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Vector-valued algebras and variants of amenability

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ABSTRACT

Let $\{A_x : x \in X\}$ be a collection of complex Banach algebras indexed by the compact Hausdorff space X. We investigate the weak- and pseudo-amenability of certain algebras $\mathcal{A}$ of A_x-valued functions in relation to the corresponding properties of the A_x.

RESUMEN

Sea $\{A_x : x \in X\}$ una colección de álgebras de Banach complejas indexadas por un espacio de Hausdorff compacto X. Investigamos la amenabilidad débil y la seudo-amenabilidad de ciertas álgebras $\mathcal{A}$ de funciones con valores en A_x en relación a las propiedades correspondientes de los A_x.

Keywords and Phrases: Function algebra, weak amenability, pseudo-amenability.

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1 Preliminaries

Suppose that $\mathcal{P}$ is a property which a Banach space A might possess. A reasonable question to ask about $\mathcal{P}$ is of the sort "What constructions on Banach spaces preserve $\mathcal{P}$?" To clarify this, we take a specific example: Let A be a complex Banach algebra and suppose that we are interested in amenability. It is then well known that $\mathcal{P}$, the property of being amenable, is preserved by quotients: If A is an amenable Banach algebra, and $I \subset A$ is a closed ideal, then A/I is also amenable. It is also well-known that amenability is preserved by projective tensor products: if A and B are both amenable, then so is $A\widehat{\otimes}B$. (See *e.g.* the survey paper [15, Prop. 2.3.2].) Loosely speaking, we can ask whether $\mathcal{P}$ is preserved "downwards" (through quotients), or "sideways" (through tensor products); amenability is preserved in both of these directions. We could also ask whether $\mathcal{P}$ is preserved by actions in two directions: *e.g.* if for a closed ideal I of a Banach algebra A, both I and A/I satisfy $\mathcal{P}$, does A also satisfy $\mathcal{P}$? (This is the 3-space problem.)

This short discussion leads us to ask the following: Can a property $\mathcal{P}$ be preserved "upwards"? We make this airy question more explicit: Let X be a set (usually, a topological space), and let $\{A_x : x \in X\}$ be a collection of Banach spaces indexed by X, over a common scalar field $\mathbb{K}$, either $\mathbb{R}$ or $\mathbb{C}$. Suppose that A_x possesses property $\mathcal{P}$ for each $x \in X$. Let $\mathcal{A} \subset \prod\{A_x : x \in X\}$ be a Banach space of functions under the pointwise operations, and let $\sigma \in \mathcal{A}$, so that $\sigma(x) \in A_x$ for all $x \in X$. What conditions on X, $\mathcal{A}$, and the A_x (aside from possessing $\mathcal{P}$) might be sufficient to insure that $\mathcal{A}$ also has $\mathcal{P}$? Again using amenability as an example, we see that if X is an infinite compact Hausdorff space, then $\mathcal{A} = \ell^1(X)$, the space of absolutely summable complex-valued functions on X under the pointwise operations, is not amenable: The amenability of each $A_x = \mathbb{C}$ is not passed on to $\mathcal{A}$, since $\mathcal{A}$ does not have a bounded approximate identity. On the other hand, $\mathcal{A} = c_0(X)$, the closure in the sup-norm of the space of $\mathbb{C}$-valued functions with finite support, is amenable. This suggests that, at the very least, $\mathcal{A}$ should have some conditions on it, and perhaps also that the collection $\{A_x : x \in X\}$ should satisfy some additional unifying property, aside from just having all A_x possess $\mathcal{P}$. We might also want X to satisfy some reasonable conditions.

Obviously, there are many ways in which it might be possible to go "up" in this sense. In this paper, we focus on one type of vector-valued function space $\mathcal{A}$. We will assume unless otherwise specified that X is a compact Hausdorff space and that $\{A_x : x \in X\}$ is a collection of complex Banach spaces; we take $\mathcal{A} \subset \prod\{A_x : x \in X\}$ to be a Banach space of functions under the pointwise operations which satisfies the conditions:

C1) For each $x \in X$, $A_x = p_x(\mathcal{A}) = \{\sigma(x) : \sigma \in \mathcal{A}\}$, that is, $\mathcal{A}$ is said to be full; p_x is the evaluation map at x, so that $p_x(\sigma) = \sigma(x)$.

C2) For each $\sigma \in \mathcal{A}$, the norm map $x \mapsto \|\sigma(x)\|$ is upper semicontinuous on X; hence σ is bounded and norm-attaining, with

$$\|\sigma\| = \sup\{\|\sigma(x)\| : x \in X\} = \|\sigma(x_0)\|$$

for some $x_0 \in X$.

C3) $\mathcal{A}$ is a $C(X)$-module under the pointwise operations.

C4) $\mathcal{A}$ is complete in the sup-norm.

We will call a space $\mathcal{A}$ which satisfies C1) - C4) an upper semicontinuous function space with fibers A_x, and abbreviate it to "function space".

If in addition to C1) - C4) it is also the case that

C5) Each A_x is a Banach algebra, and $\mathcal{A}$ is closed under pointwise multiplication (so that $\mathcal{A}$ is a Banach algebra). We call such an $\mathcal{A}$ a *function algebra*. Evidently, a function algebra $\mathcal{A}$ is commutative if and only if each fiber A_x is commutative.

Examples of such function spaces (algebras) can be found in [11, Section 2], and also (using the language of section spaces of bundles of Banach spaces and Banach algebras) in [5] and [14]. In particular, if A is a Banach algebra and $\{A_x : x \in X\}$ is a collection of Banach algebras, such function algebras include $C(X, A)$, the space of continuous A-valued functions on X, and $c_0(X, \{A_x\})$, the closure in the sup norm of the functions $\sigma \in \prod\{A_x : x \in X\}$ with finite support. A brief, and quite incomplete, bibliographical note on such function spaces can be found at the end of [11, Section 2].

For a more general setting, the reader may also wish to consult [1]. Using slightly different language, that paper studies algebras $\mathcal{A}$ of vector-valued functions over a completely regular Hausdorff space X. These functions take their values in associative topological algebras $\{A_x : x \in X\}$, and the algebra $\mathcal{A}$ is assumed to satisfy C1) and C3) above, without the completeness or norm conditions of C2), C4), or C5).

Heritability has been explored previously (using either the language of function spaces or section spaces of bundles of Banach spaces), for example in [4] (where $\mathcal{A}$ is simply a function space), [7], and [8]. Of particular interest here are papers concerned with how some variants of amenability can be inherited by function algebras $\mathcal{A}$, *e.g.* amenability itself ([9]), module amenability ([11]), and character amenability ([12]); in all these papers, appropriate uniform boundedness conditions were shown sufficient to guarantee that the property under consideration was preserved by $\mathcal{A}$. By using the existence of certain conditions intrinsic to Banach algebras sufficient to establish the various properties $\mathcal{P}$ of interest, it was possible to avoid the homological definitions of the properties. In this paper we will investigate the preservation to $\mathcal{A}$ of weak amenability in the A_x by employing a similar work-around. We will also investigate the preservation to $\mathcal{A}$ of pseudo-amenability; the results in this case are not so satisfying.

We note several important properties of function spaces (algebras) $\mathcal{A}$.

I) The evaluation map $\sigma \mapsto \sigma(x)$ from $\mathcal{A}$ to A_x is a quotient map. Indeed, we have $A_x \simeq \mathcal{A}/\overline{I_x\mathcal{A}}$, where $I_x \subset C(X)$ is the maximal ideal of functions f such that $f(x) = 0$, and $\overline{I_x\mathcal{A}}$ is the closed span in $\mathcal{A}$ of elements of the form $f\sigma$ ($\sigma \in \mathcal{A}$, $f \in I_x$). The correspondence is given by $\sigma(x) \leftrightarrow \sigma + \overline{I_x\mathcal{A}}$.

II) Let $\mathcal{B}$ be a function subspace of $\mathcal{A}$, *i.e.* a closed subspace of $\mathcal{A}$ which is also a $C(X)$-module, and set $B_x = p_x(\mathcal{B}) = \{\sigma(x) : \sigma \in \mathcal{B}\} \subset A_x$. Then $B_x \subset A_x$ is a closed subspace, and $\mathcal{B}_x = \{\sigma \in \mathcal{A} : \sigma(x) \in B_x\}$ is a function subspace (necessarily full) of $\mathcal{A}$; $\mathcal{B}_x$ has fibers B_x and A_y (if $y \neq x$). Moreover, (*) $\mathcal{B} = \bigcap\{\mathcal{B}_x : x \in X\} = \{\sigma \in \mathcal{A} : \sigma(x) \in B_x \text{ for all } x \in X\}$ and $p_x(\mathcal{B}) = B_x$. In particular, if $\mathcal{B}$ and $\mathcal{C}$ are function subspaces of $\mathcal{A}$ such that $B_x = C_x$ for all x, then $\mathcal{B} = \mathcal{C}$. [Two caveats: 1) We need $\mathcal{B}$ and $\mathcal{C}$ to be subspaces of a common function space $\mathcal{A}$; it is not enough to have function spaces $\mathcal{B}$ and $\mathcal{C}$ over X which have fibers $B_x = C_x$ for all $x \in X$, as the example $\mathcal{B} = C(X)$ and $\mathcal{C} = c_0(X)$ shows. 2) In order for (*) to hold, we also need to specify $\mathcal{B}$, and hence both its fibers B_x and the function subspaces $\mathcal{B}_x \subset \mathcal{A}$. Merely specifying some subspaces $B_x \subset A_x$ is insufficient if we wish the fibers of $\mathcal{B} = \bigcap\{\mathcal{B}_x : x \in X\}$ to be the B_x. For example, consider the case $X = [0,1]$, $\mathcal{B} = C(X)$, and $B_x = 0$ if x is rational, and $B_x = \mathbb{C}$ otherwise. Then $\mathcal{B}_x = \{f \in \mathcal{B} : f(x) = 0\}$ if x is rational, and $\mathcal{B}_x = \mathcal{B}$ otherwise. But $\bigcap\{\mathcal{B}_x : x \in X\} = \{0\} \subset \mathcal{B}$.]

III) Let $\mathcal{A}$ be a function algebra with fibers A_x. Then $\mathcal{A}$ is a $C(X)$-(bi)module, and we let $J = J_{\mathcal{A}} \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ be the closed span of elements of the form $(f\sigma \otimes \tau) - (\sigma \otimes f\tau) = [(f \otimes \overline{\mathbf{1}}) - (\overline{\mathbf{1}} \otimes f)](\sigma \otimes \tau)$, where $\overline{\mathbf{1}} \in C(X)$ is the identity, *i.e.* the function with constant value 1. We call J the $C(X)$-balanced kernel in $\mathcal{A}\widehat{\otimes}\mathcal{A}$. It is easy to check that J is both an ideal and a $C(X)\widehat{\otimes}C(X)$ submodule in $\mathcal{A}\widehat{\otimes}\mathcal{A}$. Then there is a function algebra $\mathcal{A} \otimes_X \mathcal{A}$ with fibers $A_x\widehat{\otimes}A_x$ and a $C(X)$-isometric isomorphism $q : (\mathcal{A}\widehat{\otimes}\mathcal{A})/J \to \mathcal{A} \otimes_X \mathcal{A}$, where $[q(\sigma \otimes \tau + J)](x) = (\sigma \odot \tau)(x) = \sigma(x) \otimes \tau(x)$. The isometry is given by

$$\left\|\sum_k \sigma_k \odot \tau_k\right\|_{\mathcal{A}\otimes_X\mathcal{A}} = \sup_{x\in X}\left\|\sum_k \sigma_k(x) \otimes \tau_k(x)\right\|_{A_x\widehat{\otimes}A_x} = \left\|\sum_k \sigma_k \otimes \tau_k + J\right\|_{(\mathcal{A}\widehat{\otimes}\mathcal{A})/J}.$$

Of these properties, I) and II) can be found in various locations in [5, Chap. 9]; and III) is [16, Thm. 1.2 and Prop. 1.5].

We now proceed to our studies of weak amenability and pseudo-amenability.

2 Weak amenability and heritability

Recall that if A is a Banach algebra and M is a Banach A-bimodule, then M^* can be made into Banach A-bimodule in a standard fashion via the actions

$$\langle m^*a, m\rangle = \langle m^*, am\rangle \quad \text{and} \quad \langle am^*, m\rangle = \langle m^*, ma\rangle\,,$$

where $a \in A, m \in M$, and $m^* \in M^*$. A derivation $D : A \to M$ is a continuous linear map such that $D(ab) = aD(b) + D(a)b$. The derivation $D : A \to M$ is said to be *inner* if

$$D(a) = \delta_m(a) = am - ma$$

for some $m \in M$.

Definition 2.1. *Let A be a complex Banach algebra. Say that A is weakly amenable if each derivation $D : A \to A^*$ is inner.*

Recalling that A is said to be amenable if, for any Banach A-bimodule M, each derivation $D : A \to M^*$ is inner [13], it is clear that an amenable algebra is also weakly amenable. If A is commutative, then $Fa = aF$ for each $a \in A$ and $F \in A^*$, so for a commutative algebra A to be weakly amenable is to say that A has no non-zero derivations to A^*.

Note that amenability and weak amenability can also be expressed in homological terms, the details of which are not necessary here. In the event that A is commutative there are, however, conditions intrinsic to A which are equivalent to its weak amenability. For the remainder of this section we will assume (at the possible loss of some unnecessary generality) that all algebras are commutative, and employ these conditions to investigate the heritability of weak amenability for function algebras $\mathcal{A}$.

We first note a necessary condition for $\mathcal{A}$ to be weakly amenable.

Proposition 2.2. *Suppose that the function algebra $\mathcal{A}$, defined over X, is weakly amenable. Then each fiber A_x is weakly amenable.*

Proof. Recall that $A_x \simeq \mathcal{A}/\overline{I_x\mathcal{A}}$, and use the fact that quotients of weakly amenable algebras by closed ideals are themselves weakly amenable. (See [6, Prop. 2.1] or [15, Prop. 2.5.3].) □

To exhibit conditions on the fibers A_x sufficient to make the function algebra $\mathcal{A}$ weakly amenable, we start with some notation modified from [6]: If A is a complex commutative Banach algebra, then we take $A^\#$ to be A with its standard adjunction of identity. (So, $A^\# = A \oplus \mathbb{C}\mathbf{1}$, with the ℓ^1-norm, and multiplication $(a \oplus \lambda\mathbf{1})(b \oplus \mu\mathbf{1}) = (ab + \lambda b + \mu a) \oplus \lambda\mu\mathbf{1}$. We will abuse notation only slightly and write $a + \lambda = a + \lambda\mathbf{1} \in A^\#$.) Let $K_A^\# \subset A^\#\widehat{\otimes}A^\#$ be the closed ideal which is the kernel of the multiplication map $\pi^{\#}: A^{\#}\widehat{\otimes}A^{\#} \to A^{\#}, (a+\lambda)\otimes(b+\mu) \mapsto ab+\lambda b+\mu a+\lambda\mu$, and let $\pi : A\widehat{\otimes}A \to A$ be the restriction of $\pi^{\#}$ to $A\widehat{\otimes}A \subset A^{\#}\widehat{\otimes}A^{\#}$. We set $K_A^0 = K_A^{\#} \cap (A\widehat{\otimes}A)$; note that this makes sense since A is complemented in $A^{\#}$ by a projection of norm 1, and hence $A\widehat{\otimes}A$ is an actual subset of $A^{\#}\widehat{\otimes}A^{\#}$ ([17, Prop. 2.4]). Since $A\widehat{\otimes}A$ is an ideal in $A^{\#}\widehat{\otimes}A^{\#}$, we have $K_A^0 \subset A\widehat{\otimes}A$ is also a closed ideal. In particular, for $u = \sum_k (a_k+\lambda_k)\otimes(b_k+\mu_k) \in A\widehat{\otimes}A$, we have $u \in K_A^{\#}$ if and only if

$$\sum_k (a_kb_k + \lambda_k b_k + \mu_k a_k + \lambda_k\mu_k) = 0 \in A^{\#}.$$

Especially, we have $\sum_k \lambda_k\mu_k = 0 \in \mathbb{C}$. Likewise, an element $u \in K_A^0$ is of the form $\sum_k a_k \otimes b_k$, with $\sum_k a_kb_k = 0 \in A$.

For later use, we note the following:

Lemma 2.3. *Let A be a Banach algebra. Then $\overline{(K_A^0)^2} \subset \overline{K_A^{\#}(A\widehat{\otimes}A)}$.*

Proof. By definition, K_A^0 is a subset of both $K_A^{\#}$ and $A\widehat{\otimes}A$, so that if $z, z' \in K_A^0$, we consider that $z \in K_A^{\#}$ and $z' \in A\widehat{\otimes}A$, so that $zz' \in \left(K_A^0\right)^2 \subset K_A^{\#}(A\widehat{\otimes}A)$; now use linearity and density. □

The following result characterizes the weak amenability of a commutative Banach algebra A. Recall that a Banach algebra is said to be essential provided that $A^2 = span\{ab : a, b \in A\}$ is dense in A.

Theorem 2.4 ([6, Thm. 3.2])**.** *Let A be a commutative complex Banach algebra. Then the following are equivalent:*

1) *A is weakly amenable;*
2) *A is essential and $\overline{(K_A^0)^2} = \overline{K_A^{\#}(A\widehat{\otimes}A)}$.*

Note that there are other equivalences established in the cited theorem; the one here is sufficient for our purpose.

Suppose now that $\mathcal{A}$ is a function algebra such that each A_x is weakly amenable. We will show that $\mathcal{A}$ is also weakly amenable. The first task is to show that $\mathcal{A}$ is essential.

Proposition 2.5. *Let $\mathcal{A}$ be a function algebra over X, and suppose that each A_x is essential. Then $\mathcal{A}$ is essential, and conversely.*

Proof. This result (a Stone-Weierstrass theorem for function algebras) is a variant of [5, Cor. 4.3], and can also be found in [11]. However, it is worth looking at a proof, using our current language.

Let $\sigma \in \mathcal{A}$, and let $\varepsilon > 0$. For each $x \in X$, we can find $t_x = \sum_{k=1}^{m_x} a_{x,k}b_{x,k} \in A_x^2$ such that $\|\sigma(x) - t_x\| < \varepsilon$. From condition C1), above, we can choose $\tau_{x,k}, \tau'_{x,k} \in \mathcal{A}$ such that $\tau_{x,k}(x) = a_{x,k}$, $\tau'_{x,k}(x) = b_{x,k}$. Set $\nu_x = \sum_{k=1}^{m_x} \tau_{x,k}\tau'_{x,k} \in \mathcal{A}^2$. Since $\|\sigma(x) - \nu_x(x)\| < \varepsilon$, it follows from C2) that there is a neighborhood V_x of x such that whenever $y \in V_x$, we have $\|\sigma(y) - \nu_x(y)\| < \varepsilon$. Take a finite subcover $\{V_j\} = \{V_{x_j} : j = 1, \ldots, n\}$ of the V_x, and let $\{f_j : j = 1, \ldots, n\} \subset C(X)$ be a partition of unity subordinate to the V_j, so that for each $j = 1, \ldots, n$, we have: $0 \leq f_j \leq 1$, f_j is supported on V_j, and $\sum_j f_j = \overline{\mathbf{1}}$.

For any $y \in X$, we then have

$$\left\|\sigma(y) - \sum_j f_j(y)\nu_j(y)\right\| = \left\|\sum_j f_j(y)\left[\sigma(y) - \nu_j(y)\right]\right\| \leq \sum_{j \text{ s.t. } y \in V_j} f_j(y) \left\|\sigma(y) - \nu_j(y)\right\| < \varepsilon,$$

so that $\left\|\sigma - \sum_j f_j \nu_j\right\| < \varepsilon$, and therefore σ is in the closure of $\mathcal{A}^2$.

The converse is an immediate consequence of C1). $\square$

This shows, in particular, that the property of being essential is preserved by function algebras. Especially, if each A_x has an approximate identity, so also does $\mathcal{A}$; if the approximate identities in the fibers A_x of $\mathcal{A}$ are uniformly bounded, then the approximate identity in $\mathcal{A}$ is bounded. (See [9] and [12].)

It is straightforward to check that both $\overline{(K^0_{\mathcal{A}})^2}$ and $\overline{K^{\#}_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})}$ are closed $C(X)\widehat{\otimes}C(X)$-submodules of, and ideals in, $\mathcal{A}\widehat{\otimes}\mathcal{A}$.

Lemma 2.6. *Let $\mathcal{A}$ be a function algebra such that each A_x is weakly amenable, and let $J \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ be the $C(X)$-balanced kernel. Then $J \subset \overline{(K^0_{\mathcal{A}})^2}$.*

Proof. Note that since each A_x is weakly amenable, we have each A_x^2 is dense in A_x, so that $\mathcal{A}^2$ is dense in $\mathcal{A}$. By definition, J is the closed span in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ of elements of the form $[(\overline{\mathbf{1}} \otimes f) - (f \otimes \overline{\mathbf{1}})](\sigma \otimes \tau)$. But since $\mathcal{A}^2$ is dense in $\mathcal{A}$, $\sigma \otimes \tau$ can be written as a limit of elements of form $(\sum_k \sigma'_k \sigma''_k) \otimes \left(\sum_j \tau'_j \tau''_j\right) = \sum_{k,j} \sigma'_k \sigma''_k \otimes \tau'_j \tau''_j = \sum_{j,k} \left(\sigma'_k \otimes \tau'_j\right)\left(\sigma''_k \otimes \tau''_j\right)$. Restricting ourselves for the moment to elements of the form $\sigma \otimes \tau = \sigma_1\sigma_2 \otimes \tau_1\tau_2 = (\sigma_1 \otimes \tau_1)(\sigma_2 \otimes \tau_2) \in \mathcal{A}\widehat{\otimes}\mathcal{A}$, and noting that $(\overline{\mathbf{1}} \otimes f) - (f \otimes \overline{\mathbf{1}})$ is in the kernel of the multiplication map $f \otimes g \mapsto fg$ from $C(X)\widehat{\otimes}C(X)$ to $C(X)$, we can write

$$\left[(\overline{\mathbf{1}} \otimes f) - (f \otimes \overline{\mathbf{1}})\right](\sigma \otimes \tau) = \lim_{\mu} \left[(\overline{\mathbf{1}} \otimes f) - (f \otimes \overline{\mathbf{1}})\right](\sigma_1 \otimes \tau_1) h_\mu (\sigma_2 \otimes \tau_2),$$

where $\{h_\mu\}$ is a bounded approximate identity for $\ker \pi_{C(X)}$. (Such an $\{h_\mu\}$ exists because $C(X)$ is amenable, so that the above-mentioned kernel J has a bounded approximate identity; see [10, p. 254].)

It is evident that both $\left[(\overline{\mathbf{1}} \otimes f) - (f \otimes \overline{\mathbf{1}})\right](\sigma_1 \otimes \tau_1) \in K^0_{\mathcal{A}}$, and $h_\mu(\sigma_2 \otimes \tau_2) \in K^0_{\mathcal{A}}$, so that $\left[(\overline{\mathbf{1}} \otimes f) - (f \otimes \overline{\mathbf{1}})\right](\sigma_1 \otimes \tau_1) h_\mu (\sigma_2 \otimes \tau_2) \in \overline{(K^0_{\mathcal{A}})^2}$. The rest follows by linearity, density, and the boundedness of $\{h_\mu\}$. $\square$

It follows from Lemma 2.3 that we also have $J \subset \overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}$.

We obtain from the above that $\mathcal{G} = \overline{(K_{\mathcal{A}}^0)^2}/J$ and $\mathcal{H} = \overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}/J$ are function subalgebras of $\mathcal{A} \otimes_X \mathcal{A}$, with fibers G_x, $H_x \subset A_x\widehat{\otimes}A_x$, respectively.

Now, consider $\mathcal{H} = \overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}/J$. Recalling the discussion preceding Lemma 2.3 about the multiplication maps $\pi^{\#}$ and π, we see that a typical element $u \in \overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}$ is a limit of sums of elements of the form

$$\left[\sum_k (\sigma_k + \lambda_k) \otimes (\tau_k + \mu_k)\right] \left[\sum_j \alpha_j \otimes \beta_j\right] \in \mathcal{A}\widehat{\otimes}\mathcal{A},$$

(where the first sum is in $K_{\mathcal{A}}^{\#}$ and the second is in $\mathcal{A}\widehat{\otimes}\mathcal{A}$), with $\pi^{\#}(u) = \pi(u) = 0 \in \mathcal{A}$. If u is such a limit, the image of u in $\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}/J$ under the quotient map is therefore a (uniform) limit of sums of functions of the type

$$\left[\sum_k (\sigma_k + \lambda_k) \odot (\tau_k + \mu_k)\right] \left[\sum_j \alpha_j \odot \beta_j\right] = \sum_{k,j} (\sigma_k\alpha_j + \lambda_k\alpha_j) \odot (\tau_k\beta_j + \mu_k\beta_j) \in \mathcal{A}\widehat{\otimes}_X\mathcal{A},$$

where

$$\begin{aligned}
\pi\left(\left[\sum_{k,j} (\sigma_k\alpha_j + \lambda_k\alpha_j) \odot (\tau_k\beta_j + \mu_k\beta_j)\right](x)\right) &= \pi\left(\sum_{k,j} [\sigma_k(x)\alpha_j(x) + \lambda_k\alpha_j(x)] \otimes [\tau_k(x)\beta_j(x) + \mu_k\beta_j(x)]\right) \\
&= \pi\left(\left[\sum_k (\sigma_k(x) + \lambda_k) \otimes (\tau_k(x) + \mu_k)\right]\left[\sum_j \alpha_j(x) \otimes \beta_j(x)\right]\right) \\
&= 0 \in A_x,
\end{aligned}$$

for each $x \in X$. Thus, $(u + J)(x) \in \overline{K_x^{\#}(A_x\widehat{\otimes}A_x)}$, for each $x \in X$, where $K_x^{\#} = K_{A_x}^{\#}$, and so the fibers $H_x = p_x(\overline{K_x^{\#}(A_x\widehat{\otimes}A_x)}/J) \subset p_x(\mathcal{A} \otimes_X \mathcal{A})$ of $\mathcal{H}$ are subspaces of the $\overline{K_x^{\#}(A_x\widehat{\otimes}A_x)}$ for each $x \in X$.

On the other hand, an element $v \in \overline{K_x^{\#}(A_x\widehat{\otimes}A_x)}$ is the limit in $A_x\widehat{\otimes}A_x$ of sums of elements of the form

$$\left[\sum_k (a_k + \lambda_k) \otimes (b_k + \mu_k)\right] \left[\sum_j c_j \otimes d_j\right] = \sum_{k,j} (a_kc_j + \lambda_kc_j) \otimes (b_kd_j + \mu_kd_j) \in K_x^{\#}(A_x\widehat{\otimes}A_x).$$

For each such element v, we can choose $\alpha_{k,j}, \beta_{k,j} \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\alpha_{k,j}(x) = a_kc_j + \lambda_kc_j$, $\|\alpha_{k,j}\| = \|a_kc_j + \lambda_kc_j\|$, and $\beta_{k,j}(x) = b_kd_j + \mu_kd_j$, $\|\beta_{kj}\| = \|b_kd_j + \mu_kd_j\|$ (see [14, Prop. 1.1]).

Note that by the definition of the norm in projective tensor products, we have

$$\left\|\sum_k (a_k+\lambda_k)\otimes(b_k+\mu_k)\right\| \le \sum_k \|a_k+\lambda_k\|\,\|b_k+\mu_k\| < \infty,$$

and similarly for $\left\|\sum_j c_j \otimes d_j\right\|$. Then

$$\begin{aligned}\left\|\sum_{k,j}\alpha_{k,j}\otimes\beta_{k,j}\right\| &\le \sum_{k,j}\|\alpha_{k,j}\|\,\|\beta_{k,j}\| = \sum_{k,j}\|a_kc_j+\lambda_kc_j\|\,\|b_kd_j+\mu_kd_j\| \\ &\le \sum_{k,j}\|a_k+\lambda_k\|\,\|c_j\|\,\|b_k+\mu_k\|\,\|d_j\| \\ &= \left(\sum_k \|(a_k+\lambda_k)\otimes(b_k+\mu_k)\|\right)\left(\sum_j \|c_j\otimes d_j\|\right) < \infty\end{aligned}$$

and so we have $\sum_{k,j}\alpha_{k,j}\otimes\beta_{k,j}\in\mathcal{A}\widehat{\otimes}\mathcal{A}$. Moreover,

$$\begin{aligned}\pi\left[\sum_{k,j}\alpha_{k,j}\otimes\beta_{k,j}+J\right](x) &= \pi\left[\sum_{k,j}\alpha_{k,j}\odot\beta_{k,j}\right](x) \\ &= \pi\left[\sum_{k,j}(a_kc_j+\lambda_kc_j)\otimes(b_kd_j+\mu_kd_j)\right] = 0\in A_x\end{aligned}$$

so that $\sum_{k,j}\alpha_{k,j}\odot\beta_{k,j}\in\mathcal{H}_x$. Thus, $p_x(\mathcal{H}_x) = H_x$ is dense in $\overline{K_x^{\#}(A_x\widehat{\otimes}A_x)}$; coupled with the preceding, we have $H_x = \overline{K_x^{\#}(A_x\widehat{\otimes}A_x)}$.

By similar arguments, we have $p_x(\mathcal{G}) = G_x = \overline{(K_x^0)^2}$, so that $\mathcal{G} = \overline{(K_{\mathcal{A}}^0)^2}/J = \bigcap_x\{z\in(\mathcal{A}\widehat{\otimes}\mathcal{A})+J : z(x)\in\overline{(K_x^0)^2} = G_x\} = \bigcap\{\mathcal{G}_x : x\in X\}$.

But now, since A_x is weakly amenable for each x, the fibers G_x of $\overline{(K_{\mathcal{A}}^0)^2}/J\subset\mathcal{A}\otimes_X\mathcal{A}$ and H_x of $\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}/J\subset\mathcal{A}\otimes_X\mathcal{A}$ are identical, so that $\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}/J = \overline{(K_{\mathcal{A}}^0)^2}/J\subset\mathcal{A}\otimes_X\mathcal{A}$.

We have shown:

Lemma 2.7. *Let $\mathcal{A}$ be a commutative function algebra such that each fiber A_x is weakly amenable. Then the quotient algebras $\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}/J\subset\mathcal{A}\otimes_X\mathcal{A}$ and $\overline{(K_{\mathcal{A}}^0)^2}/J\subset\mathcal{A}\otimes_X\mathcal{A}$ are identical.*

Corollary 2.8. *Let $\mathcal{A}$ be a commutative function algebra with weakly amenable fibers A_x. Then $\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})} = \overline{(K_{\mathcal{A}}^0)^2}$.*

Proof. Elementary algebra: Let $z\in\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}$. Then from the preceding Lemma, there exists $w\in\overline{(K_{\mathcal{A}}^0)^2}$ such that $z+J = w+J$. Hence $z-w\in J\subset\overline{(K_{\mathcal{A}}^0)^2}$, so that $z\in\overline{(K_{\mathcal{A}}^0)^2}+w = \overline{(K_{\mathcal{A}}^0)^2}$. Similarly, $\overline{(K_{\mathcal{A}}^0)^2}\subset\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}$. □

Theorem 2.9. *Suppose that $\mathcal{A}$ is a commutative function algebra such that each fiber A_x is weakly amenable. Then $\mathcal{A}$ is weakly amenable.*

Proof. Apply Theorem 2.4 to the preceding results. □

Corollary 2.10. *Suppose that X is a compact Hausdorff space, and that A and $\{A_x : x \in X\}$ are commutative and weakly amenable. Then so are $C(X, A)$ and $c_0(X, \{A_x\})$. If X is locally compact and Hausdorff, and A is commutative and weakly amenable, then so is $C_0(X, A)$, the space of continuous A-valued functions disappearing at infinity.*

Proof. We need only address the last assertion. Let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of X. Then $C_0(X, A)$ is $C(X)$-isometrically isomorphic to the function algebra $\overline{I_\infty C(X_\infty, A)}$, where I_∞ is the ideal in $C(X_\infty)$ of functions which disappear at ∞; and it is easily checked that $\overline{I_\infty C(X_\infty, A)}$ is a function algebra with fibers $A_x = A$, if $x \neq \infty$, $A_\infty = \{0\}$. □

Corollary 2.11. *Let $\mathcal{A}$ be a function algebra all of whose fibers are commutative C^*-algebras. Then $\mathcal{A}$ is weakly amenable.*

Proof. A C^*-algebra is weakly amenable [2, Thm. 5.6.77]. □

To the authors' knowledge, it is an open question as to whether a function algebra $\mathcal{A}$ with fibers that are all C^*-algebras is itself a C^*-algebra. That is easily seen to be the case if $\mathcal{A}$ is adjoint-closed, but the conclusion is not apparent if $\mathcal{A}$ is not assumed to be adjoint-closed.

We note that, in say [9], and similarly in [11] and [12], in order to induce amenability of its fibers A_x upward to a function algebra $\mathcal{A}$, we had to find someway of spreading the necessary boundedness conditions on each A_x across X to all of $\mathcal{A}$. In [9], for instance, we accomplished this by assuming that the bounded approximate identities on each A_x were uniformly bounded across the A_x. In the present situation, a necessary (and sufficient) condition for weak amenability of the fibers A_x of $\mathcal{A}$ is that each fiber be essential and that $\overline{K_x^\#(A_x \widehat{\otimes} A_x)} = \overline{(K_x^0)^2}$. And, as it turns out, Proposition 2.5 and the passing from $\mathcal{A}\widehat{\otimes}\mathcal{A}$ to the quotient $\mathcal{A} \otimes_X \mathcal{A}$ are the tools which spread that property across X to all of $\mathcal{A}$.

3 Pseudo-amenability and heritability

In the preceding section we mentioned the presence of conditions involving boundedness or essentialness of fibers which were sufficient to induce the heritability of the relevant conditions from fibers upward to function algebras. What happens if we eliminate boundedness conditions from the fibers? The answer, as we see in the following, is not nearly so satisfactory, at least as far as we are able to demonstrate. In this section, we make no assumptions about commutativity.

Recall the definition:

Definition 3.1 ([3]). *A (complex) Banach algebra A is said to be pseudo-amenable if there is a net $\{u_\lambda\} \subset A\widehat{\otimes}A$ (called an approximate diagonal) such that for each $a \in A$ we have $\|u_\lambda a - au_\lambda\| \to 0$ (in $A\widehat{\otimes}A$) and $\|\pi(u_\lambda)a - a\| \to 0$ (in A), where $\pi : A\widehat{\otimes}A \to A$ is the multiplication map.*

As an immediate consequence of this definition, we note:

Proposition 3.2. *Let $\mathcal{A}$ be a pseudo-amenable function algebra. Then each fiber A_x is pseudo-amenable.*

Proof. Note that $A_x \simeq \mathcal{A}/\overline{I_x\mathcal{A}}$ and that pseudo-amenability is preserved by quotients (see [3, Prop. 2.2]). $\square$

If $\{A_x : x \in X\}$ is a collection of pseudo-amenable algebras over the compact Hausdorff space X, it is shown in [3, Prop. 2.1] that each of the algebras $c_0(X, \{A_x\})$ and $\ell^p(X, \{A_x\})$, $1 \leq p < \infty$, is pseudo-amenable, where $\ell^p(X, \{A_x\}) \subset \prod\{A_x : x \in X\}$ is the space of choice functions σ over X such that $\|\sigma\| = \left(\sum_x \|\sigma(x)\|^p\right)^{1/p} < \infty$. While all of these are algebras and $C(X)$-modules, of course, only $c_0(X, \{A_x\})$ is a function algebra in our sense. Can we extend the pseudo-amenability result for $c_0(X, \{A_x\})$ to arbitrary function algebras with values in the A_x?

We obtain a partial answer. Recall that an elementary member of $\mathcal{A} \otimes_X \mathcal{A}$ is of the form $x \mapsto (\sigma \odot \tau)(x) = \sigma(x) \otimes \tau(x)$, where $\sigma, \tau \in \mathcal{A}$. Recall also that $J \subset \ker \pi$, where $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ is the multiplication map; thus (again abusing notation only slightly) $\pi : \mathcal{A} \otimes_X \mathcal{A} \to \mathcal{A}$ is well-defined.

Definition 3.3. *Let $\mathcal{A}$ be a function algebra over the compact Hausdorff space X with fibers A_x. Say that $\mathcal{A}$ is quotient pseudo-amenable if there exists a net $\{\nu_\lambda\} \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that for any $\sigma \in \mathcal{A}$ we have both*

$$\|(\nu_\lambda + J)\sigma - \sigma(\nu_\lambda + J)\|_{\mathcal{A}\otimes_X\mathcal{A}} = \sup_x \|\nu_\lambda(x)\sigma(x) - \sigma(x)\nu_\lambda(x)\| \to 0$$

and

$$\|\pi(\nu_\lambda + J)\sigma - \sigma\|_{\mathcal{A}} = \sup_x \|[\pi(\nu_\lambda)](x)\sigma(x) - \sigma(x)\| \to 0.$$

There is a slightly stronger version of Proposition 3.2:

Proposition 3.4. *Suppose that the function algebra $\mathcal{A}$ is quotient pseudo-amenable. Then each fiber A_x is pseudo-amenable.*

Proof. Let $\{\nu_\lambda\} \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ be a net which makes $\mathcal{A}$ pseudo-amenable. For any $x_0 \in X$ and $\sigma \in \mathcal{A}$, we have

$$\|(\nu_\lambda + J)\sigma - \sigma(\nu_\lambda + J)\| = \sup_{x\in X} \|[\nu_\lambda(x)]\sigma(x) - \sigma(x)[\nu_\lambda(x)]\| \geq \|[\nu_\lambda(x_0)]\sigma(x_0) - \sigma(x_0)[\nu_\lambda(x_0)]\| \to 0$$

and similarly for the other necessary convergence. $\square$

Before we proceed to the next result, we gather some notation. Suppose that $\mathcal{A}$ is a function algebra with pseudo-amenable fibers $\{A_x : x \in X\}$ and respective approximate diagonals $\{u_{\lambda_x} : \lambda_x \in \Lambda_x\}$. Set $\Lambda = \prod\{\Lambda_x : x \in X\}$, and write $\lambda(x) = \lambda_x$ (to avoid having subscripts be nested too deeply). Order Λ pointwise, *i.e.* $\lambda' \geq \lambda$ if and only if $\lambda'(x) \geq \lambda(x)$ for each $x \in X$. Given $\lambda \in \Lambda$, for each $x \in X$ we can choose and fix $\nu_{\lambda(x)} \in \mathcal{A} \otimes_X \mathcal{A}$ such that $\nu_{\lambda(x)}(x) = u_{\lambda(x)}$ and such that $\|\nu_{\lambda(x)}(x)\| = \|u_{\lambda(x)}\|$; again, the existence of such $\nu_{\lambda(x)}$ is guaranteed by Prop. 1.1 of [14]. Then by the definition of pseudo-amenability, for each $x \in X$ and $\sigma \in \mathcal{A}$, we have

$$\left\|\nu_{\lambda(x)}(x)\sigma(x) - \sigma(x)\nu_{\lambda(x)}(x)\right\| = \left\|u_{\lambda(x)}\sigma(x) - \sigma(x)u_{\lambda(x)}\right\| \to 0$$

and

$$\left\|[\pi(\nu_{\lambda(x)})](x)\sigma(x) - \sigma(x)\right\| = \left\|\pi(u_{\lambda(x)})\sigma(x) - \sigma(x)\right\| \to 0,$$

both as $\lambda(x)$ increases in Λ_x.

Theorem 3.5. *Let $\mathcal{A}$ be a function algebra over the compact Hausdorff space X with fibers A_x, and suppose that each A_x is pseudo-amenable. Then $\mathcal{A}$ is quotient pseudo-amenable.*

Proof. We use the methods of Lemma 4 and Cor. 3 of [11]. Let $F = \{\sigma_k : k = 1, \ldots, n\} \in \mathcal{A}$ and $m \in \mathbb{N}$ be given. Fix $\sigma = \sigma_k \in F$ and $x \in X$. Choose $\nu_{\lambda(x)}$ as above, and choose $\lambda_{m,k}(x) \in \Lambda_x$ such that if $\lambda(x) \geq \lambda_{m,k}(x)$, then both

$$\left\|\nu_{\lambda(x)}(x)\sigma(x) - \sigma(x)\nu_{\lambda(x)}(x)\right\| = \left\|u_{\lambda(x)}\sigma(x) - \sigma(x)u_{\lambda(x)}\right\| < 1/m$$

and

$$\left\|[\pi(\nu_{\lambda(x)})](x)\sigma(x) - \sigma(x)\right\| = \left\|\pi(u_{\lambda(x)})\sigma(x) - \sigma(x)\right\| < 1/m.$$

Then if $\lambda_m \in \Lambda$ is such that $\lambda_m \geq \max\{\lambda_{m,k} : k = 1, \ldots, n\}$ (*i.e.* $\lambda_m(x) \geq \max\{\lambda_{m,k}(x) : k = 1, \ldots, n\}$ for each $x \in X$), the above inequalities hold (for λ_m) for each $\sigma \in F$ and $x \in X$.

We now employ the upper semicontinuity of the norm functions in both $\mathcal{A}$ and $\mathcal{A} \otimes_X \mathcal{A}$. For $x \in X$, choose a neighborhood $V_x(F, m)$ such that if $y \in V_x(F, m)$ then both

$$\left\|\nu_{\lambda(x)}(x)\sigma(x) - \sigma(x)\nu_{\lambda(x)}(x)\right\| < 1/m$$

and

$$\left\|[\pi(\nu_{\lambda(x)})](x)\sigma(x) - \sigma(x)\right\| < 1/m$$

for all $\sigma \in F$.

Now, X is compact, so we can choose $\{x_j : j = 1, \ldots, s\}$ such that $\{V_j\} = \{V_{x_j}(F, m) : j = 1, \ldots, s\}$ also covers X. As in Proposition 2.5, let $\{f_j : j = 1, \ldots, s\}$ a partition of unity subordinate to the

V_j, and define $\xi = \xi(F,m)$ by $\xi = \sum_{j=1}^{s} f_j \nu_{\lambda_m(x_j)} \in \mathcal{A} \otimes_X \mathcal{A}$. Then for $y \in X$ and $\sigma \in F$, and setting $p = \|\xi(y)\sigma(y) - \sigma(y)\xi(y)\|$, we have

$$
\begin{aligned}
p &= \left\| \sum_{j \text{ s.t. } y \in V_j} f_j(y)[\nu_{\lambda_m(x_j)}(y)\sigma(y) - \sigma(y)\nu_{\lambda_m(x_j)}(y)] \right\| \\
&\leq \sum_{j \text{ s.t. } y \in V_j} f_j(y) \left\| \nu_{\lambda_m(x_j)}(y)\sigma(y) - \sigma(y)\nu_{\lambda_m(x_j)}(y) \right\| < \sum_{j \text{ s.t. } y \in V_j} f_j(y) \cdot 1/m \leq 1/m,
\end{aligned}
$$

so that $\|\xi\sigma - \sigma\xi\| = \sup_y \left\| \nu_{\lambda_m(x_j)}(y)\sigma(y) - \sigma(y)\nu_{\lambda_m(x_j)}(y) \right\| < 1/m$ (in $\mathcal{A} \otimes_X \mathcal{A}$).

Similarly, we have $\|\pi(\xi)\sigma - \sigma\| = \sup_x \|[\pi(\xi)](y)\sigma(y) - \sigma(y)\| < 1/m$ (in $\mathcal{A}$).

Finally, set $\Psi = \{(F,m) : F \subset \mathcal{A}$ is finite and $m \in \mathbb{N}\}$, and order Ψ by $(F',m') > (F,m)$ if $F' \supset F$ and $m' > m$. By the preceding, for each $(F,m) \in \Psi$ there exists $\xi = \xi(F,m) \in \mathcal{A} \otimes_X \mathcal{A}$ such that for each $\sigma \in F$ we have both $\|\xi\sigma - \sigma\xi\| < 1/m$ and $\|\pi(\xi)\sigma - \sigma\| < 1/m$. In particular, for a given $\sigma_0 \in \mathcal{A}$ and $m_0 \in \mathbb{N}$, there exists $(F_0, m_0) \in \Psi$, with $\sigma_0 \in F_0$, such that if $(F',m') > (F_0,m_0)$ then both $\|\xi'\sigma_0 - \sigma_0\xi'\| < 1/m' < 1/m_0$ and $\|\pi(\xi')\sigma_0 - \sigma_0\| < 1/m'$, where $\xi' = \xi'(F',m')$ is constructed as above. Therefore $\{\xi = \xi(F,m) : F \subset \mathcal{A}$ is finite and $m \in \mathbb{N}\}$ is an approximate diagonal for $\mathcal{A}$. □

Thus, $\mathcal{A}$ is quotient pseudo-amenable if and only if each A_x is itself pseudo-amenable.

Proposition 3.6. *Suppose that $\mathcal{A}$ is a function algebra over X, and that each fiber A_x is abelian and pseudo-amenable. Then $\mathcal{A}$ is weakly amenable.*

Proof. An abelian pseudo-amenable algebra is weakly amenable [3, Cor. 3.7]. Therefore $\mathcal{A}$ is weakly amenable; see Theorem 2.9. □

Naturally, Theorem 3.5 is a weaker result than we would like, especially given other amenability results on function algebras. We suspect that the main obstacle in general is that for pseudo-amenability we can not employ any boundedness conditions. (Indeed, in [15], pseudo-amenability is introduced as "amenability without boundedness.") The reader will note that in the proof of pseudo-amenability of $c_0(X,\{A_x\})$ (and the other spaces $\ell^p(X,\{A_x\})$) in [3], crucial use is made of the facts that elements $\sigma \in c_0(X,\{A_x\})$ with finite support are dense in the space and that there are projections from $c_0(X,\{A_x\})$ into its subspaces consisting of functions with finite support. This of course need not be the case for general function algebras.

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