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A sub-elliptic system with strongly coupled critical terms and concave nonlinearities

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ABSTRACT

In this work, we study the Nehari manifold and its application to the following sub-elliptic system involving strongly coupled critical terms and concave nonlinearities:

$$\begin{cases} -\Delta_{\mathbb{G}} u = \dfrac{\eta_1\alpha_1}{2^*}\,|u|^{\alpha_1-2}|v|^{\beta_1}u + \dfrac{\eta_2\alpha_2}{2^*}\,|u|^{\alpha_2-2}|v|^{\beta_2}u & \\ \qquad + \lambda\, g(z)\,|u|^{q-2}u, & z\in\Omega, \\ -\Delta_{\mathbb{G}} v = \dfrac{\eta_1\beta_1}{2^*}\,|u|^{\alpha_1}|v|^{\beta_1-2}v + \dfrac{\eta_2\beta_2}{2^*}\,|u|^{\alpha_2}|v|^{\beta_2-2}v & \\ \qquad + \mu\, h(z)\,|v|^{q-2}v, & z\in\Omega, \\ \quad u = v = 0, & z\in\partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{G}$ with smooth boundary, $-\Delta_{\mathbb{G}}$ is the sub-Laplacian on a Carnot group $\mathbb{G}$; η_1,η_2, λ,μ, are positive, $\alpha_1+\beta_1=2^*$, $\alpha_2+\beta_2=2^*$, $1<q<2$, $2^*=\frac{2Q}{Q-2}$ is the critical Sobolev exponent, and Q is the homogeneous dimension of $\mathbb{G}$. By exploiting the Nehari manifold and variational methods, we prove that the system has at least two positive solutions.

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RESUMEN

En este trabajo, estudiamos la variedad de Nehari y su aplicación al siguiente sistema sub-elíptico que involucra términos críticos fuertemente acoplados y nolinealidades cóncavas:

$$\begin{cases} -\Delta_{\mathbb{G}} u = \dfrac{\eta_1 \alpha_1}{2^*}\,|u|^{\alpha_1 - 2}|v|^{\beta_1} u + \dfrac{\eta_2 \alpha_2}{2^*}\,|u|^{\alpha_2 - 2}|v|^{\beta_2} u & \\ \qquad + \lambda\, g(z)\,|u|^{q-2}u, & z \in \Omega, \\ -\Delta_{\mathbb{G}} v = \dfrac{\eta_1 \beta_1}{2^*}\,|u|^{\alpha_1}|v|^{\beta_1 - 2} v + \dfrac{\eta_2 \beta_2}{2^*}\,|u|^{\alpha_2}|v|^{\beta_2 - 2} v & \\ \qquad + \mu\, h(z)\,|v|^{q-2}v, & z \in \Omega, \\ \quad u = v = 0, & z \in \partial\Omega, \end{cases}$$

donde Ω es un conjunto abierto acotado de $\mathbb{G}$ con frontera suave, $-\Delta_{\mathbb{G}}$ es el sub-Laplaciano en un grupo de Carnot $\mathbb{G}$; η_1, η_2, λ, μ, son positivas, $\alpha_1 + \beta_1 = 2^*$, $\alpha_2 + \beta_2 = 2^*$, $1 < q < 2$, $2^* = \frac{2Q}{Q-2}$ es el exponente crítico de Sobolev, y Q es la dimensión homogénea de $\mathbb{G}$. Usando la variedad de Nehari y métodos variacionales, demostramos que el sistema tiene al menos dos soluciones positivas.

Keywords and Phrases: Sub-Laplacian, concave-convex nonlinearities, strongly coupled critical terms, Nehari manifold.

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1 Introduction

In this paper, we are concerned with the sub-Laplacian system involving strongly coupled critical terms and concave nonlinearities on the Carnot group $\mathbb{G}$ given below

$$\begin{cases} -\Delta_{\mathbb{G}} u = \frac{\eta_1 \alpha_1}{2^*}|u|^{\alpha_1-2}|v|^{\beta_1}u + \frac{\eta_2 \alpha_2}{2^*}|u|^{\alpha_2-2}|v|^{\beta_2}u + \lambda g(z)|u|^{q-2}u, & z \in \Omega, \\ -\Delta_{\mathbb{G}} v = \frac{\eta_1 \beta_1}{2^*}|u|^{\alpha_1}|v|^{\beta_1-2}v + \frac{\eta_2 \beta_2}{2^*}|u|^{\alpha_2}|v|^{\beta_2-2}v + \mu h(z)|v|^{q-2}v, & z \in \Omega, \\ u = v = 0, & z \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{G}$ with smooth boundary, $-\Delta_{\mathbb{G}}$ is the sub-Laplacian on a Carnot group $\mathbb{G}$. λ, μ, are positive, $2^* = \frac{2Q}{Q-2}$ is the critical Sobolev exponent, and Q is the homogeneous dimension of $\mathbb{G}$. We consider the following conditions:

$(\mathcal{A}_0)$ $Q \geq 4$, $1 < q < 2$, $0 < \eta_i < \infty$, $\alpha_i, \beta_i > 1$ and $\alpha_i + \beta_i = 2^*$ $(i = 1, 2)$,

and we give the following assumptions on the weight functions g and h:

$(\mathcal{A}_1)$ $g, h \in L^{\frac{2^*}{2^*-q}}(\Omega)$, $g^{\pm} = \max\{\pm g, 0\} \neq 0$ in $\bar{\Omega}$ and $h^{\pm} = \max\{\pm h, 0\} \neq 0$ in $\bar{\Omega}$.

$(\mathcal{A}_2)$ There exist $a_0, r_0 > 0$ such that $B_d(0, r_0) \subset \Omega$ and $g(z), h(z) \geq a_0$ for all $z \in B_d(0, r_0)$.

Here $B_d(z, r)$ denotes the quasi-ball with center at z and radius r with respect to the gauge d.

$|u|^{\alpha_i-2}u|v|^{\beta_i}$ and $|u|^{\alpha_i}|v|^{\beta_i-2}v$, $i = 1, 2$ are called strongly coupled terms. We now recall some known results concerning the elliptic system involving the strongly coupled critical terms. When $\mathbb{G}$ is the ordinary Euclidean space $(\mathbb{R}^N, +)$, $\eta_1 = \eta_2 = 1$, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $g = h \equiv 1$, problem (1.1) becomes the following Laplacian elliptic system:

$$\begin{cases} -\Delta u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^{\beta}u + \lambda|u|^{q-2}u, & \text{in } \Omega, \\ -\Delta v = \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \mu|v|^{q-2}v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The authors in [10] proved that the system (1.2) admits at least two positive solutions. Later, Hsu [9] obtained the same results for the p-Laplacian elliptic system. There are other multiplicity results or critical elliptic equations involving concave–convex nonlinearities, see for example [1, 2]. Contrary to the nonlinear elliptic problem with the Laplacian or p-Laplacian in Euclidean space that have been widely investigated, the situation seems to be in a developing state for the sub-Laplacian problem on Carnot groups. Recently, great attention has been devoted to nonlinear elliptic problems involving critical nonlinearities, in the context of Carnot group, see for example [11, 13, 20] and references therein. To the best of our knowledge, there is no result so far concerning sub-elliptic system involving strongly coupled critical terms nonlinearities with sign-changing weight functions on Carnot group.

We look for weak solutions of (1.1) in the product space $\mathcal{H} := S_0^1(\Omega) \times S_0^1(\Omega)$, endowed with the norm

$$\|(u,v)\|_{\mathcal{H}} = \left(\|u\|^2_{S_0^1(\Omega)} + \|v\|^2_{S_0^1(\Omega)}\right)^{\frac{1}{2}}, \quad \forall (u,v) \in \mathcal{H},$$

where the Folland-Stein space $S_0^1(\Omega) = \{u \in L^2(\Omega) : \int_\Omega |\nabla_{\mathbb{G}} u|^2 dz < \infty\}$, is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\left(\|u\|_{S_0^1(\Omega)}\right) = \left(\int_\Omega |\nabla_{\mathbb{G}} u|^2 dz\right)^{\frac{1}{2}}, \quad \forall u \in S_0^1(\Omega).$$

By using the Nehari manifold and fibering map analysis, we establish the existence of at least two positive solutions for a sub-elliptic system (1.1) when (λ, μ) belongs to certain subset of $\mathbb{R}^2_+$. Since the embedding $S_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact, then the corresponding energy functional does not satisfy the Palais-Smale condition in general. Therefore, it is difficult to obtain the critical points of energy functional by simple arguments, which are based on the compactness of the Sobolev embedding. To overcome this difficulty, we extract a Palais-Smale sequence in the Nehari manifold and show that the weak limit of this sequence is the required solution of problem (1.1). The best constant of the Sobolev inequality was studied on graded groups in [15]. But in that paper, the best constant was expressed in variational form.

We consider the following scalar critical equation:

$$-\Delta_{\mathbb{G}} u = |u|^{2^*-2} u \quad \text{in } \mathbb{G}. \tag{1.3}$$

For equation (1.3), it is well known (see *e.g.* [3, 11]) that positive solutions have the following decay:

$$U(z) \sim \frac{C}{d(z)^{Q-2}} \quad \text{as } d(z) \to \infty, \tag{1.4}$$

where d is the gauge norm on $\mathbb{G}$. This result applies, in particular, to the extremals of the Sobolev inequality on Carnot groups (whose existence was proved in [8, 17], *i.e.*, to the functions U that achieve the best constant for the embedding $S_0^1(\mathbb{G}) \hookrightarrow L^{2^*}(\mathbb{G})$, that is,

$$S_{\mathbb{G}} := \inf_{u \in S_0^1(\mathbb{G}) \setminus \{0\}} \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 dz}{\left(\int_{\mathbb{G}} |u|^{2^*} dz\right)^{\frac{2}{2^*}}} = \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} U|^2 dz}{\left(\int_{\mathbb{G}} |U|^{2^*} dz\right)^{\frac{2}{2^*}}}.$$

We underline that the knowledge of the exact asymptotic behavior of Sobolev minimizers turns out to be a crucial ingredient in order to obtain existence results for Brézis-Nirenberg type problems, whenever the explicit form of Sobolev minimizers is not known, as in the present Carnot case. The knowledge of the behavior of Sobolev minimizers turns out to be crucial also for the system, due to the relation between the extremals for the best constant $S_{\eta,\alpha,\beta}$ associated to the system and the Sobolev constant $S_{\mathbb{G}}$ (see Theorem 2.1 below).

The energy functional $I_{\eta,\alpha,\beta} : \mathcal{H} \longrightarrow \mathbb{R}$ associated to (1.1) is given by

$$I_{\eta,\alpha,\beta}(u,v) = \frac{1}{2}\|(u,v)\|^2_{\mathcal{H}} - \frac{1}{2^*}K_\eta(u,v) - \frac{1}{q}\Psi_{\lambda,\mu}(u,v), \quad \forall (u,v) \in \mathcal{H},$$

where

$$K_\eta(u,v) = \int_\Omega \left(\eta_1|u|^{\alpha_1}|v|^{\beta_1} + \eta_2|u|^{\alpha_2}|v|^{\beta_2}\right) dz, \quad \Psi_{\lambda,\mu}(u,v) = \int_\Omega \left(\lambda g(z)|u|^q + \mu h(z)|v|^q\right) dz.$$

It is easy to check that $I_{\eta,\alpha,\beta} \in C^1(\mathcal{H},\mathbb{R})$ and the critical point of $I_{\eta,\alpha,\beta}$ is the weak solution of (1.1). We call a solution (u,v) positive if both u and v are positive, (u,v) is nontrivial if $u \not\equiv 0$ or $v \not\equiv 0$.

Definition 1.1. *A pair of functions $(u,v) \in \mathcal{H}$ is said to be a weak solution of problem* (1.1) *if*

$$\begin{aligned}\int_\Omega (\nabla u \nabla \phi + \nabla v \nabla \psi)\ \mathrm{d}x = &\int_\Omega \left(\frac{\eta_1\alpha_1}{2^*}|u|^{\alpha_1-2}|v|^{\beta_1}u\phi + \frac{\eta_2\alpha_2}{2^*}|u|^{\alpha_2-2}|v|^{\beta_2}u\phi\right) dx \\ &+ \int_\Omega \left(\frac{\eta_1\beta_1}{2^*}|u|^{\alpha_1}|v|^{\beta_1-2}v\psi + \frac{\eta_2\beta_2}{2^*}|u|^{\alpha_2}|v|^{\beta_2-2}v\psi\right) dx \\ &+ \int_\Omega \left(\lambda g(x)|u|^{q-2}u\phi + \mu h(x)|v|^{q-2}v\psi\right) dx \quad \text{for all } (\phi,\psi) \in \mathcal{H}.\end{aligned} \tag{1.5}$$

Define the set

$$\begin{aligned}\mathfrak{D}_\sigma &:= \left\{(\lambda,\mu) \in \mathbb{R}^+ \times \mathbb{R}^+ \backslash \{(0,0)\} : 0 < \mu\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \sigma\right\}, \quad \text{and} \\ \Lambda &:= \frac{2^*-2}{2^*-q}\left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)}\right)^{\frac{2-q}{2^*-2}} S_{\mathbb{G}}^{\frac{2^*-q}{2^*-2}}.\end{aligned} \tag{1.6}$$

So, the main result of this paper can be included in the following theorem.

Theorem 1.2. *Let $\mathbb{G}$ be a Carnot group. Assume that $(\mathcal{A}_0)$, $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold. Then, we have the following results:*

(i) *If $(\lambda,\mu) \in \mathfrak{D}_\Lambda$, then* (1.1) *has at least one positive solution in $\mathcal{H}$.*

(ii) *There exists a constant $\Lambda_* > 0$ such that system* (1.1) *has at least two distinct positive solutions in $\mathcal{H}$ for all $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_*}$.*

The paper is organized into three sections. In Section 2, we recall some basic definitions of Sobolev space on Carnot groups and we give some useful auxiliary lemmas. In Section 3, we investigate the Palais-Smale condition for the energy functional $I_{\eta,\alpha,\beta}$. Finally, the proof of Theorem 1.2 is given in Sections 4 and 5.

2 Preliminaries

In this section we recall some basic facts on the Carnot groups. For a complete treatment, we refer to the classical papers [6, 7]. We also quote for an overview on general homogeneous Lie group.

Let $\mathbb{G} = \left(\mathbb{R}^N, \circ\right)$ be a homogeneous group, *i.e.*, a Lie group equipped with a family $\{\delta_\gamma\}_{\gamma>0}$ of dilations, acting on $z \in \mathbb{R}^N$ as follows

$$\delta_\gamma\left(z^{(1)}, \ldots, z^{(r)}\right) = \left(\gamma^1 z^{(1)}, \gamma^2 z^{(2)}, \ldots, \gamma^r z^{(r)}\right),$$

where $z^{(k)} \in \mathbb{R}^{N_k}$ for every $k \in \{1, \ldots, r\}$ and $N = \sum_{k=1}^r N_k$. Then, the structure $\mathbb{G} := \left(\mathbb{R}^N, \circ, \{\delta_\gamma\}_{\gamma>0}\right)$ is called a homogeneous group with homogeneous dimension

$$Q := \sum_{k=1}^r k \cdot N_k.$$

Note that the number Q is naturally associated to the family $\{\delta_\gamma\}_{\gamma>0}$ since, for every $\gamma > 0$, the Jacobian of the map $z \mapsto \delta_\gamma(z)$ equals γ^Q. From now on, we shall assume throughout that $Q \geq 3$. We remark that, if $Q \leq 3$, then $\mathbb{G}$ is necessarily the ordinary Euclidean space $\mathbb{G} = \left(\mathbb{R}^Q, +\right)$.

Let $\mathfrak{g}$ be the Lie algebra of left invariant vector fields on $\mathbb{G}$ and assume that $\mathfrak{g}$ is stratified, *i.e.*, $\mathfrak{g} = \bigoplus_{k=1}^r V_k$ with $[V_1, V_k] = V_{k+1}$, for $1 \leq k \leq r-1$ and $[V_1, V_r] = \{0\}$. Under these assumptions, we call $\mathbb{G}$ a Carnot group. Here the integer r is called the step of $\mathbb{G}$, $\dim(V_k) = N_k$ and the symbol $[V_1, V_k]$ denotes the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$, where $X \in V_1$ and $Y \in V_k$. Let $X = \{X_1, X_2, \ldots, X_m\}$ be a basis of V_1 with $m = \dim(V_1)$. From Proposition 1.2.29 of [14], the left invariant vector field X_i $(k = 1, \ldots, m)$ has an explicit form as follows:

$$X_i = \frac{\partial}{\partial x_i^{(1)}} + \sum_{l=2}^{k} \sum_{r=1}^{\dim(V_l)} a_{i,r}^{(l)}\left(x^{(1)}, \ldots, x^{(l-1)}\right) \frac{\partial}{\partial x_r^{(l)}},$$

where $a_{i,r}^{(l)}$ is a homogeneous (with respect to δ_γ) polynomial function of degree $l-1$. Then, once a basis $X_1, X_2, \ldots, X_m$ of the horizontal layer is fixed, we define, for any function $u : \mathbb{G} \to \mathbb{R}$ for which the partial derivatives $X_j u$ exist, the horizontal gradient of u, denoted by $\nabla_{\mathbb{G}} u$, as the horizontal section

$$\nabla_{\mathbb{G}} u := (X_1 u, X_2 u, \ldots, X_m u).$$

Moreover, if $\phi = (\phi_1, \phi_2, \ldots, \phi_m)$ is an horizontal section such that $X_j \phi_j \in L^1_{loc}(\mathbb{G})$ for $j = 1, \ldots, m$, we define $\operatorname{div}_{\mathbb{G}} \phi$ as the real-valued function

$$\operatorname{div}_{\mathbb{G}}(\phi) := -\sum_{j=1}^m X_j^* \phi_j = \sum_{j=1}^m X_j \phi_j.$$

From the above results, the second-order differential operator

$$\Delta_{\mathbb{G}} := \sum_{j=1}^{m} X_j^2.$$

is called the (canonical) sub-Laplacian on $\mathbb{G}$. The sub-Laplacian $\Delta_{\mathbb{G}}$ is a left invariant homogeneous hypoelliptic differential operator, thanks to Hörmander's theorem, and $\Delta_{\mathbb{G}} u = \operatorname{div}_{\mathbb{G}}(\nabla_{\mathbb{G}} u)$. In addition, we can check that $\nabla_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}$ are left-translation invariant with respect to the group action τ_z and δ_γ-homogeneous, respectively, of degree one and two, that is, $\nabla_{\mathbb{G}}(u \circ \tau_z) = \nabla_{\mathbb{G}} u \circ \tau_z, \nabla_{\mathbb{G}}(u \circ \delta_\gamma) = \gamma \nabla_{\mathbb{G}} u \circ \delta_\gamma$, and $\Delta_{\mathbb{G}}(u \circ \tau_z) = \Delta_{\mathbb{G}} u \circ \tau_z$, $\Delta_{\mathbb{G}}(u \circ \delta_\gamma) = \gamma^2 \Delta_{\mathbb{G}} u \circ \delta_\gamma$, where the left translation $\tau_z : \mathbb{G} \to \mathbb{G}$ is defined as

$$x \mapsto \tau_z x := z \circ x, \quad \forall x, z \in \mathbb{G}.$$

Moreover, there exists a homogeneous norm d on $\mathbb{G}$ such that

$$\Gamma(z) = \frac{C}{d(z)^{Q-2}}, \quad \forall z \in \mathbb{G},$$

is a fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at 0, for a suitable constant $C > 0$. By definition, the homogeneous norm d on $\mathbb{G}$ is a continuous smooth function, away from the origin, such that $d(\delta_\gamma(z)) = \gamma d(z)$ for every $\gamma > 0$ and $z \in \mathbb{G}, d\left(z^{-1}\right) = d(z)$ and $d(z) = 0$ iff $z = 0$.

We will give some results which will be used to prove the existence in multiple critical cases. Let U be a fixed positive minimizer for the best constant $S_{\mathbb{G}}$ and define the family

$$U_\varepsilon(z) = \varepsilon^{\frac{2-Q}{2}} U\left(\delta_{\frac{1}{\varepsilon}}(z)\right), \quad \forall \varepsilon > 0. \tag{2.1}$$

The functions U_ε are also minimizers for $S_{\mathbb{G}}$ and, up to a normalization, they satisfy

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} U_\varepsilon|^2 dz = \int_{\mathbb{G}} |U_\varepsilon(z)|^{2^*} dz = S_{\mathbb{G}}^{\frac{Q}{2}}, \quad \forall \varepsilon > 0.$$

For any $0 < \eta_i < \infty$ $(i = 1, 2)$, $\alpha_i, \beta_i > 1$ with $\alpha_i + \beta_i = 2^*$, by the Young inequality, the following best Sobolev-type constants are well defined and crucial for the study of (1.1):

$$\begin{aligned} S_{\eta,\alpha,\beta} :&= \inf_{(u,v)\in\mathcal{H}\setminus\{(0,0)\}} \frac{\int_{\mathbb{G}} \left(|\nabla_{\mathbb{G}} u|^2 + |\nabla_{\mathbb{G}} v|^2\right) dz}{\left(\int_{\mathbb{G}} \left(\eta_1 |u|^{\alpha_1} |v|^{\beta_1} + \eta_2 |u|^{\alpha_2} |v|^{\beta_2}\right) dx\right)^{2/2^*}} \\ &= \inf_{(u,v)\in\mathcal{H}\setminus\{(0,0)\}} \|(u,v)\|^2 \left(\int_{\mathbb{G}} \left(\eta_1 |u|^{\alpha_1} |v|^{\beta_1} + \eta_2 |u|^{\alpha_2} |v|^{\beta_2}\right) dx\right)^{-2/2^*}. \end{aligned} \tag{2.2}$$

For any $t \geq 0$, we define the function

$$\mathfrak{h}(t) := \frac{1+t^2}{(\eta_1 t^{\beta_1} + \eta_2 t^{\beta_2})^{\frac{2}{2^*}}}. \tag{2.3}$$

Since $\mathfrak{h}$ is continuous on $(0,\infty)$ such that $\lim\limits_{t\to 0^+} \mathfrak{h}(t) = \lim\limits_{t\to+\infty} \mathfrak{h}(t) = +\infty$, then there exists $t_0 > 0$ a minimal point of function $\mathfrak{h}$, that is,

$$\mathfrak{h}(t_0) = \min_{t\geq 0} \mathfrak{h}(t) > 0. \tag{2.4}$$

Summarizing, we have the following relationship between $S_{\mathbb{G}}$ and $S_{\eta,\alpha,\beta}$.

Theorem 2.1. *Assume that $(\mathcal{A}_0)$ hold, then*

(i) $S_{\eta,\alpha,\beta} = \mathfrak{h}(t_0)\, S_{\mathbb{G}}$.

(ii) $S_{\eta,\alpha,\beta}$ *has the minimizers* $\big(U_\varepsilon(z),\, t_0 U_\varepsilon(z)\big)$, *for* $\varepsilon > 0$, *where* $U_\varepsilon(z)$ *are defined as in* (2.1).

Proof. Suppose $\kappa \in S_0^1(\mathbb{G})$. Choosing $(u,v) = (\kappa, t_0\kappa)$ in (2.2) we have

$$\frac{1+t_0^2}{\left(\eta_1 t_0^{\beta_1} + \eta_2 t_0^{\beta_2}\right)^{\frac{2}{2^*}}} \frac{\displaystyle\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\kappa|^2 dz}{\left(\displaystyle\int_{\mathbb{G}} |\kappa|^{2^*} dz\right)^{2/2^*}} \geq S_{\eta,\alpha,\beta}. \tag{2.5}$$

Taking the infimum as $\kappa \in S_0^1(\mathbb{G})$ in (2.5), we have

$$\mathfrak{h}(t_0) S_{\mathbb{G}} \geq S_{\eta,\alpha,\beta}. \tag{2.6}$$

Let $\{(u_n, v_n)\} \subset \mathcal{H}$ be a minimizing sequence of $S_{\eta,\alpha,\beta}$ and define $w_n = s_n v_n$, where

$$s_n := \left(\left(\int_{\mathbb{G}} |v_n|^{2^*}\ \mathrm{d}z\right)^{-1} \int_{\mathbb{G}} |u_n|^{2^*}\ \mathrm{d}z\right)^{\frac{1}{2^*}}.$$

Then

$$\int_{\mathbb{G}} |w_n|^{2^*}\ \mathrm{d}z = \int_{\mathbb{G}} |u_n|^{2^*}\ \mathrm{d}z. \tag{2.7}$$

From the Young inequality and (2.6) it follows that

$$\begin{aligned} \int_{\mathbb{G}} |u_n|^{\alpha_i} |w_n|^{\beta_i}\ \mathrm{d}z &\leq \frac{\alpha_i}{2^*} \int_{\mathbb{G}} |u_n|^{2^*}\ \mathrm{d}z + \frac{\beta_i}{2^*} \int_{\mathbb{G}} |w_n|^{2^*}\ \mathrm{d}z \\ &= \int_{\mathbb{G}} |u_n|^{2^*}\ \mathrm{d}z = \int_{\mathbb{G}} |w_n|^{2^*}\ \mathrm{d}z, \quad i = 1,2. \end{aligned} \tag{2.8}$$

Consequently,

$$\frac{\|(u_n,v_n)\|^2}{\left(\int_{\mathbb{G}}\left(\eta_1|u_n|^{\alpha_1}|v_n|^{\beta_1}+\eta_2|u_n|^{\alpha_2}|v_n|^{\beta_2}\right)dx\right)^{2/2_s^*}}\geq$$
$$\frac{\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u_n|^2dz}{\left(\left(\eta_1 s_n^{-\beta_1}+\eta_2 s_n^{-\beta_2}\right)\int_{\mathbb{G}}|u_n|^{2^*}\right)^{\frac{2}{2^*}}}+\frac{s_n^{-2}\int_{\mathbb{G}}|\nabla_{\mathbb{G}}w_n|^2dz}{\left(\left(\eta_1 s_n^{-\beta_1}+\eta_2 s_n^{-\beta_2}\right)\int_{\mathbb{G}}|w_n|^{2^*}\,\mathrm{d}z\right)^{\frac{2}{2^*}}}\geq \mathfrak{h}(s_n^{-1})S_{\mathbb{G}}\geq \mathfrak{h}(t_0)S_{\mathbb{G}}.$$

As $n\to\infty$ we have

$$S_{\eta,\alpha,\beta}\geq \mathfrak{h}(t_0)S_{\mathbb{G}},$$

which together with (2.6) implies that

$$S_{\eta,\alpha,\beta}= \mathfrak{h}(t_0)S_{\mathbb{G}}.$$

By (2.2) and (2.1), $S_{\eta,\alpha,\beta}$ has the minimizers $(U_\varepsilon(x),t_0U_\varepsilon(x))$. □

Let $R>0$ be such that $B_d(0,R)\subset\Omega$ (we can suppose $0\in\Omega$, due to the group translation invariance) and let a cut-off function $\varphi\in C_0^\infty(B_d(0,R))$, $0\leq\varphi\leq 1$, $\varphi=1$ in $B_d\left(0,\frac{R}{2}\right)$ and $\varphi=0$ in $\mathbb{G}\backslash B_d(0,R)$. Set

$$u_\varepsilon(z)=\varphi(z)U_\varepsilon(z).$$

Then, from [11, Lemma 3.3], we obtain the required results.

Lemma 2.2. *The functions u_ε satisfy the following estimates, as $\varepsilon\to 0$:*

$$\int_\Omega|\nabla_{\mathbb{G}}u_\varepsilon|^2dz=S_{\mathbb{G}}^{\frac{Q}{2}}+O\left(\varepsilon^{Q-2}\right),\quad \int_\Omega|u_\varepsilon|^{2^*}dz=S_{\mathbb{G}}^{\frac{Q}{2}}+O\left(\varepsilon^{Q}\right),$$

and

$$\int_\Omega|u_\varepsilon|^2dz=\begin{cases}C\varepsilon^2+O\left(\varepsilon^{Q-2}\right), & \text{if } Q>4,\\ C\varepsilon^2|\ln\varepsilon|+O\left(\varepsilon^2\right), & \text{if } Q=4.\end{cases}$$

Moreover, similarly as the proof of [12, Lemma 6.1], we get the following results.

Lemma 2.3. *The following estimates hold as $\varepsilon\to 0$:*

$$\int_\Omega|u_\varepsilon|^q dz\geq\begin{cases}O\left(\varepsilon^{Q+\frac{(2-Q)q}{2}}\right), & \text{if } \frac{Q}{Q-2}<q<2,\\ O\left(\varepsilon^{Q+\frac{(2-Q)q}{2}}|\ln(\varepsilon)|\right), & \text{if } q=\frac{Q}{Q-2},\\ O\left(\varepsilon^{\frac{(Q-2)}{2}}\right), & \text{if } 1\leq q<\frac{Q}{Q-2}.\end{cases}$$

3 The Palais-Smale condition

In this section, we use the second concentration-compactness principle and concentration-compactness principle at infinity to prove that the $(PS)_c$ condition holds.

Definition 3.1. *Let $c \in \mathbb{R}$ and $I_{\eta,\alpha,\beta} \in C^1(\mathcal{H}, \mathbb{R})$.*

(i) *A sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is called a Palais–Smale sequence at the level c $(PS)_c$-sequence, in short, for the functional $I_{\eta,\alpha,\beta}$ if $I_{\eta,\alpha,\beta}(u_n, v_n) \to c$ and $I'_{\eta,\alpha,\beta}(u_n, v_n) \to 0$ as $n \to \infty$.*

(ii) *We say that $I_{\eta,\alpha,\beta}$ satisfies the $(PS)_c$ condition if any $(PS)_c$-sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ for $I_{\eta,\alpha,\beta}$ has a convergent subsequence in E.*

Since $g, h \in L^{\frac{2^*}{2^*-q}}(\Omega)$, we obtain from the Hölder and Sobolev inequalities that, for all $u \in S_0^1(\Omega)$,

$$\int_\Omega g(z)|u|^q dz \le \left(\int_\Omega |g(z)|^{\frac{2^*}{2^*-q}} dz\right)^{\frac{2^*-q}{2^*}} \left(\int_\Omega |u|^{2^*} dz\right)^{\frac{q}{2}} \le \|g\|_{L^{\frac{2^*}{2^*-q}}} S_{\mathbb{G}}^{-\frac{q}{2}} \|u\|^q_{S_0^1(\Omega)}. \tag{3.1}$$

Similarly, one can get

$$\int_\Omega h(z)|v|^q dz \le \left(\int_\Omega |h(z)|^{\frac{2^*}{2^*-q}} dz\right)^{\frac{2^*-q}{2^*}} \left(\int_\Omega |v|^{2^*} dz\right)^{\frac{\psi}{2}} \le \|h\|_{L^{\frac{2^*}{2^*-q}}} S_{\mathbb{G}}^{-\frac{q}{2}} \|v\|^q_{S_0^1(\Omega)}. \tag{3.2}$$

Hence, in view of (3.1) and (3.2), we can obtain

$$\Psi_{\lambda,\mu}(u, v) \le \left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right) S_{\mathbb{G}}^{-\frac{q}{2}} \|(u, v)\|^q_{\mathcal{H}}. \tag{3.3}$$

Moreover, the Young inequality and (3.1), (3.2) imply that

$$\begin{aligned}\Psi_{\lambda,\mu}(u, v) \le{}& \frac{1}{Q}\frac{2^*q}{2^*-q}\|(u, v)\|_{\mathcal{H}} \\ &+ \frac{2-q}{2} S_{\mathbb{G}}^{-\frac{q}{2-q}} \left(\frac{2^*-q}{2^*-2}\right)^{\frac{q}{2-q}} \left[\left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}} + \left(\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}}\right].\end{aligned} \tag{3.4}$$

Lemma 3.2. *Let $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ be a $(PS)_c$-sequence of $I_{\eta,\alpha,\beta}$ with $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$. Then $I'_{\eta,\alpha,\beta}(u, v) = 0$ and*

$$I_{\eta,\alpha,\beta}(u, v) \ge -\frac{(2^*-q)(2-q)}{2q2^*} S_{\mathbb{G}}^{-\frac{q}{2-q}} \left(\frac{2^*-q}{2^*-2}\right)^{\frac{q}{2-q}} \left[\left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}} + \left(\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}}\right].$$

Proof. Since $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a $(PS)_c$-sequence of $I_{\eta,\alpha,\beta}$ with $(u_n,v_n) \rightharpoonup (u,v)$ weakly in $\mathcal{H}$, it is easy to check that $I'_{\eta,\alpha,\beta}(u,v)=0$, and then $\left\langle I'_{\eta,\alpha,\beta}(u,v),(u,v)\right\rangle = 0$, that is,

$$\|(u,v)\|_{\mathcal{H}} = K_\eta(u,v) + \Psi_{\lambda,\mu}(u,v).$$

Then from (3.4), we have

$$\begin{aligned} I_{\eta,\alpha,\beta}(u,v) &= \frac{1}{Q}\|(u,v)\|_{\mathcal{H}} - \frac{2^*-q}{2^*q}\Psi_{\lambda,\mu}(u,v) \\ &\geq -\frac{(2^*-q)(2-q)}{2q2^*} S_{\mathbb{G}}^{-\frac{q}{2-q}} \left(\frac{2^*-q}{2^*-2}\right)^{\frac{q}{2-q}} \left[\left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}} + \left(\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}}\right]. \end{aligned}$$

This ends the proof of lemma. □

Lemma 3.3. *Assume that $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a $(PS)_c$-sequence of $I_{\eta,\alpha,\beta}$ and the condition $(\mathcal{A}_1)$ holds. Then $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$.*

Proof. Assume by contradiction that $\|(u_n,v_n)\|_{\mathcal{H}} \to +\infty$. Set

$$(\tilde{u}_n,\tilde{v}_n) = \left(\frac{u_n}{\|(u_n,v_n)\|_{\mathcal{H}}}, \frac{v_n}{\|(u_n,v_n)\|_{\mathcal{H}}}\right).$$

Then, $\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} = 1$, and

$$\begin{cases} (\tilde{u}_n,\tilde{v}_n) \rightharpoonup (u,v) \text{ weakly in } \mathcal{H}, \\ (\tilde{u}_n,\tilde{v}_n) \to (u,v) \text{ strongly in } \left(L^r(\Omega)\right)^2, \forall r \in [1,2^*), \\ (\tilde{u}_n(z),\tilde{v}_n(z)) \to (u(z),v(z)) \text{ a.e. in } \Omega. \end{cases} \tag{3.5}$$

Set $\bar{u}_n := \tilde{u}_n - u$, $\bar{v}_n := \tilde{v}_n - v$, there exists a positive constant $C>0$ such that

$$\int_\Omega |\bar{u}_n|^{2^*} dz < C, \quad \int_\Omega |\bar{v}_n|^{2^*} dz < C, \tag{3.6}$$

and by (3.5), one has that for any $\varepsilon > 0$, there exists $r_0 > 0$ such that

$$\int_{B_d(0,r_0)} |\bar{u}_n|^{2^*} dz < \varepsilon, \quad \int_{B_d(0,r_0)} |\bar{v}_n|^{2^*} dz < \varepsilon, \tag{3.7}$$

for n large enough, where $B_d(0,r_0) = \{z \in \mathbb{G} : d(0,z) \leq r_0\}$ is a ball with center at 0 and radius r_0 with respect to the gauge d. Moreover, since $g,h \in L^{\frac{2^*}{2^*-q}}(\Omega)$, for the above constant r_0, we have

$$\int_{\Omega\setminus B_d(0,r_0)} |g(z)|^{\frac{2^*}{2^*-q}} dz < \varepsilon, \quad \int_{\Omega\setminus B_d(0,r_0)} |h(z)|^{\frac{2^*}{2^*-q}} dz < \varepsilon. \tag{3.8}$$

Then, by (3.6), (3.7), (3.8) and Hölder inequality, we get

$$\begin{aligned}
\Psi_{\lambda,\mu}(\bar{u}_n,\bar{u}_n) &= \int_{\Omega\setminus B_d(0,r_0)} (\lambda g(z)|\bar{u}_n|^q + \mu h(z)|\bar{v}_n|^q)\,dz + \int_{B_d(0,r_0)} (\lambda g(z)|\bar{u}_n|^q + \mu h(z)|\bar{v}_n|q^q)\,dz \\
&\le \lambda \left(\int_{\Omega\setminus B_d(0,r_0)} |g|^{\frac{2^*}{2^*-q}}\,dz\right)^{\frac{2^*-q}{2^*}} \left(\int_{\Omega\setminus B_d(0,r_0)} |\bar{u}_n|^{2^*}\,dz\right)^{\frac{q}{2^*}} \\
&+ \mu \left(\int_{\Omega\setminus B_d(0,r_0)} |h|^{\frac{2^*}{2^*-q}}\,dz\right)^{\frac{2^*-q}{2^*}} \left(\int_{\Omega\setminus B_d(0,u_0)} |\bar{v}_n|^{2^*}\,dz\right)^{\frac{q}{2^*}} \\
&+ \lambda \left(\int_{B_d(0,r_0)} |g|^{\frac{2^*}{2^*-q}}\,dz\right)^{\frac{2^*-q}{2^*}} \left(\int_{B_d(0,r_0)} |\bar{u}_n|^{2^*}\,dz\right)^{\frac{q}{2^*}} \\
&+ \mu \left(\int_{B_d(0,r_0)} |h|^{\frac{2^*}{2^*-q}}\,dz\right)^{\frac{2^*-q}{2^*}} \left(\int_{B_d(0,u_0)} |\bar{v}_n|^{2^*}\,dz\right)^{\frac{q}{2^*}} \\
&\le C_1\varepsilon^{\frac{2^*-q}{2^*}} + C_2 2\varepsilon^{\frac{q}{2^*}},
\end{aligned}$$

which yields that $\Psi_{\lambda,\mu}(\bar{u}_n,\bar{v}_n) \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \Psi_{\lambda,\mu}(\tilde{u}_n,\tilde{v}_n) = \lim_{n\to\infty} \Psi_{\lambda,\mu}(\bar{u}_n,\bar{v}_n) + \Psi_{\lambda,\mu}(u,v) = \Psi_{\lambda,\mu}(u,v). \tag{3.9}$$

On the other hand, since $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a $(PS)_c$-sequence of $I_{\eta,\alpha,\beta}$ and $u_n = \|(u_n,v_n)\|_{\mathcal{H}} \cdot \tilde{u}_n$, $v_n = \|(u_n,v_n)\|_{\mathcal{H}} \cdot \tilde{v}_n$, we deduce that

$$\begin{aligned}
\frac{1}{2}\|(u_n,v_n)\|_{\mathcal{H}}\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} &= \frac{1}{2^*}\|(u_n,v_n)\|_{\mathcal{H}}^{2^*} K_\eta(\tilde{u}_n,\tilde{v}_n) \\
&+ \frac{1}{q}\|(u_n,v_n)\|_{\mathcal{H}}^{q} \Psi_{\lambda,\mu}(\tilde{u}_n,\tilde{v}_n) + o_n(1),
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\|(u_n,v_n)\|_{\mathcal{H}}\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} &= \|(u_n,v_n)\|_{\mathcal{H}}^{2^*} K_\eta(\tilde{u}_n,\tilde{v}_n) \\
&+ \|(u_n,v_n)\|_{\mathcal{H}}^{q} \Psi_{\lambda,\mu}(\tilde{u}_n,\tilde{v}_n) + o_n(1).
\end{aligned} \tag{3.11}$$

From (3.9), (3.10), (3.11), $1 < q < 2$ and $\|(u_n,v_n)\|_{\mathcal{H}} \to +\infty$, one has

$$\lim_{n\to\infty} \|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} = \frac{2(2^*-q)}{q(2^*-2)} \lim_{n\to\infty} \frac{\Psi_{\lambda,\mu}(\bar{u}_n,\bar{v}_n)}{\|(u_n,v_n)\|_{\mathcal{H}}^{2-q}} = 0,$$

which contradicts $\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} = 1$. Therefore, $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. □

Lemma 3.4. *$\mathcal{I}_{\lambda,\alpha,\beta}$ satisfies the $(PS)_c$ condition in $\mathcal{H}$, with c satisfying*

$$0 < c < c_\infty := \frac{1}{Q} S_{\eta,\alpha,\beta}^{\frac{Q}{2}} - C_0 \left[\left(\lambda \|g\|_{L^{\frac{2^*}{2^*-q}}} \right)^{\frac{2}{2-q}} + \left(\mu \|h\|_{L^{\frac{2^*}{2^*-q}}} \right)^{\frac{2}{2-q}} \right] \tag{3.12}$$

where $C_0 = C_0(q,Q) := \frac{(2^*-q)(2-q)}{2q2^*} S_{\mathbb{G}}^{-\frac{q}{2-q}} \left(\frac{2^*-q}{2^*-2} \right)^{\frac{q}{2-q}}$ is a positive constant depending only on q, Q and $S_{\mathbb{G}}$.

Proof. Let $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ be a $(PS)_c$-sequence for $I_{\eta,\alpha,\beta}$ with $c \in (0, c_\infty)$. It follows from Lemma 3.3 that $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. Then, there exists a subsequence still denoted by $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ and $(u, v) \in \mathcal{H}$ such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$, and

$$\begin{cases} u_n \rightharpoonup u, \quad v_n \rightharpoonup v \text{ weakly in } L^{2^*}(\Omega), \\ u_n \to u, \quad v_n \to v \text{ strongly in } L^r(\Omega), \ \forall\, 1 \le r < 2^*, \\ u_n(z) \to u(z), \quad v_n(z) \to v(z) \text{ a.e. in } \Omega. \end{cases} \tag{3.13}$$

Hence, from (3.13), it is easy to verify that $I'_{\eta,\alpha,\beta}(u,v) = 0$ and

$$\lim_{n\to\infty} \Psi_{\lambda,\mu}(u_n, v_n) = \Psi_{\lambda,\mu}(u,v). \tag{3.14}$$

Set $\tilde{u}_n = u_n - u, \tilde{v}_n = v_n - v$. By Brézis-Lieb lemma [18], we get

$$\|(u_n, v_n)\|_{\mathcal{H}} = \|(u,v)\|_{\mathcal{H}} + \|(\tilde{u}_n, \tilde{v}_n)\|_{\mathcal{H}} + o_n(1), \tag{3.15}$$

$$\int_\Omega |u_n|^{2^*} dz = \int_\Omega |u|^{2^*} dz + \int_\Omega |\tilde{u}_n|^{2^*} dz + o_n(1), \tag{3.16}$$

$$\int_\Omega |v_n|^{2^*} dz = \int_\Omega |v|^{2^*} dz + \int_\Omega |\tilde{v}_n|^{2^*} dz + o_n(1), \tag{3.17}$$

and

$$\int_\Omega |u_n|^{\alpha_i} |v_n|^{\beta_i} dz = \int_\Omega |u|^{\alpha_i} |v|^{\beta_i} dz + \int_\Omega |\tilde{u}_n|^{\alpha_i} |\tilde{v}_n|^{\beta_i} dz + o_n(1). \tag{3.18}$$

So, (3.16), (3.17) and (3.18) yield

$$K_\eta(u_n, v_n) = K_\eta(u,v) + K_\eta(\tilde{u}_n, \tilde{v}_n) + o_n(1). \tag{3.19}$$

Then, using (3.14), (3.15) and (3.19), we have

$$c = \frac{1}{2} \|(\tilde{u}_n, \tilde{v}_n)\|_{\mathcal{H}} - \frac{1}{2_s^*} K_\eta(\tilde{u}_n, \tilde{v}_n) + I_{\eta,\alpha,\beta}(u,v) + o_n(1), \tag{3.20}$$

and

$$o_n(1) = \|(\bar{u}_n, \bar{v}_n)\|_{\mathcal{H}} - K_\eta(\bar{u}_n, \bar{v}_n). \tag{3.21}$$

We may assume that

$$\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} \to l, \quad K_\eta\left(\tilde{u}_n,\tilde{v}_n\right) \to l \geq 0 \quad \text{as } n\to\infty.$$

If $l=0$, the proof is completed. Assume that $l>0$, then from (3.21), we have

$$S_{\eta,\alpha,\beta} l^{\frac{2}{2^*}} = S_{\eta,\alpha,\beta}\left(\lim_{n\to\infty} K_\eta\left(\tilde{u}_n,\tilde{v}_n\right)\right)^{\frac{2}{2^*}} \leq \lim_{n\to\infty}\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} = l,$$

which implies that $l \geq S_{\eta,\alpha,\beta}^{\frac{Q}{2}}$. Hence, from (3.20) and Lemma 3.2, we have

$$\begin{aligned} c &= I_{\eta,\alpha,\beta}(u_n,v_n)+o_n(1) = \left(\frac{1}{2}-\frac{1}{2^*}\right) l + I_{\eta,\alpha,\beta}(u,v)+o_n(1) \\ &\geq \frac{1}{Q} S_{\eta,\alpha,\beta}^{\frac{Q}{2}} - C_0\left[\left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}} + \left(\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}}\right], \end{aligned} \tag{3.22}$$

which contradicts $c<c_\infty$. The proof is completed. □

4 Nehari manifold

Now we focus our attention on Problem (1.1) by using the Nehari manifold approach. For this reason, we introduce the Nehari manifold

$$\mathcal{N}_{\eta,\alpha,\beta} = \left\{w\in\mathcal{H}\backslash\{0\} : \left\langle I'_{\eta,\alpha,\beta}(w), w\right\rangle = 0\right\}.$$

where $w=(u,v)$ and $\|w\|_{\mathcal{H}} = \|(u,v)\|_{\mathcal{H}}$. Note that $\mathcal{N}_{\eta,\alpha,\beta}$ contains all nonzero solution of (1.1), and $w\in\mathcal{N}_{\eta,\alpha,\beta}$ if and only if

$$\|w\|_{\mathcal{H}} = K_\eta(w) + \Psi_{\lambda,\mu}(w). \tag{4.1}$$

Lemma 4.1. *$I_{\eta,\alpha,\beta}$ is coercive and bounded below on $\mathcal{N}_{\eta,\alpha,\beta}$.*

Proof. Let $w\in\mathcal{N}_{\eta,\alpha,\beta}$ by (3.3) and (4.1). We find

$$\begin{aligned} I_{\eta,\alpha,\beta}(w) &= \frac{2^*-2}{22^*}\|w\|_{\mathcal{H}} - \frac{2^*-2}{q2^*}\Psi_{\lambda,\mu}(w) \\ &\geq \frac{2^*-2}{22^*}\|w\|_{\mathcal{H}} - \frac{2^*-q}{q2^*}\left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*-q}{2^*}}}\right) S_{\mathbb{G}}^{-\frac{q}{2}}\|w\|_{\mathcal{H}}^q. \end{aligned} \tag{4.2}$$

Since $1<q<2$, we see that $I_{\eta,\alpha,\beta}$ is coercive and bounded below on $\mathcal{N}_{\eta,\alpha,\beta}$. This achieves the proof of the lemma. □

Define $\Phi(w) := \left\langle I'_{\eta,\alpha,\beta}(w), w\right\rangle$, then for all $w = (u, v) \in \mathcal{N}_{\eta,\alpha,\beta}$, we have

$$\begin{aligned}\langle \Phi'(w), w\rangle &= 2\|w\|_{\mathcal{H}} - 2^* K_\eta(w) - q\Psi_{\lambda,\mu}(w)\\ &= (2-q)\|w\|_{\mathcal{H}} - (2^*-q)\, K_\eta(w)\\ &= (2-2^*)\,\|w\|_{\mathcal{H}} + (2^*-q)\,\Psi_{\lambda,\mu}(w).\end{aligned} \tag{4.3}$$

Now, similar to the method used in [16], we split $\mathcal{N}_{\eta,\alpha,\beta}$ into three disjoint parts:

$$\begin{aligned}\mathcal{N}^+_{\eta,\alpha,\beta} &:= \{w \in \mathcal{N}_{\eta,\alpha,\beta} : \langle \Phi'(w), w\rangle > 0\},\\ \mathcal{N}^0_{\eta,\alpha,\beta} &:= \{w \in \mathcal{N}_{\eta,\alpha,\beta} : \langle \Phi'(w), w\rangle = 0\},\\ \mathcal{N}^-_{\eta,\alpha,\beta} &:= \{w \in \mathcal{N}_{\eta,\alpha,\beta} : \langle \Phi'(w), w\rangle < 0\}.\end{aligned} \tag{4.4}$$

Note that $\mathcal{N}_{\eta,\alpha,\beta}$ contains every nonzero solution of problem (1.1). In order to study the properties of Nehari manifolds. We now present some properties of $\mathcal{N}^+_{\eta,\alpha,\beta}, \mathcal{N}^0_{\eta,\alpha,\beta}$ and $\mathcal{N}^-_{\eta,\alpha,\beta}$ to state our main results.

Lemma 4.2. *Assume that $w_0 = (u_0, v_0)$ is a local minimizer for $I_{\eta,\alpha,\beta}$ on the set $\mathcal{N}_{\eta,\alpha,\beta} \setminus \mathcal{N}^0_{\eta,\alpha,\beta}$. Then $I'_{\eta,\alpha,\beta}(w_0) = 0$ in $\mathcal{H}^{-1}$, where $\mathcal{H}^{-1}$ denotes the dual space of the space $\mathcal{H}$.*

Proof. The proof is similar as that of [21, Lemma 3.4] and the details are omitted. □

Lemma 4.3. *$\mathcal{N}^0_{\eta,\alpha,\beta} = \emptyset$ for all $(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+$ with*

$$0 < \lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \Lambda$$

where Λ is given in (1.6).

Proof. We argue by contradiction. Assume that there exist $\lambda, \mu \in (0, +\infty)$ with

$$0 < \lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \Lambda$$

such that $\mathcal{N}^0_{\eta,\alpha,\beta} \neq \emptyset$. Then, for $w \in \mathcal{N}^0_{\eta,\alpha,\beta}$, by (4.3), we have

$$\|w\|_{\mathcal{H}} = \frac{2^*-q}{2-q} K_\eta(w) \tag{4.5}$$

and

$$\|w\|_{\mathcal{H}} = \frac{2^*-q}{2^*-2} \Psi_{\lambda,\mu}(w). \tag{4.6}$$

From the Young inequality, we have that

$$K_\eta(w) \leq (\eta_1 + \eta_2)\, S_{\mathbb{G}}^{-\frac{2^*}{2}} \|w\|^{2^*}_{\mathcal{H}},$$

and (4.5) yields

$$\|w\|_{\mathcal{H}} \geq \left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)} S_{\mathbb{G}}^{\frac{2^*}{2}}\right)^{\frac{1}{2^*-2}}. \tag{4.7}$$

On the other hand, from (3.3) and (4.6), it follows that

$$\|w\|_{\mathcal{H}} \leq \left(\frac{2^*-q}{2^*-2}\left(\lambda\|g\|_{L^{\frac{2*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right) S_{\mathbb{G}}^{\frac{-q}{2}}\right)^{\frac{1}{2-q}}. \tag{4.8}$$

Therefore, in view of (4.7) and (4.8), we obtain

$$\lambda\|g\|_{L^{\frac{2*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} \geq \frac{2^*-2}{2^*-q}\left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)}\right)^{\frac{2-q}{2^*-2}} S_{\mathbb{G}}^{\frac{2^*-q}{2^*-2}} := \Lambda,$$

which is a contradiction. This completes the proof of Lemma. □

By Lemmas 4.2 and 4.3 , for $(\lambda,\mu) \in \mathfrak{D}_\Lambda$, we can write $\mathcal{N}_{\eta,\alpha,\beta} = \mathcal{N}^+_{\eta,\alpha,\beta} \cup \mathcal{N}^-_{\eta,\alpha,\beta}$ and define

$$c_{\eta,\alpha,\beta} = \inf_{w\in\mathcal{N}_{\eta,\alpha,\beta}} I_{\eta,\alpha,\beta}(w); \quad c^+_{\eta,\alpha,\beta} = \inf_{w\in\mathcal{N}^+_{\eta,\alpha,\beta}} I_{\eta,\alpha,\beta}(w); \quad c^-_{\eta,\alpha,\beta} = \inf_{w\in\mathcal{N}^-_{\eta,\alpha,\beta}} I_{\eta,\alpha,\beta}(w).$$

Lemma 4.4. *Assume that $(\mathcal{A}_0)$, hold. Then, we have the following results:*

(i) $c_{\eta,\alpha,\beta} \leq c^+_{\eta,\alpha,\beta} < 0$ *for all* $(\lambda,\mu) \in \mathfrak{D}_\Lambda$.

(ii) *There exists a constant* $C_0 = C_0(\lambda, q, Q, S_{\mathbb{G}}, \Lambda) > 0$ *such that* $c^-_{\eta,\alpha,\beta} \geq C_0 > 0$, *for all* $(\lambda,\mu) \in \mathfrak{D}_{\frac{q}{2}\Lambda}$.

Proof. (i) For $w \in \mathcal{N}^+_{\eta,\alpha,\beta} \subset \mathcal{N}_{\eta,\alpha,\beta}$, by (4.3), we have

$$\|w\|_{\mathcal{H}} > \frac{2^*-q}{2-q} K_\eta(w),$$

and so

$$\begin{aligned} I_{\eta,\alpha,\beta}(w) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|w\|_{\mathcal{H}} - \left(\frac{1}{2^*}-\frac{1}{q}\right) K_\eta(w) \\ &\leq \left(\frac{q-2}{2q} + \frac{2^*-q}{2^*q}\frac{2-q}{2^*-q}\right)\|w\|_{\mathcal{H}} = -\frac{(2-q)(2^*-2)}{22^*q}\|w\|_{\mathcal{H}} < 0. \end{aligned}$$

Thus, from the definition of $c_{\eta,\alpha,\beta}$ and $c^+_{\eta,\alpha,\beta}$, we can deduce that $c_{\eta,\alpha,\beta} \leq c^+_{\eta,\alpha,\beta} < 0$.

(ii) For $w \in \mathcal{N}^-_{\eta,\alpha,\beta}$, similar to (4.7), we have

$$\|w\|_{\mathcal{H}} > \left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)} S_{\mathbb{G}}^{\frac{2^*}{2}}\right)^{\frac{1}{2^*-2}}. \tag{4.9}$$

In view of (4.2) and (4.9), we get

$$
\begin{aligned}
I_{\eta,\alpha,\beta}(w) &\geq \|w\|_{\mathcal{H}}^{q}\left(\frac{2^*-2}{22^*}\|w\|_{\mathcal{H}}^{2-q}-\frac{2^*-q}{q2^*}\left(\lambda\|g\|_{L^{\frac{2}{2^*-q}}}+\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right)S_{\mathbb{G}}^{-\frac{q}{2}}\right)\\
&\geq \|w\|_{\mathcal{H}}^{q}\left(\frac{2^*-2}{22^*}\left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)}\right)^{\frac{2-q}{2^*-2}}S_{\mathbb{G}}^{\frac{2^*(2-q)}{2(2^*-2)}}\right.\\
&\quad\left.-\frac{2^*-q}{q2^*}\left(\lambda\|g\|_{L^{\frac{2}{2^*-q}}}+\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right)S_{\mathbb{G}}^{-\frac{q}{2}}\right).
\end{aligned}
$$

So, if namely,

$$
0<\lambda\|g\|_{L^{\frac{2}{2^*-q}}}+\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}<\frac{q}{2}\frac{2^*-2}{2^*-q}\left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)}\right)^{\frac{2-q}{2^*-2}}S_{\mathbb{G}}^{\frac{2^*-q}{2^*-2}}=\frac{q}{2}\Lambda,
$$

we get

$$
\begin{aligned}
I_{\eta,\alpha,\beta}(w) \geq &\left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)}S_{\mathbb{G}}^{\frac{2^*}{2}}\right)^{\frac{q}{2^*-2}}\left(\frac{2^*-2}{22^*}\left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)}\right)^{\frac{2-q}{2^*-2}}S_{\mathbb{G}}^{\frac{2^*(2-q)}{2(2^*-2)}}\right.\\
&\left.-\frac{2^*-q}{q2^*}\left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}}+\mu\|h\|_{L^{\frac{2^*}{2^*-4}}}\right)S_{\mathbb{G}}^{-\frac{q}{2}}\right):=C_0\left(\lambda,q,Q,S_{\mathbb{G}},\Lambda\right)>0,
\end{aligned}
$$

and this completes the proof. □

For each $w\in\mathcal{H}\backslash\{0\}$, we have $K_\eta(w)>0$ and let

$$
t_{\max}=\left(\frac{(2-q)\|w\|_{\mathcal{H}}}{(2^*-q)\,K_\eta(w)}\right)^{\frac{1}{2^*-2}}>0.
$$

So, we get the following result.

Lemma 4.5. *Let $(\lambda,\mu)\in\mathfrak{D}_\Lambda$. For every $w\in\mathcal{H}$ with $K_\eta(w)>0$, the following results hold:*

(i) *If $\Psi_{\lambda,\mu}(w)\leq 0$, then there is a unique $t^->t_{\max}$ such that $(t^-w)\in\mathcal{N}^-_{\eta,\alpha,\beta}$ and*

$$
I_{\eta,\alpha,\beta}(t^-w)=\sup_{t\geq 0}I_{\eta,\alpha,\beta}(tw).
$$

(ii) *If $\Psi_{\lambda,\mu}(w)>0$, then there exist unique t^+ and t^- with $0<t^+<t_{\max}<t^-$ such that $(t^+w)\in\mathcal{N}^+_{\eta,\alpha,\beta}$ and $(t^-w)\in\mathcal{N}^-_{\eta,\alpha,\beta}$. Moreover,*

$$
I_{\eta,\alpha,\beta}(t^+w)=\inf_{0\leq t\leq t_{\max}}I_{\eta,\alpha,\beta}(tw),\quad I_{\eta,\alpha,\beta}(t^-w)=\sup_{t\geq 0}I_{\eta,\alpha,\beta}(tw).
$$

Proof. The proof is similar to [5, Lemma 2.6], and is omitted here. □

5 Proof of the main results

In this section, we provide the proofs of the main results of this work. Before giving the proof of Theorem 1.2, we need the following lemma.

Lemma 5.1. *Assume that $(\mathcal{A}_0)$, hold. Then, we have the following results:*

(i) *If $(\lambda,\mu)\in\mathfrak{D}_\Lambda$, then there exists a $(PS)_{c_{\eta,\alpha,\beta}}$-sequence $\{(u_n,v_n)\}_{n\in\mathbb{N}}\subset\mathcal{N}_{\eta,\alpha,\beta}$ for $I_{\eta,\alpha,\beta}$.*

(ii) *If $(\lambda,\mu)\in\mathfrak{D}_{\frac{q}{2}\Lambda}$, then there exists a $(PS)_{c^-_{\eta,\alpha,\beta}}$-sequence $\{(u_n,v_n)\}_{n\in\mathbb{N}}\subset\mathcal{N}^-_{\eta,\alpha,\beta}$ for $I_{\eta,\alpha,\beta}$.*

Proof. The proof is almost the same as Proposition 9 in [19]. □

Now we establish the existence of a local minimizer of $I_{\eta,\alpha,\beta}$ on $\mathcal{N}^+_{\eta,\alpha,\beta}$.

Theorem 5.2. *Assume that $(\mathcal{A}_0)$, hold. If $(\lambda,\mu)\in\mathfrak{D}_\Lambda$, then $I_{\eta,\alpha,\beta}$ has a minimizer $(u_1,v_1)\in\mathcal{N}^+_{\eta,\alpha,\beta}$ such that (u_1,v_1) is a nonnegative solution of (1.1) and*

$$I_{\eta,\alpha,\beta}(u_1,v_1)=c_{\eta,\alpha,\beta}=c^+_{\eta,\alpha,\beta}<0.$$

Proof. In view of the Lemma 5.1 (i), there exists a minimizing sequence $\{(u_n,v_n)\}_{n\in\mathbb{N}}\subset\mathcal{N}_{\eta,\alpha,\beta}$ such that

$$\lim_{n\to\infty} I_{\eta,\alpha,\beta}(u_n,v_n)=c_{\eta,\alpha,\beta}\quad\text{and}\quad\lim_{n\to\infty} I'_{\eta,\alpha,\beta}(u_n,v_n)=0. \tag{5.1}$$

Since $I_{\eta,\alpha,\beta}$ is coercive on $\mathcal{N}_{\eta,\alpha,\beta}$, we get that $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. Passing to a subsequence, still denoted by $\{(u_n,v_n)\}_{n\in\mathbb{N}}$, we can assume that there exists $(u_1,v_1)\in\mathcal{H}$ such that $(u_n,v_n)\rightharpoonup(u_1,v_1)$ weakly in $\mathcal{H}$ and

$$\begin{cases} u_n\rightharpoonup u_1,\quad v_n\rightharpoonup v_1 \text{ weakly in } L^{2^*}(\Omega),\\ u_n\to u_1,\quad v_n\to v_1 \text{ strongly in } L^r(\Omega),\ \forall r\in[1,2^*),\\ u_n(z)\to u_1(z),\quad v_n(z)\to v_1(z) \text{ a.e. in } \Omega. \end{cases} \tag{5.2}$$

By the proof of Lemma 3.3 and (5.2), we get

$$\lim_{n\to\infty}\Psi_{\lambda,\mu}(u_n,v_n)=\Psi_{\lambda,\mu}(u_1,v_1). \tag{5.3}$$

From (5.1), (5.2) and (5.3), it is easy to prove that (u_1,v_1) is a weak solution of (1.1). Moreover, the fact that $(u_n,v_n)\in\mathcal{N}_{\eta,\alpha,\beta}$ implies that

$$\Psi_{\lambda,\mu}(u_n,v_n)=\frac{q(2^*-2)}{2(2^*-q)}\|(u_n,v_n)\|_{\mathcal{H}}-\frac{q2^*}{2^*-q}I_{\eta,\alpha,\beta}(u_n,v_n). \tag{5.4}$$

Let $n\to\infty$ in (5.4), by (5.3) and $c_{\eta,\alpha,\beta}<0$, we deduce that

$$\Psi_{\lambda,\mu}(u_1,v_1)\geq-\frac{q2^*}{2^*-q}c_{\eta,\alpha,\beta}>0,$$

which implies that $(u_1, v_1) \in \mathcal{H}$ is a nontrivial solution of (1.1).

Now, we prove that $(u_n, v_n) \to (u_1, v_1)$ strongly in $\mathcal{H}$ and that $I_{\eta,\alpha,\beta}(u_1, v_1) = c_{\eta,\alpha,\beta}$. By applying Fatou's lemma and $(u_1, v_1) \in \mathcal{N}_{\eta,\alpha,\beta}$, one has

$$\begin{aligned} c_{\eta,\alpha,\beta} \le I_{\eta,\alpha,\beta}(u_1, v_1) &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\|(u_1, v_1)\|_{\mathcal{H}} - \frac{2^* - q}{q2^*}\Psi_{\lambda,\mu}(u_1, v_1) \\ &\le \liminf_{n\to\infty}\left[\left(\frac{1}{2} - \frac{1}{2^*}\right)\|(u_n, v_n)\|_{\mathcal{H}} - \frac{2^* - q}{q2^*}\Psi_{\lambda,\mu}(u_n, v_n)\right] \le \lim_{n\to\infty} I_{\eta,\alpha,\beta}(u_n, v_n) = c_{\eta,\alpha,\beta}. \end{aligned}$$

This yields $I_{\eta,\alpha,\beta}(u_1, v_1) = c_{\eta,\alpha,\beta}$ and $\lim_{n\to\infty}\|(u_n, v_n)\|_{\mathcal{H}} = \|(u_1, v_1)\|_{\mathcal{H}}$. The standard argument shows that $(u_n, v_n) \to (u_1, v_1)$ strongly in $\mathcal{H}$.

Next, we claim that $(u_1, v_1) \in \mathcal{N}^+_{\eta,\alpha,\beta}$. In fact, if $(u_1, v_1) \in \mathcal{N}^-_{\eta,\alpha,\beta}$, by Lemma 4.5 (ii), there are unique $t_1^+$and $t_1^- > 0$ such that $\left(t_1^+ u_1, t_1^+ v_1\right) \in \mathcal{N}^+_{\eta,\alpha,\beta}$, $\left(t_1^- u_1, t_1^- v_1\right) \in \mathcal{N}^-_{\eta,\alpha,\beta}$ and $t_1^+ < t_1^- = 1$. Since $\frac{d}{dt} I_{\eta,\alpha,\beta}\left(t_1^+ u_1, t_1^+ v_1\right) = 0$ and $\frac{d^2}{dt^2} I_{\eta,\alpha,\beta}\left(t_1^+ u_1, t_1^+ v_1\right) > 0$, there exists $t_1^* \in \left(t_1^+, t_1^-\right)$such that $I_{\eta,\alpha,\beta}\left(t_1^+ u_1, t_1^+ v_1\right) < I_{\eta,\alpha,\beta}(t_1^* u_1, t_1^* v_1)$. By Lemma 4.5, it follows that

$$I_{\eta,\alpha,\beta}\left(t_1^+ u_1, t_1^+ v_1\right) < I_{\eta,\alpha,\beta}(t_1^* u_1, t_1^* v_1) \le I_{\eta,\alpha,\beta}\left(t_1^- u_1, t_1^- v_1\right) = I_{\eta,\alpha,\beta}(u_1, v_1),$$

which contradicts $I_{\eta,\alpha,\beta}(u_1, v_1) = c_{\eta,\alpha,\beta}$. Moreover, since $I_{\eta,\alpha,\beta}(u_1, v_1) = I_{\eta,\alpha,\beta}(|u_1|, |v_1|)$ and $(|u_1|, |v_1|) \in \mathcal{N}^+_{\eta,\alpha,\beta}$, we may assume that (u_1, v_1) is a nonnegative nontrivial solution of system (1.1). By means of Bony's maximum principle [4], such solution turn out to be strictly positive. □

Now we establish the existence of a local minimizer of $I_{\eta,\alpha,\beta}$ on $\mathcal{N}^-_{\eta,\alpha,\beta}$.

Lemma 5.3. *Assume that $(\mathcal{A}_0)$ hold. Then, there exist $(u_0, v_0) \in \mathcal{H}\backslash\{(0,0)\}$ and $\Lambda_5 > 0$ such that for all $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_5}$, the following holds:*

$$\sup_{t\ge 0} I_{\eta,\alpha,\beta}(tu_0, tv_0) < c_\infty, \tag{5.5}$$

where c_∞ is a constant given in (3.12). In particular, $c^-_{\eta,\alpha,\beta} < c_\infty$ for all $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_5}$.

Proof. Without loss of generality, we assume that $0 \in \Omega$. Let $R \in (0, r_0)$ be such that the quasi-ball $B_d(0, R) \subset \Omega$, and let a cut-off function $\varphi \in C_0^\infty(B_d(0, R))$ satisfying $0 \le \varphi \le 1$, $\varphi = 1$ in $B_d\left(0, \frac{R}{2}\right)$ and $\varphi = 0$ in $\mathbb{G}\backslash B_d(0, R)$. Here r_0 is given in $(\mathcal{A}_2)$. Now, let $u_\varepsilon(z) = \varphi(z)U_\varepsilon(z)$ and consider the function

$$J_\eta(t) = \frac{t^2}{2}\left(1 + t_0^2\right)\|u_\varepsilon\|^2_{S_0^1(\Omega)} - \frac{t^{2^*}}{2^*}\left(\eta_1 t^{\beta_1} + \eta_2 t^{\beta_2}\right)\int_\Omega |u_\varepsilon|^{2^*} dz, \tag{5.6}$$

where t_0 be given in Theorem 2.1. By Lemma 2.2 and the definition of $S_{\eta,\alpha,\beta}$, we obtain that

$$
\begin{aligned}
\sup_{t\geq 0} J_\eta(t) &\leq \left(\frac{1}{2}-\frac{1}{2^*}\right)\left(\frac{\left(1+t_0^2\right)\|u_\varepsilon\|^2_{S_0^1(\Omega)}}{\left(\eta_1 t^{\beta_1}+\eta_2 t^{\beta_2}\right)^{\frac{2}{2^*}}\left(\int_\Omega |u_\varepsilon|^{2^*}dz\right)^{\frac{2}{2^*}}}\right)^{\frac{2^*}{2^*-2}} \\
&\leq \frac{1}{Q}\left(\mathfrak{h}\left(t_0\right)\frac{\|u_\varepsilon\|^2_{S_0^1(\Omega)}}{\left(\int_\Omega |u_\varepsilon|^{2^*}dz\right)^{\frac{2}{2^*}}}\right)^{\frac{\rho}{2}} = \frac{1}{Q}\left(\mathfrak{h}\left(t_0\right)\frac{S_{\mathbb{G}}^{\frac{Q}{2}} + +O(\varepsilon^{Q-2})}{\left(S_{\mathbb{G}}^{\frac{Q}{2}}+O(\varepsilon^{Q})\right)^{\frac{2}{2^*}}}\right)^{\frac{Q}{2}} \\
&= \frac{1}{Q}\left(\mathfrak{h}\left(t_0\right)S_{\mathbb{G}}\right)^{\frac{Q}{2}} + c_1\varepsilon^{Q-2} = \frac{1}{Q}S_{\eta,\alpha,\beta}^{\frac{Q}{2}} + c_1\varepsilon^{Q-2},
\end{aligned}
\tag{5.7}
$$

where c_1 is a positive constant and the following fact has been used:

$$
\sup_{t\geq 0}\left(\frac{t^2}{2}A-\frac{t^{2^*}}{2^*}B\right) = \frac{1}{Q}\left(\frac{A}{B^{\frac{Q-2}{Q}}}\right)^{\frac{Q}{2}}, \quad \forall A,B>0.
$$

Choosing $\Lambda_1>0$ such that $0<\lambda\|g\|_{L^{\frac{2^*-q}{2^*}}}+\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}<\Lambda_1$, by the definitions of $I_{\eta,\alpha,\beta}$, there exists $t_m\in(0,1)$ such that

$$
I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right) \leq \frac{t^2}{2}\left(1+t_0^2\right)\|u_\varepsilon\|^2_{S_0^1(\Omega)} < c_\infty, \quad \forall t<t_m,
$$

and one has

$$
\sup_{0\leq t<t_m} I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right) < c_\infty, \tag{5.8}
$$

for all $\lambda,\mu\in(0,+\infty)$ with

$$
0<\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}}+\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}<\Lambda_1.
$$

Moreover, by the definitions of $I_{\eta,\alpha,\beta}$ and $(u_\varepsilon, t_0u_\varepsilon)$, using the condition $(\mathcal{A}_2)$, Lemma 2.3 and (5.7), we have

$$
\begin{aligned}
\sup_{t\geq t_m} I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right) &= \sup_{t\geq t_m}\left(J_\lambda(t)-\frac{t^q}{q}\int_\Omega\left(\lambda g(z)+\mu h(z)t_0^q\right)|u_\varepsilon|^q dz\right) \\
&\leq \frac{1}{Q}S_{\eta,\alpha,\beta}^{\frac{Q}{2}} + c_1\varepsilon^{Q-2} - \frac{t_m^q}{q}a_0\left(\lambda+\mu t_0^q\right)\int_\Omega |u_\varepsilon|^q dz \\
&\leq \frac{1}{Q}S_{\eta,\alpha,\beta}^{\frac{Q}{2}} + c_1\varepsilon^{Q-2} \\
&\quad - \frac{t_m^q}{q}a_0\left(\lambda+\mu t_0^q\right)\begin{cases} c_2\varepsilon^{Q-\frac{(Q-2)q}{2}}, & \text{if } q>\frac{Q}{Q-2}, \\ c_3\varepsilon^{Q-\frac{(Q-2)q}{2}}|\ln\varepsilon|, & \text{if } q=\frac{Q}{Q-2}, \\ c_4\varepsilon^{\frac{(Q-2)q}{2}}, & \text{if } q<\frac{Q}{Q-2}, \end{cases}
\end{aligned}
\tag{5.9}
$$

where c_2, c_3, c_4 are positive constants.

(i) If $1 < q < \frac{Q}{Q-2}$, then by $Q \geq 4$ one can get that $q\frac{Q-2}{2} < \frac{Q}{2} \leq Q-2$. Thus, for $\varepsilon > 0$ small enough, we can choose $\Lambda_2 > 0$ such that

$$\sup_{t\geq t_m} I_{\eta,\alpha,\beta}(tu_\varepsilon, tt_0u_\varepsilon) \leq \frac{1}{Q}S^{\frac{q}{2}}_{\eta,\alpha,\beta} + c_1\varepsilon^{Q-2} - \frac{t_0^q}{q}a_0c_4\varepsilon^{\frac{(Q-2)q}{2}} < c_\infty,$$

for all $\lambda, \mu \in (0, +\infty)$, with $0 < \lambda\|g\|_{L^{\frac{2}{2^*-q}}} + \mu\|h\|_{L^{\frac{2}{2^*-q}}} < \Lambda_2$.

(ii) If $\frac{Q}{Q-2} \leq q < 2$, we have $Q > 4$ and $q \geq \frac{Q}{Q-2} > \frac{4}{Q-2}$, which implies that

$$Q - \frac{(Q-2)q}{2} - (Q-2) = 2 - \frac{(Q-2)q}{2} = \frac{4-(Q-2)q}{2} = \frac{(Q-2)\left(\frac{4}{Q-2} - q\right)}{2} < 0.$$

Then for ε small enough, by a similar argument in (i), we can choose $\Lambda_3 > 0$ such that

$$\sup_{t\geq t_m} I_{\eta,\alpha,\beta}(tu_\varepsilon, tt_0u_\varepsilon) < c_\infty,$$

for all $\lambda, \mu \in (0, +\infty)$ with $0 < \lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \Lambda_3$.

Set $\Lambda_4 = \min\{\Lambda_2, \Lambda_3\}$, from cases (i) and (ii), for all $\lambda, \mu \in (0, +\infty)$ with

$$0 < \sup_{t\geq t_m} I_{\eta,\alpha,\beta}(tu_\varepsilon, tt_0u_\varepsilon) < c_\infty. \tag{5.10}$$

Thus, taking $\Lambda_5 = \min\{\Lambda_1, \Lambda_4\}$, (5.8) and (5.10) induce that $\sup_{t\geq 0} I_{\eta,\alpha,\beta}(tu_\varepsilon, tt_0u_\varepsilon) < c_\infty$ holds for all $\lambda, \mu \in (0, +\infty)$ with $0 < \lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \Lambda_5$.

Finally, we prove that $c^-_{\eta,\alpha,\beta} < c_\infty$ for all $\lambda, \mu \in (0, +\infty)$ with $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_5}$. Recall that $(u_0, v_0) := (u_\varepsilon, t_0u_\varepsilon)$. It is easy to see that $K_\eta(u_\varepsilon, t_0u_\varepsilon) > 0$. Then, combining (5.5) with Lemma 4.5, and using the definition of $c^-_{\eta,\alpha,\beta}$, we obtain that there exists $t_2^- > 0$ such that $\left(t_2^-u_0, t_2^-v_0\right) \in \mathcal{N}^-_{\eta,\alpha,\beta}$ and

$$c^-_{\eta,\alpha,\beta} \leq I_{\eta,\alpha,\beta}\left(t_2^-u_0, t_2^-v_0\right) \leq \sup_{t\geq 0} I_{\eta,\alpha,\beta}(tu_0, tv_0) < c_\infty,$$

for all $\lambda, \mu \in (0, +\infty)$ with $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_5}$. The proof is now complete. □

Theorem 5.4. *Under the assumptions of Theorem 1.2. If $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_*}$, then the functional $I_{\eta,\alpha,\beta}$ has a minimizer $(u_2, v_2) \in \mathcal{N}^-_{\eta,\alpha,\beta}$ and it satisfies $I_{\eta,\alpha,\beta}(u_2, v_2) = c^-_{\eta,\alpha,\beta}$, and (u_2, v_2) is a positive solution of* (1.1)*, where $\Lambda_* = \min\left\{\Lambda_5, \frac{q}{2}\Lambda\right\}$.*

Proof. By Lemma 5.1 (ii), there exists a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\eta,\alpha,\beta}$ in $\mathcal{H}$ for $I_{\eta,\alpha,\beta}$, for all $(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+$satisfying

$$0 < \lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \frac{q}{2}\Lambda.$$

In the light of Lemmas 5.3, 3.4 and 5.1 (ii), for $0 < \lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \Lambda_*$, the functional $I_{\eta,\alpha,\beta}$ satisfies $(PS)_{c^-_{\eta,\alpha,\beta}}$ condition for $c^-_{\eta,\alpha,\beta} > 0$. Since $I_{\eta,\alpha,\beta}$ is coercive on $\mathcal{N}_{\eta,\alpha,\beta}$, we can deduce that $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{N}_{\eta,\alpha,\beta}$ and $\mathcal{H}$. So, there exists a subsequence still denoted by $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ and $(u_2, v_2) \in \mathcal{N}^-_{\eta,\alpha,\beta}$ such that $(u_n, v_n) \to (u_2, v_2)$ strongly in $\mathcal{H}$, and $I_{\eta,\alpha,\beta}(u_2, v_2) = c^-_{\eta,\alpha,\beta} > 0$, $I'_{\eta,\alpha,\beta}(u_2, v_2) = 0$ for all $(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+$ with

$$0 < \lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \Lambda_*.$$

Finally, arguing as in the proof of Theorem 5.2, we have that (u_2, v_2) is a positive solution of the system (1.1). □

Proof of Theorem 1.2. By Theorem 5.2, we obtain that for all $(\lambda, \mu) \in \mathfrak{D}_\Lambda$, Problem (1.1) has a positive solution $(u_1, v_1) \in \mathcal{N}^+_{\eta,\alpha,\beta}$. By Theorem 5.4, we obtain a second positive solution $(u_2, v_2) \in \mathcal{N}^-_{\eta,\alpha,\beta}$ for all $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_*} \subset \mathfrak{D}_\Lambda$. Since $\mathcal{N}^+_{\eta,\alpha,\beta} \cap \mathcal{N}^-_{\eta,\alpha,\beta} = \emptyset$, this implies that (u_1, v_1) and (u_2, v_2) are distinct.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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