right) + 1 - \zeta_{\{1\}} - \zeta_{\{1,2\}}.$$

Writing $\varphi(u) = u^3 + \alpha u^2 + \beta u + \gamma$*, let* $q = \left(\frac{\beta}{3} - \frac{\alpha^2}{9}\right)$ *and let* $r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3$*, so that* $\varphi(u) = u^3 + 3qu - 2r$*. It follows that* $\zeta_{[q]} = \zeta_{\{1,2\}}$ *and that* q *has spike form fundamental square roots* $\zeta_{\{1\}} - \frac{3}{2}\zeta_{\{2\}}$ *and* $\zeta_{\{2\}} - \frac{3}{2}\zeta_{\{1\}}$*, the (uncountably many) zeros of* $\varphi(u)$ *include the following:*

$$\begin{aligned} u_1 &= -1 + \frac{1}{3}\zeta_{\{1\}} + \zeta_{\{1,2\}}, \\ u_2 &= -1 + \frac{4}{3}\zeta_{\{1\}} - \frac{3}{2}\zeta_{\{2\}}, \\ u_3 &= -1 - \frac{7}{6}\zeta_{\{1\}} + \zeta_{\{2\}}. \end{aligned}$$

These zeros are easily confirmed by evaluating the polynomial.

3.3 A general cubic formula

For convenience in symbolic computation, a general cubic formula is now obtained as a corollary of Theorem 3.2.

Theorem 3.16 (General Zeon Cubic Formula)**.** *Let* $\varphi(u) = u^3 + \alpha u^2 + \beta u + \gamma \in \mathbb{C}\mathfrak{Z}[u]$*, let* $q = \frac{1}{3}\beta - \frac{1}{9}\alpha^2$ *and let* $r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3$*. Suppose* $\mathfrak{C}q \neq 0$ *and set* $\Delta_\varphi = q^3 + r^2$*. Suppose* Δ_φ *has a square root* δ*. Letting* $s_1 \in (r+\delta)^{1/3}$ *and* $s_2 \in (r-\delta)^{1/3}$*, it follows that* $\varphi(u)$ *has zeros given by*

$$\begin{aligned} u_1 &= (s_1 + s_2) - \frac{\alpha}{3}, \\ u_2 &= -\frac{1}{2}(s_1 + s_2) - \frac{\alpha}{3} + \frac{i\sqrt{3}}{2}(s_1 - s_2), \\ u_3 &= -\frac{1}{2}(s_1 + s_2) - \frac{\alpha}{3} - \frac{i\sqrt{3}}{2}(s_1 - s_2). \end{aligned}$$

Proof. First, the general cubic equation $\varphi(u) = u^3 + \alpha u^2 + \beta u + \gamma = 0$ is depressed by the substitution $u \mapsto z - \alpha/3$ as follows

$$\begin{aligned} \varphi(z - \alpha/3) &= (z - \alpha/3)^3 + \alpha(z - \alpha/3)^2 + \beta(z - \alpha/3) + \gamma \\ &= z^3 + \left(\beta - \frac{\alpha^2}{3}\right)z + \frac{2\alpha^3}{27} - \frac{\alpha\beta}{3} + \gamma \\ &= z^3 + \left(\beta - \frac{\alpha^2}{3}\right)z - 2\left(\frac{\alpha\beta}{6} - \frac{\alpha^3}{27} - \frac{\gamma}{2}\right). \end{aligned}$$

Since $\mathfrak{C}q \neq 0$, the zeros of $\varphi(z - \alpha/3)$ are given by the depressed cubic formula of Theorem 3.2. In particular, the zeros are given by

$$z = A^{1/3} - \left(\frac{\beta}{3} - \frac{\alpha^2}{9}\right) A^{-1/3},$$

corresponding to the cube roots of

$$A = \left(\frac{\alpha\beta}{6} - \frac{\gamma}{2} - \frac{\alpha^3}{27}\right) \pm \sqrt{\left(\frac{\beta}{3} - \frac{\alpha^2}{9}\right)^3 + \left(\frac{\alpha\beta}{6} - \frac{\alpha^3}{27} - \frac{\gamma}{2}\right)^2}.$$

Letting $q = \beta/3 - \alpha^2/9$ and $r = (\alpha\beta - 3\gamma)/6 - \alpha^3/27$, we have $z = A^{1/3} - qA^{-1/3}$, where

$$A = r \pm \sqrt{q^3 + r^2}.$$

Letting δ be a square root of $\Delta_\varphi = q^3 + r^2$, we have $A = r \pm \delta$. Next, observe that $(r + \delta)(r - \delta) = r^2 - \delta^2 = -q^3$, so that

$$(r + \delta)^{-1} = -(r - \delta)q^{-3}.$$

It follows that $qA^{-1/3} = -(r - \delta)^{1/3}$. Hence, the first zero of the depressed cubic is $z_1 = s_1 + s_2$, where $s_1 = (r + \delta)^{1/3}$ and $s_2 = (r - \delta)^{1/3}$. Letting x_0 be a fixed cube root of A, it follows that $e^{i2\pi/3}x_0$ and $e^{i4\pi/3}x_0$ are the remaining cube roots, where $e^{i4\pi/3} = (e^{i2\pi/3})^{-1}$. Thus, the remaining zeros of the depressed cubic are

$$z_2 = e^{i2\pi/3}s_1 + e^{i4\pi/3}s_2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) s_1 + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) s_2 = -\frac{1}{2}(s_1 + s_2) + i\frac{\sqrt{3}}{2}(s_1 - s_2)$$

and

$$z_3 = e^{i4\pi/3}s_1 + e^{i2\pi/3}s_2 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) s_1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) s_2 = -\frac{1}{2}(s_1 + s_2) - i\frac{\sqrt{3}}{2}(s_1 - s_2).$$

Translating by $\alpha/3$ gives the zeros $u_j = z_j - \alpha/3$ of the general cubic for $j = 1, 2, 3$. □

4 Classification

As we have seen since beginning with Example 3.1, there can be multiple possible approaches to finding solutions of zeon cubic equations. It would be helpful to have a method for determining which methods are appropriate for a given zeon cubic. For that, we turn to a zeon extension of the cubic discriminant.

We recall that given a monic cubic polynomial (with real coefficients) $f(x) = x^3+ax^2+bx+c \in \mathbb{R}[x]$, the *cubic discriminant* of $f(x)$ is defined to be

$$\Delta_f = 18abc - 4a^3b + a^2b^2 - 4b^3 - 27c^2. \tag{4.1}$$

Letting $q = \frac{b}{3} - \frac{a^2}{9}$ and $r = \frac{1}{6}(ab - 3c) - \frac{a^3}{27}$, the discriminant is given by

$$\Delta_f = -4(3q)^3 - 27(2r)^2 = -108(q^3 + r^2).$$

Traditionally, the cubic discriminant is used to characterize the zeros of $f(x)$. In particular, the following properties are well known.

- When $\Delta_f = 0$, the cubic has a repeated root.
- When $\Delta_f < 0$, the cubic has three distinct real roots.
- When $\Delta_f > 0$, the cubic has one real root and a conjugate pair of complex roots.

We extend the cubic discriminant to zeon cubic polynomials by defining $\Delta_\varphi = q^3 + r^2$. In view of Theorems 3.2, and 3.5, the following classification is sensible for cubic polynomials over $\mathbb{C}\mathfrak{Z}$.

Theorem 4.1 (Classification). *Let* $\varphi(u) = u^3 + \alpha u^2 + \beta u + \gamma \in \mathbb{C}\mathfrak{Z}[u]$. *Let* $\Delta_\varphi = q^3 + r^2$, *where*

$$q = \frac{1}{3}\beta - \frac{1}{9}\alpha^2, \quad r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3.$$

Then the following hold.

(1) *If* $\mathfrak{C}\Delta_\varphi \neq 0$, *then* φ *has three spectrally simple zeros. When* $\mathfrak{C}q \neq 0$, *the zeros are given by the cubic formula of Theorem 3.16. When* $\mathfrak{C}q = 0$, *the zeros are obtained from Algorithm 1 using the scalar zeros of* f_φ.

(2) *If* $\mathfrak{C}\Delta_\varphi = 0$, *then* φ *either has no zeros or has uncountably many zeros.*

Proof. Observing that $\Delta_{f_\varphi} = -108\mathfrak{C}\Delta_\varphi$, we see that the scalar polynomial f_φ has three distinct complex zeros when the discriminant is nonzero. Hence, φ has three spectrally simple zeon zeros when $\mathfrak{C}\Delta_\varphi \neq 0$.

It is clear that $\mathfrak{C}\Delta_\varphi = 0$ implies $\mathfrak{C}q \neq 0 \Leftrightarrow \mathfrak{C}r \neq 0$. It follows that the induced complex polynomial $f_\varphi(z - \mathfrak{C}\alpha/3) = z^3 + 3\mathfrak{C}qz - 2\mathfrak{C}r$ has a repeated root, λ_0. Thus, φ has no zeros or uncountably many zeros. If the repeated root λ_0 has multiplicity 2, there exists a spectrally simple zero μ of $\varphi(u)$ and uncountably many other zeros having common scalar part $\lambda_0 - \mathfrak{C}\alpha/3$. If λ_0 is a zero of multiplicity three and φ has zeros, then all zeros of $\varphi(u)$ have common scalar part $\lambda_0 - \mathfrak{C}\alpha/3$. $\square$

Example 4.2. *Consider the zeon cubic polynomial* $\varphi(u) = u^3 + \zeta_{\{1,2\}}u - (1 + 3\zeta_{\{2\}})$. *The zeon cubic discriminant of* φ *is* $\Delta_\varphi = \frac{1}{4} + \zeta_{\{2,3\}}$, *which is invertible. Hence,* φ *has three spectrally simple zeros. However, since* $q = \frac{1}{3}\zeta_{\{1,2\}}$ *is nilpotent, the cubic formula of Theorem 3.2 fails.*

The scalar zeros of $f_\varphi(z) = z^3 - 1$ *are* $\left\{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i\right\}$. *Applying Algorithm 1, rational approximations of the spectrally simple zeros are as follows:*

$$\lambda_1 = 1 - \frac{1}{3}\zeta_{\{1,2\}} + \frac{2}{3}\zeta_{\{2,3\}},$$
$$\lambda_2 = \left(-\frac{1}{2} + \frac{181i}{209}\right) + \left(\frac{1}{6} + \frac{125i}{433}\right)\zeta_{\{1,2\}} - \left(\frac{1}{3} - \frac{153i}{265}\right)\zeta_{\{2,3\}},$$
$$\lambda_3 = \left(-\frac{1}{2} - \frac{181i}{209}\right) + \left(\frac{1}{6} - \frac{153i}{530}\right)\zeta_{\{1,2\}} - \left(\frac{1}{3} + \frac{153i}{265}\right)\zeta_{\{2,3\}}.$$

4.1 Cubic polynomials over $\mathbb{C}\mathfrak{Z}_2$

In this section, the special case of cubic polynomials over $\mathbb{C}\mathfrak{Z}_2$ are considered. When φ is a cubic polynomial in $\mathbb{C}\mathfrak{Z}_2[u]$, there are no indeterminate cases.

Proposition 4.3. *Let* $\varphi(u) = u^3 + 3qu - 2r \in \mathbb{C}\mathfrak{Z}_2[u]$. *Let* $\Delta_\varphi = q^3 + r^2$, *where*

$$q = \frac{1}{3}\beta - \frac{1}{9}\alpha^2, \quad r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3.$$

(1) *If* Δ_φ *is invertible, then* $\varphi(u)$ *has three spectrally simple zeon zeros. The zeros are given by the cubic formula of Theorem 3.2 if* $\mathfrak{C}q \neq 0$. *Otherwise, the zeros are obtained from Algorithm 1 using the scalar zeros of* f_φ.

(2) *If* Δ_φ *is a nonzero null monomial of grade 2, then*

   (a) $\varphi(u)$ *has one spectrally simple zero and a set of spectrally non-simple zeros if* q *is invertible;*

   (b) $\varphi(u)$ *has no zeros if* q *is a nonzero nilpotent in* $\mathbb{C}\mathfrak{Z}_2$.

(3) *If* Δ_φ *is a nonzero nilpotent of minimal grade 1, then* $\varphi(u)$ *has no zeros.*

(4) *If* $\Delta_\varphi = 0$, *then*

   (a) $\varphi(u)$ *has a spectrally simple zeon zero and a set of spectrally non-simple zeon zeros if* r *is invertible;*

   (b) $\varphi(u)$ *has a set of spectrally non-simple zeros if* $r = a\zeta_{[2]}$ *for* $a \in \mathbb{C}$;

   (c) $\varphi(u)$ *has no zeros if* $r \neq 0$ *is nilpotent and not a null monomial of grade 2.*

Proof. The results follow from Theorems 3.2 and 3.5 along with the following observations.

(1) Nilpotent cube roots do not exist in $\mathbb{C}\mathfrak{Z}_2$.

(2) In $\mathbb{C}\mathfrak{Z}_2$, nilpotent square roots only exist for null monomials $a\zeta_{[2]}$.

To prove 2(b), suppose $\Delta_\varphi = a\zeta_{[2]}$ for nonzero $a \in \mathbb{C}$. If q is a nonzero nilpotent in $\mathbb{C}\mathfrak{Z}_2$, then $q^3 = 0$ so that $\Delta_\varphi = r^2$. It follows that $r = b\zeta_{\{1\}} + c\zeta_{\{2\}}$ for nonzero $b, c \in \mathbb{C}$. Any zeros of φ must also be nilpotent. Hence, any zero $z \in \mathbb{C}\mathfrak{Z}_2$ must satisfy

$$\varphi(z) = z^3 + 3qz - 2r = 3qz - 2(b\zeta_{\{1\}} + c\zeta_{\{2\}}) = 0,$$

where the minimal grade of qz is either 0 or 2. In either case, we have a contradiction.

Part 3 follows from the fact that a nilpotent of minimal grade 1 has no square roots.

Next, 4(b) is established as follows. If $r = a\zeta_{[2]}$ and $\Delta_\varphi = 0$, then $\mathfrak{C}q = 0$ so that $q^3 = 0$. If $q = s\zeta_i$ for any nonzero s, then

$$\varphi\left(\frac{2a}{3s}\zeta_{[2]\setminus\{i\}}\right) = \left(\frac{2a}{3}\zeta_{[2]\setminus\{i\}}\right)^3 + 3s\zeta_{\{i\}}\left(\frac{2a}{3s}\zeta_{[2]\setminus\{i\}}\right) - 2a\zeta_{[2]} = 0 + 2a\zeta_{[2]} - 2a\zeta_{[2]} = 0.$$

Turning to 4(c), suppose $r = a\zeta_{\{1\}} + b\zeta_{\{2\}}$ where $a, b \in \mathbb{C}$ are not both zero. If a and b are both nonzero, then $r^2 = 2ab\zeta_{[2]}$. Thus $\Delta_\varphi = 0$ requires $q^3 = -r^2$, which is impossible in $\mathbb{C}\mathfrak{Z}_2$. We conclude that $r = a\zeta_{\{i\}}$ for nonzero $a \in \mathbb{C}$ and $i \in \{1, 2\}$; further, we see that q is nilpotent. Hence, if $z \in \mathbb{C}\mathfrak{Z}_2$ is a zero of φ, it follows that

$$\varphi(z) = z^3 + 3qz - 2r = 3qz - 2a\zeta_{\{i\}} = 0,$$

where the minimal grade of qz is either 0 or 2. Again, we have a contradiction. □

5 Conclusion & avenues for further research

Zeros of cubic polynomials over $\mathbb{C}\mathfrak{Z}$ have been classified up to two indeterminate cases. In those indeterminate cases, sufficient conditions have been provided for existence of spectrally nonsimple zeon zeros. In the special case of cubic polynomials over $\mathbb{C}\mathfrak{Z}_2$, the zeros have been completely classified.

One obvious goal of future work is the consideration of zeros of quartic zeon polynomials over $\mathbb{C}\mathfrak{Z}$, particularly since the quartic is the highest order polynomial equation that can be solved by radicals in the general case. Based on existing results, a quartic polynomial $\varphi(u) = u^4 + \alpha u^3 + \beta u^2 + \gamma u + \delta$ having one spectrally simple zeon zero ω can be reduced by polynomial division to the product

$(u-\omega)\psi(u)$, where $\psi(u)$ is a monic cubic polynomial in $\mathbb{C}\mathfrak{Z}[u]$. The classification of cubic zeros established here can then be applied to $\psi(u)$. If $\varphi(u)$ has two simple zeros, the zeon quadratic formula can be applied to the remaining factor. If $\varphi(u)$ splits, all zeros can be found using Algorithm 1. If all zeros of $\varphi(u)$ are spectrally nonsimple, additional tools are needed: either an effective algorithm for computing spectrally nonsimple zeros or a zeon extension of the quartic formula.

More broadly, zeros of zeon polynomials are essential for considering spectral properties of zeon matrices. Letting Ψ denote an $m \times m$ matrix with entries from $\mathbb{C}\mathfrak{Z}$, eigenvalues of Ψ are spectrally simple zeon zeros of the characteristic polynomial of Ψ. Here Ψ is appropriately regarded as a $\mathbb{C}\mathfrak{Z}$-linear operator on the module $\mathbb{C}\mathfrak{Z}^m$. The zeon combinatorial Laplacian has recently been shown to enumerate paths and cycles in finite graphs, so its spectral properties are of particular interest [12]. With zeon eigenvalues in hand Putzer's theorem can also be useful for computing zeon matrix exponentials.

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