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A note on constructing sine and cosine functions in discrete fractional calculus

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ABSTRACT

In this paper, we introduce two sets of linear fractional order h-difference equations and derive their solutions. These solutions, referred to as trigonometric functions of fractional h-discrete calculus, are proven to have properties similar to sine and cosine functions on $\mathbb{R}$. The illustrated graphs confirm these similarities.

RESUMEN

En este artículo, introducimos dos conjuntos de ecuaciones de h-diferencias lineales de orden fraccionario y derivamos sus soluciones. Probamos que estas soluciones, referidas como funciones trigonométricas del cálculo fraccionario h-discreto, tienen propiedades similares a las funciones seno y coseno en $\mathbb{R}$. Las gráficas ilustradas confirman estas similaridades.

Keywords and Phrases: Discrete trigonometric functions, nabla operator, fractional h-discrete calculus, fractional difference equations, Picard's iteration.

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1 Introduction

The linear second order differential equation

$$y''(t) + \omega^2 y(t) = 0,$$

where $t \in \mathbb{R}$ and ω is a nonzero real number, produces two linearly independent solutions. They are well-known trigonometric functions, $\sin(\omega t) = \sum_{n=0}^{\infty}(-1)^n \omega^{2n+1} \frac{t^{2n+1}}{(2n+1)!}$ and $\cos(\omega t) = \sum_{n=0}^{\infty}(-1)^n \omega^{2n} \frac{t^{2n}}{(2n)!}$. Picard's iteration method is one of the fundamental methods in applied mathematics to construct these infinite series. Motivated by this construction technique, we will use calculus on the discrete time domain $h\mathbb{N}_a = \{a, a+h, a+2h, \dots\}$, where $a \in \mathbb{R}$ and $h \in \mathbb{R}^+$, to derive corresponding sum representations for sine and cosine functions of h-discrete fractional calculus.

Discrete fractional calculus, also known as non-integer order calculus on a discrete domain, has garnered significant attention from mathematicians over the past decade. It offers a novel approach to analyzing differences (derivatives) and sums (integrals) of arbitrary (non-integer) orders within discrete settings. A recent book by Goodrich and Peterson [6] provides a comprehensive collection of pioneering results for discrete fractional calculus, with a particular focus on the case where $h = 1$. In particular, the results obtained on the domain $h\mathbb{N}_a$ extend the findings of fractional discrete calculus on $\mathbb{N}_a$. This generalization provides a more comprehensive framework for understanding and applying fractional calculus within discrete settings. For further reading on this generalized domain, we refer the reader to the following papers [1–3, 7–15].

In this article, our goal is to introduce and derive solutions for the following two sets of linear fractional order h-difference equations

$$\nabla^{\alpha}_{h,a} y(t) + \omega^2 y(t-h) = 0, \tag{1.1}$$

and

$$\nabla^{\alpha}_{h,a} y(t) + \omega^2 y(t) = 0, \tag{1.2}$$

where $t \in h\mathbb{N}_a$, $1 < \alpha < 2$. The equation (1.1) includes a time delay, while the equation (1.2) is formulated without a time delay. We also note that in [1] the authors proved that $\nabla^{\alpha}_{h} y(t)$ is getting close to $y''(t)$ when $\alpha \to 2$ and $h \to 0$. Hence in the limit position, the Eq. (1.2) is approximating to the second order differential equation, $y''(t) + \omega^2 y(t) = 0$.

The following theorem, found in [3], presents the solution to the fractional h-difference equation in terms of Mittag-Leffler type functions. A natural question arises: Do sine and cosine functions appear in this solution when $k = 2$? To explore this, we apply Picard's iteration method to derive the sine and cosine functions of fractional h-discrete calculus.

Theorem 1.1 ([3])**.** *Let $\lambda \in \mathbb{R}$, $h > 0$, $k \in N$, and $\alpha \in (k-1, k)$. The general solution of the following problem*

$$\nabla^{\alpha}_{h,0} y(t) = \lambda y(t-h), \quad t \in h\mathbb{N}_{kh}, \tag{1.3}$$

is given by

$$y(t) = C_1 \tilde{E}^h_{\lambda,\alpha,\alpha-1}(t,0) + C_2 \tilde{E}^h_{\lambda,\alpha,\alpha-2}(t,0) + \cdots + C_k \tilde{E}^h_{\lambda,\alpha,\alpha-k}(t,0),$$

where $C_1, C_2, \dots, C_k$ are constants.

This paper is organized by following the outline given below. In order to make our calculations easy to follow, we provide basic definitions in h-discrete fractional calculus and related results in the preliminary section. We use Riemann-Liouville definition for the fractional derivative. Additionally, we develop techniques to convert Eqs. (1.1) and (1.2) into sum equations to apply Picard's iteration. Section 3 focuses on Eq. (1.1), where we define a iteration formula and derive two finite sums as solutions, illustrating their graphs for various values of α between one and two. Building on Section 3, we define two infinite series and state a theorem that outlines their properties and shows them as solutions to Eq. (1.2) in Section 4. Finally, we give a short concluding remark.

2 Preliminaries

Let h be any positive real number and a be any real number. We define $h\mathbb{N}_a$ to be the set $\{a, a+h, a+2h, \dots\}$. Suppose $F : h\mathbb{N}_a \to \mathbb{R}$ is a function.

Definition 2.1 ([5])**.** *The forward and the backward h-difference operator are defined by*

$$\Delta_h F(t) = \frac{F(t+h) - F(t)}{h}, \quad t \in h\mathbb{N}_a,$$

and

$$\nabla_h F(t) = \frac{F(t) - F(t-h)}{h}, \quad t \in h\mathbb{N}_{a+h},$$

respectively.

Remark 2.2. *Throughout this paper, we suggest that the reader considers the following:*

(i) if $h = 1$, we have the backward difference operator, or nabla operator (∇)

$$\nabla F(t) = F(t) - F(t-1), \quad t \in \mathbb{N}_{a+1};$$

(ii) if $\lim\limits_{h\to 0} \dfrac{F(t) - F(t-h)}{h}$ exists, then we have $\lim\limits_{h\to 0} \nabla_h F(t) = F'(t)$.

Definition 2.3 ([3]). *For any $t,\ r \in \mathbb{R}$, the h-rising factorial function is defined by*

$$t_h^{\overline{r}} = h^r \frac{\Gamma(\frac{t}{h} + r)}{\Gamma(\frac{t}{h})},$$

whenever the quotient is well-defined. Here $\Gamma(\cdot)$ denotes the Euler gamma function.

Definition 2.4 ([3]). *Let $a \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. The nabla h-fractional sum of order α is defined by*

$$\nabla_{h,a}^{-\alpha} F(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a/h}^{t/h} (t - \rho(sh))_h^{\overline{\alpha-1}} F(sh)h, \quad t \in h\mathbb{N}_a,$$

where $h \in \mathbb{R}^+$ and $\rho(t) = t - h$.

Definition 2.5 ([3]). *The nabla h-fractional difference of order α in the sense of Riemann–Liouville is defined by*

$$\nabla_{h,a}^{\alpha} F(t) := \nabla_h^n \nabla_{h,a}^{-(n-\alpha)} F(t), \quad t \in h\mathbb{N}_{a+nh},$$

where $a \in \mathbb{R}$, $n - 1 < \alpha \leq n$, and $n \in \mathbb{N}$.

Lemma 2.6 ([3]). *Let $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$ such that $\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}$ and $\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}$ are defined. Then we have that*

(i) $\nabla_{h,a}^{-\alpha} (t - \rho(a))_h^{\overline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (t - \rho(a))_h^{\overline{\beta+\alpha}}, \quad t \in h\mathbb{N}_a.$

(ii) $\nabla_{h,a}^{\alpha} (t - \rho(a))_h^{\overline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t - \rho(a))_h^{\overline{\beta-\alpha}}, \quad t \in h\mathbb{N}_a.$

In the following sections, we use Lemma 2.6 as one of the main tools to obtain some important identities. We want to note that $\dfrac{1}{\Gamma(-n)}$ for $n \in \mathbb{N}_0$ is considered as zero. The proof of the next lemma is elementary. We omit the proof.

Lemma 2.7. *Let $b \in hN_a$. The following is valid:*

$$\nabla_h (b - t)_h^{\overline{\beta}} = -\beta (b - \rho(t))_h^{\overline{\beta-1}}, \quad t \in h\mathbb{N}_a.$$

The following equality is known as Leibniz's rule for the nabla difference operator. The proof can be adapted from its proof in time scales calculus [5].

Lemma 2.8. *For a function $G : h\mathbb{N}_a \times \mathbb{N} \to \mathbb{R}$, the following is valid:*

$$\nabla_h \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} G(t, sh)h = \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} \nabla_h G(t, sh)h + G(\rho(t), t),$$

where $t \in h\mathbb{N}_a$.

Next, we demonstrate how ∇_h and $\nabla_{h,a}^{-\alpha}$ commute in a theorem.

Theorem 2.9. *For any positive real number α, the following equality holds:*

$$\nabla_{h,a+h}^{-\alpha}\nabla_h f(t) = \nabla_h \nabla_{h,a}^{-\alpha} f(t) - \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a).$$

Proof. Using Lemma 2.7 and the summation by parts formula in h-discrete calculus, we have

$$\begin{aligned}
\nabla_{h,a+h}^{-\alpha}\nabla_h f(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=\frac{a+h}{h}}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{\alpha-1}} \nabla_h f(sh)h \\
&= \frac{(t-\rho(sh))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(sh)\Big|_{\frac{a}{h}}^{\frac{t}{h}} + \frac{(\alpha-1)}{\Gamma(\alpha)} \sum_{s=\frac{a+h}{h}}^{\frac{t}{h}} (t+h-\rho(sh))_h^{\overline{\alpha-2}} f(\rho(sh))h \\
&= h^{\alpha-1} f(t) - \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) + \frac{1}{\Gamma(\alpha-1)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-1} (t-\rho(sh))_h^{\overline{\alpha-2}} f(sh)h \\
&= -\frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) + \frac{1}{\Gamma(\alpha-1)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{\alpha-2}} f(sh)h \\
&= \nabla_h \nabla_{h,a}^{-\alpha} f(t) - \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a).
\end{aligned}$$

□

This result can be generalized for the operator ∇_h^n using the principle of mathematical induction.

Theorem 2.10. *Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$. The following equality holds.*

$$\nabla_{h,a+nh}^{-\alpha}\nabla_h^n f(t) = \nabla_h^n \nabla_{h,a}^{-\alpha} f(t) - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \nabla_h^{n-k-1} (t-kh-\rho(a))_h^{\overline{\alpha-1}} \nabla_h^k f(t)\Big|_{t=a+kh}.$$

Proof. The proof of the equality follows from Theorem 2.9 for $n = 1$ and the induction assumption for $n > 1$

$$\nabla_{h,a+nh}^{-\alpha}\nabla_h^n f(t) = \nabla_h^n \nabla_{h,a}^{-\alpha} f(t) - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \nabla_h^{n-k-1} (t-kh-\rho(a))_h^{\overline{\alpha-1}} \nabla_h^k f(t)\Big|_{t=a+kh}.$$

For $n + 1$, we have

$$\begin{aligned}
\nabla_{h,a+(n+1)h}^{-\alpha}\nabla_h^{n+1} f(t) &= \nabla_{h,a+nh+h}^{-\alpha}\nabla_h \nabla_h^n f(t) \\
&= \nabla_h \nabla_{h,a+nh}^{-\alpha}\nabla_h^n f(t) - \frac{(t-nh-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_h^k f(t)\Big|_{t=a+nh} \\
&= I.
\end{aligned}$$

Next we use the induction assumption on the quantity $\nabla_h \nabla_{h,a+nh}^{-\alpha} \nabla_h^n f(t)$ to obtain

$$
\begin{aligned}
I &= \nabla_h \left[\nabla_h^n \nabla_{h,a}^{-\alpha} f(t) - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \nabla_h^{n-k-1} (t-kh-\rho(a))_h^{\overline{\alpha-1}} \nabla_h^k f(t)\Big|_{t=a+kh} \right] \\
&- \frac{(t-nh-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_h^k f(t)\Big|_{t=a+nh} \\
&= \nabla_h^{n+1} \nabla_{h,a}^{-\alpha} f(t) - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \nabla_h^{n-k} (t-kh-\rho(a))_h^{\overline{\alpha-1}} \nabla_h^k f(t)\Big|_{t=a+kh} \\
&- \frac{(t-nh-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_h^k f(t)\Big|_{t=a+nh} \\
&= \nabla_h^{n+1} \nabla_{h,a}^{-\alpha} f(t) - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \nabla_h^{n-k} (t-kh-\rho(a))_h^{\overline{\alpha-1}} \nabla_h^k f(t)\Big|_{t=a+kh}.
\end{aligned}
$$

This completes the proof. □

We close the preliminary section with the following lemma. The identities we have in this lemma will be used in the following sections to shortened the quantities in our calculations.

Lemma 2.11. *Let $1 < \alpha < 2$. The following are valid.*

(i) $\nabla_{h,a}^{-(2-\alpha)} f(t)\Big|_{t=a} = h^{2-\alpha} f(a)$,

(ii) $\nabla_h \nabla_{h,a}^{-(2-\alpha)} f(t)|\Big|_{t=a+h} = h^{1-\alpha} \left[f(a+h) + (1-\alpha) f(a) \right].$

Proof. The proof of the part (i) follows from the definition of the fractional h-difference operator. Indeed we have,

$$
\begin{aligned}
\nabla_{h,a}^{-(2-\alpha)} f(t)\Big|_{t=a} &= \frac{1}{\Gamma(2-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{2-\alpha-1}} f(sh)h\Big|_{t=a} \\
&= \frac{h}{\Gamma(2-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{a}{h}} (a-\rho(sh))_h^{\overline{1-\alpha}} f(sh) \\
&= \frac{h}{\Gamma(2-\alpha)} (a-\rho(a))_h^{\overline{1-\alpha}} f(a) = h^{2-\alpha} f(a).
\end{aligned}
$$

For the proof of the part (ii), we use Lemma 2.8 as a tool. Hence we have

$$
\nabla_h \nabla_{h,a}^{-(2-\alpha)} f(t)|\Big|_{t=a+h} = \nabla_h \Big[\frac{1}{\Gamma(2-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{1-\alpha}} f(sh)h \Big]\Big|_{t=a+h}
$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(2-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}} \nabla_h [(t-\rho(sh))_h^{\overline{1-\alpha}}] f(sh) h \Big|_{t=a+h} + \\
&(t-h-\rho(t))_h^{\overline{1-\alpha}}] f(t) \Big|_{t=a+h} \\
&= \frac{h(1-\alpha)}{\Gamma(2-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{a}{h}+1} (a+h-\rho(sh))_h^{\overline{-\alpha}} f(sh) \\
&= \frac{h(1-\alpha)}{\Gamma(2-\alpha)} \Big[(2h)_h^{\overline{-\alpha}} f(a) + (h)_h^{\overline{-\alpha}} f(a+h) | \Big] \\
&= h^{1-\alpha} \left[f(a+h) + (1-\alpha) f(a) \right]. \qquad \square
\end{aligned}
$$

3 A fractional order h-difference equation with delay

Here we consider the following α-th order linear fractional h-difference equation

$$\nabla_h^\alpha y(t) + \omega^2 y(t-h) = 0, \tag{3.1}$$

where $1 < \alpha < 2$ and $\omega \in \mathbb{R}$.

Apply the operator $\nabla_{h,a+2h}^{-\alpha}$ to each side of the equation (3.1) to obtain

$$\nabla_{h,a+2h}^{-\alpha} \nabla_h^2 \nabla_{h,a}^{-(2-\alpha)} y(t) + \nabla_{h,a+2h}^{-\alpha} \omega^2 y(t-h) = 0.$$

Apply Theorem 2.10 to obtain

$$\nabla_h^2 \nabla_{h,a}^{-\alpha} \nabla_{h,a}^{-(2-\alpha)} y(t) - \sum_{k=0}^{1} \frac{\nabla_h^{1-k} (t-kh-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_h^k \nabla_{h,a}^{-(2-\alpha)} y(t)|_{t=a+kh} + \nabla_{h,a+2h}^{-\alpha} \omega^2 y(t-h) = 0.$$

Hence we have

$$
\begin{aligned}
\nabla_h^2 \nabla_{h,a}^{-\alpha} \nabla_{h,a}^{-(2-\alpha)} y(t) = &\frac{(t-\rho(a))_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} \nabla_{h,a}^{-(2-\alpha)} y(t)|_{t=a} + \frac{(t-h-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_h \nabla_{h,a}^{-(2-\alpha)} y(t)|_{t=a+h} \\
&- \omega^2 \nabla_{h,a+2h}^{-\alpha} y(t-h).
\end{aligned}
$$

It follows from Lemma 2.11 and the composition property for the fractional sum operators (Lemma 2 in [2]), we have

$$
\begin{aligned}
y(t) = &\frac{(t-\rho(a))_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} h^{2-\alpha} y(a) + \frac{(t-h-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h^{1-\alpha} \left[(1-\alpha) y(a) + y(a+h) \right] \\
&- \omega^2 \nabla_{h,a+2h}^{-\alpha} y(t-h).
\end{aligned} \tag{3.2}
$$

Conversely, assume that y has the representation (3.2). We first note that

$$\nabla_{h,a+2h}^{-\alpha}y(t) = \nabla_{h,a}^{-\alpha}y(t) - \frac{1}{\Gamma(\alpha)}(t-\rho(a+h)_h^{\overline{\alpha-1}}y(a+h)h - \frac{1}{\Gamma(\alpha)}(t-\rho(a))_h^{\overline{\alpha-1}}y(a)h.$$

In addition to the above equality, we use the power rule (Lemma 2.6) and the composition property for the fractional sum operators (Lemma 2 in [2]) to derive from (3.2) by applying the operator ∇_h^α to the each side of the equation to obtain

$$\begin{aligned}\nabla_h^\alpha y(t) =& \nabla_h^\alpha\Bigg[\frac{(t-\rho(a))_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}h^{2-\alpha}y(a) + \frac{(t-h-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}h^{1-\alpha}[(1-\alpha)y(a)+y(a+h)] \\ &- \omega^2\nabla_{h,a+2h}^{-\alpha}y(t-h)\Bigg] \\ &= \nabla_h^\alpha\left[\frac{(t-\rho(a))_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}h^{2-\alpha}y(a) + \frac{(t-h-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}h^{1-\alpha}\left[(1-\alpha)y(a)+y(a+h)\right]\right] \\ &- \omega^2\nabla_h^\alpha\Bigg[\nabla_{h,a}^{-\alpha}y(t-h) - \frac{1}{\Gamma(\alpha)}(t-h-\rho(a+h))_h^{\overline{\alpha-1}}y(a+h)h \\ &- \frac{1}{\Gamma(\alpha)}(t-h-\rho(a))_h^{\overline{\alpha-1}}y(a)h\Bigg] = -\omega^2 y(t-h).\end{aligned}$$

Thus, we have proved the following lemma.

Lemma 3.1. *y is a solution of the problem* (3.1)*, if and only if, y has the representation* (3.2)*.*

Next, for the simplicity in our calculations, we consider $a=0$. We define a sequence of functions on $h\mathbb{N}_0$ as follows:

$$\begin{aligned}y_0(t) &= \frac{(t+h)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}h^{2-\alpha}y(0) + \frac{(t)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}h^{1-\alpha}\left[(1-\alpha)y(0)+y(h)\right], \\ y_n(t) &= -\omega^2\nabla_{h,2h}^{-\alpha}y_{n-1}(t-h),\end{aligned}$$

for $n \in \mathbb{N}_1$.

Using this iteration formula along with the power rule (Lemma 2.6), we observe that $\sum_{n=0}^{\infty} y_n(t)$ truncates to the following finite sum

$$\begin{aligned}&h^{2-\alpha}y(0)\sum_{n=0}^{\frac{t}{h}}(-1)^n\omega^{2n}\frac{(t-(n-1)h)_h^{\overline{(n+1)\alpha-2}}}{\Gamma((n+1)\alpha-1)} \\ &\qquad + h^{1-\alpha}\left[(1-\alpha)y(0)+y(h)\right]\sum_{n=0}^{\frac{t}{h}}(-1)^n\omega^{2n}\frac{(t-nh)_h^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)}.\end{aligned}$$

To convince the reader, we present a few elements in the sequence $< y_n(t) >$ and explain why we have finite sum instead of infinite sum at the end of this process.

$$
\begin{aligned}
y_1(t) &= -\omega^2 \nabla_{h,2h}^{-\alpha} y_0(t-h) \\
&= -\omega^2 \nabla_{h,2h}^{-\alpha} \left[\frac{(t)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} h^{2-\alpha} y(0) + \frac{(t-h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h^{1-\alpha} \left[(1-\alpha)y(0) + y(h)\right] \right] \\
&= -\omega^2 \left[\frac{(t)_h^{\overline{2\alpha-2}}}{\Gamma(2\alpha-1)} h^{2-\alpha} y(0) + \frac{(t-h)_h^{\overline{2\alpha-1}}}{\Gamma(2\alpha)} h^{1-\alpha} \left[(1-\alpha)y(0) + y(h)\right] \right].
\end{aligned}
$$

We repeat this calculation for y_2 to obtain the general term y_n of the sequence. Our main tool is the power rule (Lemma 2.6).

$$
\begin{aligned}
y_1(t) &= -\omega^2 \nabla_{h,2h}^{-\alpha} y_1(t-h) \\
&= -\omega^2 \nabla_{h,2h}^{-\alpha} \left[-\omega^2 \left[\frac{(t-h)_h^{\overline{2\alpha-2}}}{\Gamma(2\alpha-1)} h^{2-\alpha} y(0) + \frac{(t-2h)_h^{\overline{2\alpha-1}}}{\Gamma(2\alpha)} h^{1-\alpha} \left[(1-\alpha)y(0) + y(h)\right] \right] \right] \\
&= \omega^4 \left[\frac{(t-h)_h^{\overline{3\alpha-2}}}{\Gamma(3\alpha-1)} h^{2-\alpha} y(0) + \frac{(t-2h)_h^{\overline{3\alpha-1}}}{\Gamma(3\alpha)} h^{1-\alpha} \left[(1-\alpha)y(0) + y(h)\right] \right].
\end{aligned}
$$

From this, the general term $y_n(t)$ follows.

$$
y_n(t) = (-1)^n \omega^{2n} \left[h^{2-\alpha} y(0) \frac{(t-(n-1)h)_h^{\overline{(n+1)\alpha-2}}}{\Gamma((n+1)\alpha-1)} + h^{1-\alpha} \left[(1-\alpha)y(0) + y(h)\right] \frac{(t-nh)_h^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)} \right].
$$

When we consider the infinite sum $\sum_{n=0}^{\infty} y_n(t)$, the terms with h-rising factorial powers become zero for $n > \frac{t}{h}$. Hence we obtain

$$
\begin{aligned}
&h^{2-\alpha} y(0) \sum_{n=0}^{\frac{t}{h}} (-1)^n \omega^{2n} \frac{(t-(n-1)h)_h^{\overline{(n+1)\alpha-2}}}{\Gamma((n+1)\alpha-1)} \\
&\qquad + h^{1-\alpha} \left[(1-\alpha)y(0) + y(h)\right] \sum_{n=0}^{\frac{t}{h}} (-1)^n \omega^{2n} \frac{(t-nh)_h^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)}.
\end{aligned}
$$

If we look closely at this sum which is the general solution of the fractional difference equation (3.1), we observe that there are two linearly independent solutions. Hence we define these two linearly independent solutions as sine and cosine functions.

We define

$$
C_h(t,\alpha,\omega) = \sum_{n=0}^{\frac{t}{h}} (-1)^n \omega^{2n} \frac{(t+h-nh)_h^{\overline{(n+1)\alpha-2}}}{\Gamma((n+1)\alpha-1)}
$$

and

$$S_h(t,\alpha,\omega)=\sum_{n=0}^{\frac{t}{h}}(-1)^n\omega^{2n+1}\frac{(t-nh)_h^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)},$$

$t\in h\mathbb{N}_0$. It turns out that $\tilde{E}^h_{-\omega^2,\alpha,\alpha-2}(t+h,0)=C_h(t,\alpha,\omega)$ and $\tilde{E}^h_{-\omega^2,\alpha,\alpha-1}(t,0)=S_h(t,\alpha,\omega)$ when we compare the above solutions with the solution representation in Theorem 1.1.

Next we list some properties of these functions.

Theorem 3.2. *The following equalities are valid.*

(i) $\Delta_h S_h(t,\alpha,\omega)=\omega C_h(t,\alpha,\omega)$.

(ii) $h^{2-\alpha}C_h(0,\alpha,\omega)=1,\qquad S_h(0,\alpha,\omega)=0.$

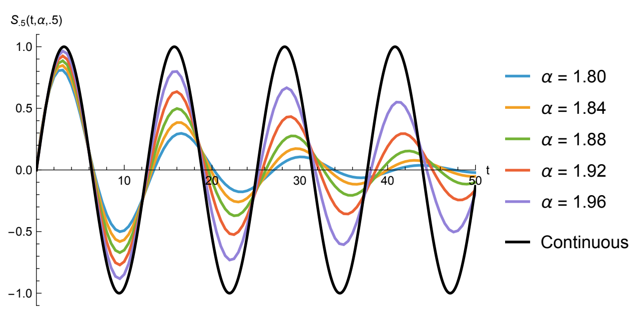
(iii) $\nabla^{\alpha}_{h,a}C_h(t,\alpha,\omega)+\omega^2C_h(t-h,\alpha,\omega)=0.$

(iv) $\nabla^{\alpha}_{h,a}S_h(t,\alpha,\omega)+\omega^2S_h(t-h,\alpha,\omega)=0.$

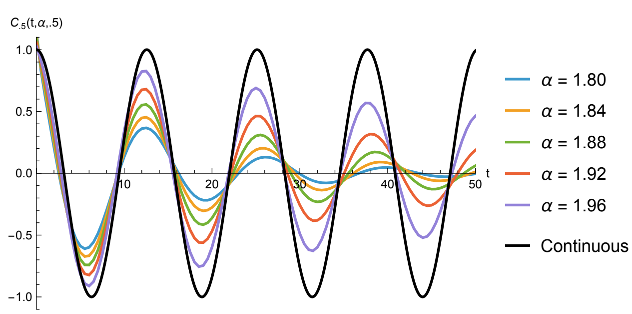
Proof. The proofs of (i) and (ii) are straightforward from the definitions of the functions C_h and S_h. The proofs of (iii) and (iv) can be found at Theorem 3.6 in [3]. □

Remark 3.3. *In Figure 1, we illustrate the graphs of $C_h(t,\alpha,\omega)$ and $S_h(t,\alpha,\omega)$ for a small value of h and for several α values between one and two.*

Figure 1: Family of graphs of $S_{.5}(t,\alpha,.5)$ and $C_{.5}(t,\alpha,.5)$.



(a) $S_{.5}(t,\alpha,.5)$



(b) $C_{.5}(t,\alpha,.5)$

4 A fractional order h-difference equation without delay

Here we consider the following α-th order linear fractional h-difference equation

$$\nabla_{h,a}^{\alpha} y(t) + \omega^2 y(t) = 0, \tag{4.1}$$

where $1 < \alpha < 2$ and $\omega \in \mathbb{R}$. We assume that $\omega^2 h^\alpha < 1$.

Here we define

$$Cos_h(t,\alpha,\omega) = (1+\omega^2 h^\alpha) \sum_{n=0}^{\infty} (-1)^n \omega^{2n} \frac{(t+h)_h^{\overline{(n+1)\alpha-2}}}{\Gamma((n+1)\alpha-1)}$$

and

$$Sin_h(t,\alpha,\omega) = (1+\omega^2 h^\alpha) \sum_{n=0}^{\infty} (-1)^n \omega^{2n+1} \frac{(t)_h^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)},$$

$t \in h\mathbb{N}_0$. These series are convergent when $\omega^2 h^\alpha < 1$.

Next we list some properties of these functions. We omit their proof since they are mainly relying on the power rule (Lemma 2.6).

Theorem 4.1. *The following equalities are valid.*

(i) $\Delta_h Sin_h(t,\alpha,\omega) = \omega Cos_h(t,\alpha,\omega)$.

(ii) $h^{2-\alpha} Cos_h(0,\alpha,\omega) = 1, \quad Sin_h(0,\alpha,\omega) = 0$.

(iii) $\nabla_{h,a}^{\alpha} Cos_h(t,\alpha,\omega) + \omega^2 Cos_h(t,\alpha,\omega) = 0$.

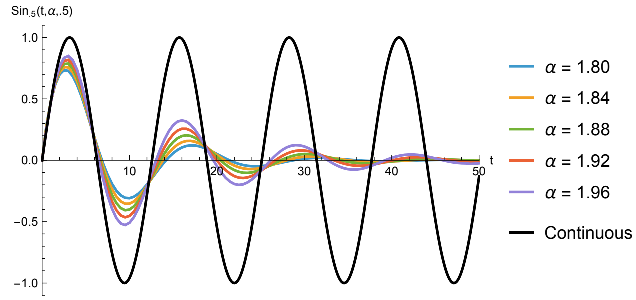
(iv) $\nabla_{h,a}^{\alpha} Sin_h(t,\alpha,\omega) + \omega^2 Sin_h(t,\alpha,\omega) = 0$.

Remark 4.2. *In Figure 2, we illustrate the graphs of $Cos_h(t,\alpha,\omega)$ and $Sin_h(t,\alpha,\omega)$ for a small value of h and for several α values between one and two.*

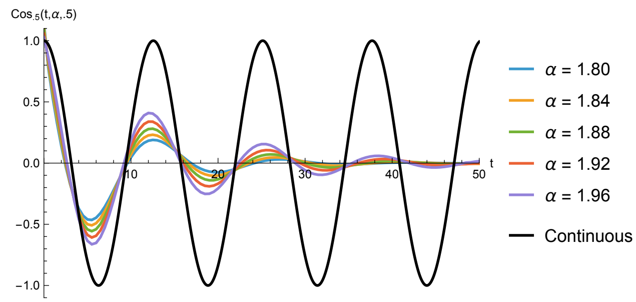
5 A concluding remark

The development of fractional calculus on the set $h\mathbb{N}_a = \{a, a+h, a+2h, \dots\}$ has shown promising results. In a recent paper [4], the pharmacokinetic (PK)-pharmacodynamic (PD) model was formulated on this time domain, with the PK component defined on an hourly basis and the PD component on a daily basis. h-discrete calculus offers the flexibility to select the right h values, enabling the construction of such an advanced model. Continuous improvement in existing models, whether in science, technology, or any other field, often hinges on the development of new theories and the refinement of analytical methods. Such a development of the theory starts with construction of the basic functions.

Figure 2: Family of graphs of $Sin_{.5}(t,\alpha,.5)$ and $Cos_{.5}(t,\alpha,.5)$.



(a) $Sin_{.5}(t,\alpha,.5)$



(b) $Cos_{.5}(t,\alpha,.5)$

In this article, we employed the widely recognized applied mathematics technique, Picard's iteration, to develop sine and cosine like functions within the framework of h-discrete fractional calculus. We constructed these functions as solutions to some linear fractional h-difference equations and illustrated their graphs. Sine and cosine functions as infinite sums can be calculated using a similar matrix method as in [2]. All these functions are potential candidates for application in various areas of mathematics. Deriving their analytical properties is just one of many open problems to explore.

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