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Existence and stability of solutions of totally nonlinear neutral Caputo q-fractional difference equations

Adusei-Poku Afful[1]
Ernest Yankson[1,✉]
Agnes Adom-Konadu[1]

[1] *Department of Mathematics, University of Cape Coast, Ghana.*
opokuafful2@gmail.com
ernest.yankson@ucc.edu.gh[✉]
agnes.donkor@ucc.edu.gh

ABSTRACT

This paper investigates the existence and stability of solutions for a class of totally nonlinear neutral Caputo q-fractional difference equations of order $0 < \alpha < 1$. By transforming the equation into an equivalent integral equation and leveraging the Krasnoselskii-Burton fixed point theorem, we establish sufficient conditions for the existence of solutions. The methodology involves decomposing the integral operator into a sum of a compact operator and a large contraction. Furthermore, suitable conditions for the stability of these solutions are derived. Our theoretical results extend and generalize previous findings in the literature. An illustrative example is provided to demonstrate the applicability of the main theorems.

RESUMEN

Este artículo investiga la existencia y estabilidad de soluciones para una clase de ecuaciones en diferencias Caputo q-fraccionarias neutrales totalmente no lineales de orden $0 < \alpha < 1$. Transformando la ecuación en una ecuación integral equivalente y aprovechando el teorema de punto fijo de Krasnoselskii-Burton, establecemos condiciones suficientes para la existencia de soluciones. La metodología involucra descomponer el operador integral en una suma de operadores compactos y una contracción grande. Más aún, derivamos condiciones apropiadas para la estabilidad de estas soluciones. Nuestros resultados teóricos extienden y generalizan hallazgos previos en la literatura. Se entrega un ejemplo ilustrativo para demostrar la aplicabilidad de los teoremas principales.

Keywords and Phrases: Existence, stability, q-fractional difference equations, Krasnoselskii-Burton fixed point, large contraction, Arzela-Ascoli's theorem.

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1 Introduction

The realm of q-calculus, also known as quantum calculus, offers a fascinating extension of classical calculus, operating without the conventional concept of limits. Its genesis can be traced back to the early twentieth century with the pioneering work of F. H. Jackson [21]. This framework provides a robust mathematical toolkit for analyzing functions that may exhibit non-smooth behavior. Subsequent developments by numerous researchers have significantly enriched the theoretical underpinnings of q-calculus and broadened its applicability [5, 17–20].

In recent decades, the intersection of q-calculus with fractional calculus has given rise to the vibrant field of q-fractional calculus, leading to the study of q-fractional difference equations. These equations have garnered considerable attention due to their capacity to model complex systems with memory and hereditary properties [2, 8–10, 17, 22–24, 30]. Fixed point theorems have emerged as indispensable tools in the analysis of q-fractional difference equations, instrumental not only in establishing the existence and uniqueness of solutions but also in examining crucial stability properties [6, 12–16, 25–28]. The work of Mesmouli, Ardjouni, and collaborators [25–28] is particularly relevant, addressing various forms of nonlinear neutral Caputo q-fractional difference equations. The Caputo q-fractional derivative, introduced by Abdeljawad and Baleanu [3], alongside supporting theoretical work [1, 7], provides essential tools for such investigations.

For $0 < q < 1$, define the time scale $\mathbb{T}_q = \{q^n, n \in \mathbb{Z}\} \cup \{0\}$, where $\mathbb{Z}$ is the set of integers. For $a = q^{n_0}$ and $n_0 \in \mathbb{Z}$, denote $\mathbb{T}_a = [a, \infty)_q = \{q^i a, i = 0, 1, 2, \dots\}$. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space and define $\mathbb{I}_\tau = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, \dots, a\}$ and $\mathbb{T}_{\tau a} = [\tau a, \infty)q = \{q^{-i}\tau a, i = 0, 1, 2, \dots\}$, where $\tau = q^d \in \mathbb{T}_q$, $d \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\mathbb{I}_\tau = \{a\}$ with $d = 0$, is the non-delay case.

Recently, Abdeljawad, Alzabut and Zhou in [2] studied the existence of solutions for the q-fractional difference equation

$$\begin{cases} {}_qC_a^\alpha x(t) = f(t, x(t), x(\tau t)), \quad t \in \mathbb{T}_a, \\ x(t) = \phi(t), \quad t \in \mathbb{I}_\tau, \end{cases} \tag{1.1}$$

where $f : \mathbb{T}_a \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and ${}_qC_a^\alpha$ represents Caputo's q-fractional difference of order $\alpha \in (0, 1)$. By employing the Krasnoselskii fixed point theorem, the authors obtained existence results.

Moreover, Mesmouli and Ardjouni in [25] studied the existence, uniqueness and stability of solutions for nonlinear neutral q-fractional difference equation

$$\begin{cases} {}_qC_a^\alpha (x(t) - g(t, x(\tau t)) = f(t, x(t), x(\tau t)), \quad t \in \mathbb{T}_a, \\ x(t) = \psi(t), \quad t \in \mathbb{I}_\tau, \end{cases} \tag{1.2}$$

where $f : \mathbb{T}_a \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{T}_a \times \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{I}_\tau \to \mathbb{R}$ and ${}_qC_a^\alpha$ represents Caputo's q-fractional difference of order $\alpha \in (0,1)$. To establish the results, the authors applied Krasnoselskii's and Banach's fixed point theorems, as well as Arzela–Ascoli's theorem.

Motivated by [2] and [25], we study the existence and stability of solutions for the totally nonlinear neutral q-fractional difference equation

$$\begin{cases} {}_qC_a^\alpha(h(x(t)) - g(t, x(\tau t))) = f(t, x(t), x(\tau t)), \quad t \in \mathbb{T}_a, \\ x(t) = \psi(t), \quad t \in \mathbb{I}_\tau, \end{cases} \tag{1.3}$$

where $h : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{T}_a \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{T}_a \times \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{I}_\tau \to \mathbb{R}$ and ${}_qC_a^\alpha$ represents Caputo's q-fractional difference of order $\alpha \in (0,1)$. To prove our main results, we employ the Krasnoselskii-Burton fixed point theorem.

The paper is structured as follows: Section 2 provides essential preliminaries, including definitions and lemmas from q-calculus and fractional difference calculus, the inversion of Equation (1.3) to its integral form, and the statement of the Krasnoselskii-Burton fixed point theorem. Section 3 is dedicated to proving the existence of solutions for Equation (1.3) under derived conditions. Section 4 presents results on the stability of these solutions. Section 5 offers an illustrative example. Finally, Section 6 presents concluding remarks.

2 Preliminaries

In this section, we give some basic notations, definitions, and properties of q-calculus and fractional difference calculus, which are used throughout this paper; see [2] and [25].

Definition 2.1 ([3]). *For a function $f : \mathbb{T}_q \to \mathbb{R}$, its nabla q-derivative of f is defined as*

$$_q\nabla f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in \mathbb{T}_q - \{0\}. \tag{2.1}$$

Definition 2.2 ([3]). *For a function $f : \mathbb{T}_q \to \mathbb{R}$, the nabla q-integral of f is defined as*

$$\int_0^t f(s)\nabla_q s = (1-q)t \sum_{i=0}^{\infty} q^i f\left(q^i t\right). \tag{2.2}$$

For $a \in \mathbb{T}_q$, (2.2) becomes

$$\int_a^t f(s)\nabla_q s = \int_0^t f(s)\nabla_q s - \int_0^a f(s)\nabla_q s. \tag{2.3}$$

Definition 2.3 ([1,3]). *The q-factorial function for $n \in \mathbb{N}$ is given by*

$$(t-s)_q^n = \prod_{i=0}^{n-1} \left(t - q^i s\right). \tag{2.4}$$

In case α is a non-positive integer, the q-factorial function is given by

$$(t-s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \frac{\left(1 - \frac{s}{t} q^i\right)}{\left(1 - \frac{s}{t} q^{i+\alpha}\right)}. \tag{2.5}$$

In the following Lemma, we present some properties of q-factorial functions.

Lemma 2.4 ([9]). *For $\alpha, \beta, a \in \mathbb{R}$, we have*

(i) $(t-s)_q^{\alpha+\beta} = (t-s)_q^\alpha (t - q^\alpha s)_q^\beta$.

(ii) $(at - as)_q^\alpha = a^\alpha (t-s)_q^\alpha$.

(iii) The nabla q-derivative of the q-factorial function with respect to t is

$$\nabla_q (t-s)_q^\alpha = \frac{1-q^\alpha}{1-q} (t-s)_q^{\alpha-1}. \tag{2.6}$$

(iv) The nabla q-derivative of the q-factorial function with respect to s is

$$\nabla_q (t-s)_q^\alpha = \frac{1-q^\alpha}{1-q} (t-qs)_q^{\alpha-1}. \tag{2.7}$$

Definition 2.5 ([3,7]). *For a function $f : \mathbb{T}_q \to \mathbb{R}$, the left q-fractional integral ${}_q\nabla_a^{-\alpha}$ of order $\alpha \neq 0, -1, -2, \ldots$ and starting at $a = q^{n_0} \in \mathbb{T}_q, n_0 \in \mathbb{Z}$, is defined by*

$${}_q\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} f(s) \nabla_q s = \frac{1-q}{\Gamma_q(\alpha)} \sum_{i=n}^{n0-1} q^i \left(q^n - q^{i+1}\right)_q^{\alpha-1} f\left(q^i\right), \tag{2.8}$$

where

$$\Gamma_q(\alpha+1) = \frac{1-q^\alpha}{1-q} \Gamma_q(\alpha), \quad \Gamma_q(1) = 1, \quad \alpha > 1. \tag{2.9}$$

Remark 2.6. *The left q-fractional integral ${}_q\nabla_a^{-\alpha}$ maps functions defined on $\mathbb{T}_q$ to functions defined on $\mathbb{T}_q$.*

Definition 2.7 ([3]). *Let $0 < \alpha \notin \mathbb{N}$. Then*

(i) the left Caputo q-fractional derivative of order α of a function f defined on $\mathbb{T}_q$ is defined by

$${}_qC_a^\alpha f(t) = \nabla_a^{-(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma_q(n-\alpha)} \int_a^t (t-qs)_q^{n-\alpha-1} \nabla_q^n f(s) \nabla_q s \tag{2.10}$$

where $n = [\alpha] + 1$. *In case* $\alpha \in \mathbb{N}$, *then* ${}_qC_a^\alpha f(t) = \nabla_q^n f(t)$.

(ii) *The left Riemann q-fractional derivative is defined by* $\left({}_q\nabla_a^\alpha f\right)(t) = \left(\nabla_q \nabla_a^{-(n-\alpha)} f\right)(t)$.

(iii) *In virtue of [3], the Riemann and Caputo q-fractional derivatives are related by*

$$\left({}_qC_a^\alpha f\right)(t) = \left({}_q\nabla_a^\alpha f\right)(t) - \frac{(t-a)_q^{-\alpha}}{\Gamma_q(1-\alpha)} f(a). \tag{2.11}$$

Lemma 2.8 ([3]). *Let* $\alpha > 0$ *and* f *be defined in a suitable domain. Then*

$${}_q\nabla_a^{-\alpha}\left({}_qC_a^\alpha f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a), \tag{2.12}$$

and if $0 < \alpha \leq 1$ *we have*

$${}_q\nabla_a^{-\alpha}\left({}_qC_a^\alpha f\right)(t) = f(t) - f(a). \tag{2.13}$$

The following identity is crucial in solving the linear q-fractional equations

$${}_q\nabla_a^{-\alpha}(x-a)_q^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma(\alpha+\mu+1)}(x-a)_q^{\mu+\alpha}, \quad (0 < a < x < b), \tag{2.14}$$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$.

We give the equivalence of Equation (1.3). So, the solvability of this equivalent equation implies the existence and stability of solutions to Equation (1.3).

Lemma 2.9. $x(t)$ *is a solution of* (1.3) *if and only if it admits the following representation*

$$\begin{aligned} x(t) = \psi(a) - H(\psi(a)) - g(a, \psi(\tau a)) + H(x(t)) + g(t, x(\tau t)) \\ + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} f(s, x(s), x(\tau s)) \nabla_q s, \quad t \in \mathbb{T}_a \end{aligned} \tag{2.15}$$

where

$$H(x(t)) = x(t) - h(x(t)). \tag{2.16}$$

Proof. Let

$$z(t) = h(x(t)) - g(t, x(\tau t)).$$

Then, we can write (3) as

$${}_qC_a^\alpha z(t) = f(t, x(t), x(\tau t)).$$

By the same way used in [2] and [25], we obtain for $t \in \mathbb{T}_{a\tau}$, the initial value problem for Equation (1.3) is equivalent to the following equation

$$z(t) = z(a) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f(s, x(s), x(\tau s)) \nabla_q s. \tag{2.17}$$

So

$$x(t) = \psi(a) - H(\psi(a)) - g(a, \psi(\tau a)) + H(x(t)) + g(t, x(\tau t)) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f(s, x(s), x(\tau s)) \nabla_q s.$$

The proof is complete. □

The space l_∞ denotes the set of real bounded sequences with respect to the usual supremum norm. We recall that l_∞ is a Banach space.

Definition 2.10. *A set $\mathbb{M}$ of sequences in l_∞ is uniformly Cauchy if for every $\epsilon > 0$, there exists an integer $\mathbb{N}^*$ such that $|x(t) - x(s)| < \epsilon$ whenever $t, s > \mathbb{N}^*$ for any $x = \{x(n)\}$ in $\mathbb{M}$.*

The following discrete version of Arzela–Ascoli's theorem has a crucial role in the proof of our main theorem.

Definition 2.11 ([29, Arzela-Ascoli])**.** *A bounded, uniformly Cauchy subset $\mathbb{M}$ of $l_\infty(\mathbb{T}_a)$ (all bounded real-valued sequences with domain $\mathbb{T}_a$) is relatively compact.*

Definition 2.12 ([11, Large contraction])**.** *Let $(\mathbb{M}, d)$ be a metric space and $B : \mathbb{M} \to \mathbb{M}$. B is said to be a large contraction if for each pair $x, y \in \mathbb{M}$ with $x \neq y$ then $d(Bx, By) < d(x, y)$ and if for each $\varepsilon > 0$ there exists $\delta < 1$ such that*

$$[x, y \in \mathbb{M}, d(x, y) \geq \varepsilon] \Rightarrow d(Bx, By) < \delta d(x, y).$$

Theorem 2.13 ([11, Krasnoselskii-Burton])**.** *Let $\mathbb{M}$ be a closed convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map $\mathbb{M}$ into $\mathbb{M}$ such that*

(i) for all $x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,

(ii) A is continuous and $A\mathbb{M}$ is contained in a compact subset of $\mathbb{M}$,

(iii) B is a large contraction.

Then there is a $z \in \mathbb{M}$ with $z = Az + Bz$.

We will use the next theorem to show the existence of solutions for Equation (1.3).

Theorem 2.14 ([4]). *Let $\|\cdot\|$ be the supremum norm, $\mathbb{M} = \{x \in \mathbb{C}(\mathbb{T}, \mathbb{R}) : \|x\| \leq R\}$, where R is a positive constant. Suppose that h is satisfying the following conditions*

(H1) h *is continuous on* $U_R = [-R, R]$.

(H2) h *is strictly increasing on* U_R.

(H3) $\sup_{s \in U_R \cap \mathbb{T}_a} {}_qC_a^\alpha h(s) \leq 1$.

(H4) $(s-r)\Big\{\sup_{i \in U_R \cap \mathbb{T}_a} {}_qC_a^\alpha h(i)\Big\} \geq h(s) - h(r) \geq (s-r)\Big\{\inf_{i \in U_R \cap \mathbb{T}_a} {}_qC_a^\alpha h(i)\Big\} \geq 0$ *for* $s, r \in U_R$ *with* $s \geq r$.

Then, the mapping H defined by Equation (2.16) *is a large contraction on* $\mathbb{M}$.

Let $\mathbb{T} = [\tau a, T_1]_q = \{q^{-i}\tau a, i = 0, 1, \ldots, n_1 + d\}$ where $T_1 = q^{-n_1-d}\tau a$ with $n_1 \in [d+3, \infty) \cap \mathbb{Z}$, and $\mathbb{C}(\mathbb{T}, \mathbb{R})$ be the set of all real bounded sequences. $\mathbb{C}(\mathbb{T}, \mathbb{R})$ is a Banach space endowed with the norm

$$\|x\| = \sup_{t \in \mathbb{T}} |x(t)|.$$

Define the set

$$\mathbb{M} = \{x \in \mathbb{C}(\mathbb{T}, \mathbb{R}) : x(t) = \psi(t) \text{ for } t \in \mathbb{I}_\tau \text{ and } \|x\| \leq R\}, \tag{2.18}$$

a non-empty bounded closed and convex subset of $\mathbb{C}(\mathbb{T}, \mathbb{R})$.

3 Existence of solutions

We prove our main results under the following assumptions:

- There exists a constant $L_f > 0$ such that for all $t \in \mathbb{T}_a$, and for all $x, y, z, w \in \mathbb{R}$,

$$|f(t, x, z) - f(t, y, w)| \leq L_f(\|x - y\| + \|z - w\|). \tag{3.1}$$

- There exists a constant $L_g > 0$ such that for all $t \in \mathbb{T}_a$, and for all $x, y \in \mathbb{R}$,

$$|g(t, x) - g(t, y)| \leq L_g\|x - y\|. \tag{3.2}$$

- There exists a constant $R > 0$, satisfying the inequality,

$$J\left[|\psi(a)| + |H(\psi(a))| + |g(a, \psi(\tau a))| + RL_g + \sigma_g + \frac{(2RL_f + \sigma_f)C(\alpha)}{\Gamma_q(\alpha)}\right] \leq R, \tag{3.3}$$

  where $C(\alpha) = \frac{(1-q)(T_1-a)_q^\alpha}{(1-q^\alpha)}$ is a positive constant depending on α and T_1, with $\sigma_f = \sup_{t \in \mathbb{T}_a} |f(t, 0, 0)|$, $\sigma_g = \sup_{t \in \mathbb{T}_a} |g(t, 0)|$ and $J \geq 3$ is a constant.

Define a mapping $S : \mathbb{M} \to \mathbb{C}$ by

$$(Sx)(t) = \psi(a) - H(\psi(a)) - g(a, \psi(\tau a)) + H(x(t)) + g(t, x(\tau t)) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f(s, x(s), x(\tau s)) \nabla_q s. \quad (3.4)$$

We express (3.4) as

$$(Sx)(t) = (Ax)(t) + (Bx)(t),$$

where the operators $A, B : \mathbb{M} \to \mathbb{C}$ are defined by

$$(Ax)(t) = \psi(a) - H(\psi(a)) - g(a, \psi(\tau a)) + g(t, x(\tau t)) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f(s, x(s), x(\tau s)) \nabla_q s, \quad (3.5)$$

and

$$(Bx)(t) = H(x(t)). \quad (3.6)$$

Lemma 3.1. *Assume that conditions* (3.1), (3.2) *and* (3.3) *hold. Then, the operator* $A : \mathbb{M} \to \mathbb{M}$ *defined in Equation* (3.5) *is compact and continuous.*

Proof. Let A be defined by Equation (3.5). In view of conditions (3.1) and (3.2), we arrive at

$$\begin{aligned} |f(t, x(t), x(\tau t))| &= |f(t, x(t), x(\tau t)) - f(t, 0, 0) + f(t, 0, 0)| \\ &\leq |f(t, x(t), x(\tau t)) - f(t, 0, 0)| + |f(t, 0, 0)| \leq 2L_f \|x\| + \sigma_f. \end{aligned}$$

and

$$|g(t, x(\tau t))| = |g(t, x(\tau t)) - g(t, 0) + g(t, 0)| \leq |g(t, x(\tau t)) - g(t, 0)| + |g(t, 0)| \leq L_g \|x\| + \sigma_g.$$

We have

$$\begin{aligned} |(Ax(t))| &= \left| \psi(a) - H(\psi(a)) - g(a, \psi(\tau a)) + g(t, x(\tau t)) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f(s, x(s), x(\tau s)) \nabla_q s \right| \\ &\leq |\psi(a)| + |H(\psi(a))| + |g(a, \psi(\tau a))| + |g(t, x(\tau t))| + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} |f(s, x(s), x(\tau s))| \nabla_q s \\ &\leq |\psi(a)| + |H(\psi(a))| + |g(a, \psi(\tau a))| + L_g \|x\| + \sigma_g + \frac{2L_f \|x\| + \sigma_f}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} \nabla_q s \\ &\leq |\psi(a)| + |H(\psi(a))| + |g(a, \psi(\tau a))| + RL_g + \sigma_g + \frac{2RL_f + \sigma_f}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} \nabla_q s. \end{aligned}$$

By the relations (2.9), (2.14) and the fact that $(t-a)_q^0 = 1$, we have

$$\frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}(t-a)_q^0 \nabla_q s = {}_q\nabla_a^\alpha (t-a)_q^0 = \frac{\Gamma_q(1)(t-a)_q^\alpha}{\Gamma_q(\alpha+1)} \leq \frac{(T_1-a)_q^\alpha}{\Gamma_q(\alpha+1)} = \frac{(1-q)(T_1-a)_q^\alpha}{(1-q^\alpha)\Gamma_q(\alpha)}, \quad t < T_1.$$

Then

$$|(Ax(t))| \leq |\psi(a)| + |H(\psi(a))| + |g(a,\psi(\tau a))| + RL_g + \sigma_g + \frac{(2RL_f+\sigma_f)C(\alpha)}{\Gamma_q(\alpha)}.$$

Thus

$$\|Ax\| \leq \frac{R}{J} \leq R.$$

Hence, $A : \mathbb{M} \to \mathbb{M}$ which implies $A(\mathbb{M})$ is uniformly bounded.

To prove the continuity of A, we consider a sequence (x_n) which converges to x such that

$$\begin{aligned}|(Ax_n)(t) - (Ax)(t)| &\leq |g(t,x_n(\tau t)) - g(t,x(\tau t))| \\ &\quad + \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}\Big|f(s,x_n(s),x_n(\tau s)) - f(s,x(s),x(\tau s))\Big|\nabla_q s \\ &\leq L_g\|x_n - x\| + \frac{L_f}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}\|x_n - x\|\nabla_q s \\ &\leq L_g\|x_n - x\| + \frac{2L_fC(\alpha)}{\Gamma_q(\alpha)}\|x_n - x\| \leq \left(L_g + \frac{2L_fC(\alpha)}{\Gamma_q(\alpha)}\right)\|x_n - x\|.\end{aligned}$$

From the above analysis, it implies that

$$\|(Ax_n)(t) - (Ax)(t)\| \leq \left(L_g + \frac{2L_fC(\alpha)}{\Gamma_q(\alpha)}\right)\|x_n - x\|.$$

Hence whenever $x_n \to x$, $Ax_n \to Ax$. This shows the continuity of A.

To prove that A is compact. We will prove that $A(\mathbb{M})$ is equicontinuous. Let $x \in \mathbb{M}$, then for any $t_1, t_2 \in \mathbb{T}_a$ with $0 \leq t_1 \leq t_2 \leq \mathbb{T}_1$, we have

$$\begin{aligned}|(Ax)(t_2) - (Ax)(t_1)| &\leq |g(t_2,x(\tau t_2))| + |g(t_1,x(\tau t_1))| \\ &\quad + \frac{1}{\Gamma_q(\alpha)}\Big|\int_a^{t_2}(t_2-qs)_q^{\alpha-1}f(s,x(s),x(\tau s))\nabla_q s \\ &\quad - \int_a^{t_1}(t_1-qs)_q^{\alpha-1}f(s,x(s),x(\tau s))\nabla_q s\Big|\end{aligned}$$

$$
\begin{aligned}
&\leq |g(t_2, x(\tau t_2))| + |g(t_1, x(\tau t_1))| \\
&+ \frac{1}{\Gamma_q(\alpha)} \int_a^{t_1} \Big|(t_2 - qs)_q^{\alpha-1} - (t_1 - qs)_q^{\alpha-1}\Big| |f(s, x(s), x(\tau s))| \nabla_q s \\
&+ \int_{t_1}^{t_2} (t_2 - qs)_q^{\alpha-1} |f(s, x(s), x(\tau s))| \nabla_q s.
\end{aligned}
$$

By the assumptions (3.1), (3.3), and Lemma 2.9, we obtain

$$
\begin{aligned}
|(Ax)(t_2) - (Ax)(t_1)| \leq{}& |g(t_2, x(\tau t_2))| + |g(t_1, x(\tau t_1))| \\
&+ \Big(2RL_f + \sigma_f\Big) \Bigg[\frac{1}{\Gamma_q(\alpha)} \int_a^{t_1} \Big|(t_2 - qs)_q^{\alpha-1} - (t_1 - qs)_q^{\alpha-1}\Big| \nabla_q s \\
&+ \frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - qs)_q^{\alpha-1} \nabla_q s \Bigg].
\end{aligned}
$$

By using (2.8), we obtain

$$
\begin{aligned}
|(Ax)(t_2) - (Ax)(t_1)| \leq{}& |g(t_2, x(\tau t_2))| + |g(t_1, x(\tau t_1))| \\
&+ (2RL_f + \sigma_f) \left[{}_q\nabla_a^{-\alpha} \left((t_2 - a)_q^0 - (t_1 - a)_q^0 \right) +_q \nabla_{t_1}^{-\alpha} (t_2 - t_1)_q^0 \right].
\end{aligned}
$$

From (2.14), it follows that

$$
\begin{aligned}
|(Ax)(t_2) - (Ax)(t_1)| \leq{}& |g(t_2, x(\tau t_2))| + |g(t_1, x(\tau t_1))| \\
&+ \frac{(2RL_f + \sigma_f)}{\Gamma_q(\alpha + 1)} \Big[(t_2 - a)_q^{\alpha} - (t_1 - a)_q^{\alpha} + (t_2 - t_1)_q^{\alpha} \Big].
\end{aligned}
$$

Hence it follows that $|(Ax)(t_2) - (Ax)(t_1)| \to 0$ as $t_1 \to t_2$. Thus that $A(\mathbb{M})$ is equicontinuous. So, the compactness of A follows by the Ascoli-Arzela theorem. □

The next Lemma, gives a relationship between the mappings H and B in the sense of large contraction.

Lemma 3.2. *Let B be defined by (3.6). Suppose that*

$$
\max \Big(|H(-R)|, |H(R)| \Big) \leq \frac{(J-1)}{J} R, \tag{3.7}
$$

and all conditions of Theorem 2.14 hold. Then $B : \mathbb{M} \to \mathbb{M}$ is a large contraction.

Proof. We will first show that B maps $\mathbb{M}$ into itself. Let $x \in \mathbb{M}$, then by (3.7) we have

$$
|(Bx)(t)| = |(Hx)(t)| \leq \max \Big\{ |H(-R)|, |H(R)| \Big\} \leq \frac{(J-1)}{J} R \leq R.
$$

Thus

$$\|Bx\| \leq R.$$

That is $Bx \in \mathbb{M}$ and consequently, we have $B : \mathbb{M} \to \mathbb{M}$.

We next show that B is a large contraction. By Theorem 2.14, if H is a large contraction on $\mathbb{M}$, then for any $x, y \in \mathbb{M}$ with $x \neq y$, we have $\|Hx - Hy\| \leq \|x - y\|$. This implies that

$$|(Bx)(t) - (By)(t)| = |(Hx)(t) - (Hy)(t)| \leq \|x - y\|.$$

Thus

$$\|Bx - By\| \leq \|x - y\|.$$

In a similar manner, one could also show that

$$\|Bx - By\| \leq \delta\|x - y\|,$$

holds if we know the existence of a $\delta \in (0, 1)$ and that for all $\epsilon > 0$,

$$[x, y \in \mathbb{M}, \|x - y\| > 0] \Rightarrow \|Hx - Hy\| \leq \delta\|x - y\|.$$

The proof is complete. □

Theorem 3.3. *Suppose the hypotheses of Lemmas 3.1 and 3.2 hold. Let* $\mathbb{M}$ *defined by* (2.18). *Then Equation* (1.3) *has a solution in* $\mathbb{M}$.

Proof. By Lemma 3.1, $A : \mathbb{M} \to \mathbb{M}$ is continuous and compact. Also, from Lemma 3.2, the mapping $B : \mathbb{M} \to \mathbb{M}$ is a large contraction. Next, we prove that if $x, y \in \mathbb{M}$, we have $\|Ax + By\| \leq R$. Let $x, y \in \mathbb{M}$ with $\|x\|, \|y| \leq R$. By (3.3) and (3.7), we obtain

$$\begin{aligned}
\|Ax + By\| &\leq \|Ax\| + \|By\| \\
&\leq \left[|\psi(a)| + |H(\psi(a))| + |g(a, \psi(\tau a))| + RL_g + \sigma_g + \frac{(2RL_f + \sigma_f)C(\alpha)}{\Gamma_q(\alpha)}\right] + \frac{(J-1)R}{J} \\
&\leq \frac{R}{J} + \frac{(J-1)R}{J} = R.
\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii-Burton theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = Az + Bz$. By Lemma 2.9, this fixed point is a solution of Equation (1.3). Hence Equation (1.3) has a solution. This completes the proof. □

4 Stability

Now, we show that the solutions of Equation (1.3) are stable by giving sufficient conditions.

Theorem 4.1. *Assume that conditions* (3.1) *and* (3.2) *hold. Also, suppose that*

$$c = \left(k + L_g + \frac{2L_f C(\alpha)}{\Gamma_q(\alpha)}\right) < 1, \tag{4.1}$$

and all conditions of Theorem 2.14 hold. Moreover, for $\epsilon > 0$*, there exists*

$$\delta = \frac{1-c}{1+k+L_g}\epsilon.$$

Then, the solutions of Equation (1.3) *are stable.*

Proof. Let x be a solution of Equation (1.3) and $\widehat{x}$ be a solution of Equation (1.3) satisfying the initial function $\widehat{x}(t) = \widehat{\psi}(t)$ on $\mathbb{I}_\tau$. For $t \in \mathbb{T}_a$, applying conditions (3.1), (3.2), (4.1) and all conditions of Theorem 2.14, yields

$$\begin{aligned}
|x(t) - \widehat{x}(t)| &\leq \left|\psi(a) - \widehat{\psi}(a)\right| + \left|H(\psi(a)) - H(\widehat{\psi}(a))\right| + \left|H(x(t)) - H(\widehat{x}(t))\right| \\
&\quad + \left|g(a, \psi(\tau a)) - g(a, \widehat{\psi}(\tau a))\right| + \left|g(t, x(\tau t)) - g(t, \widehat{x}(\tau t))\right| \\
&\quad + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} \left|f(s, x(s), x(\tau s)) - f(s, \widehat{x}(s), \widehat{x}(\tau s))\right| \nabla_q s \\
&\leq (1+k+L_g)\|\psi - \widehat{\psi}\| + \left(k + L_g + \frac{2L_f C(\alpha)}{\Gamma_q(\alpha)}\right)\|x - \widehat{x}\| \\
&\leq (1+k+L_g)\|\psi - \widehat{\psi}\| + c\|x - \widehat{x}\|.
\end{aligned}$$

Hence

$$\|x - \widehat{x}\| \leq \frac{1+k+L_g}{1-c}\|\psi - \widehat{\psi}\|$$

Then, for any $\epsilon > 0$, let $\delta = \frac{1-c}{1+k+L_g}\epsilon$, so for $\|\psi - \widehat{\psi}\| < \delta$ there is $\|x - \widehat{x}\| < \epsilon$. Therefore, the solutions of Equation (1.3) are stable. The proof is complete. □

5 Illustrative example

In this section we provide an example. Specifically, we apply Theorems 3.3 and 4.1 to the equation

$$\begin{cases} {}_{\frac{2}{3}}C_1^{\frac{3}{4}}\left(\left(\frac{1}{10}\sin(x(t))+\frac{9}{10}x(t)\right)-\frac{1}{50}\cos(t)\arctan\left(x\left(\frac{2t}{3}\right)\right)\right)=\frac{1}{100}\Big(\sin(x(t))+\cos\left(x\left(\frac{2t}{3}\right)\right)\Big)e^{-t}, \\ t\in[1,9/4]_{\frac{2}{3}}, \\ x(t)=0.02\cos(\pi t), \quad t\in\{2/3,1\}. \end{cases} \tag{5.1}$$

It follows from the equation that $q=2/3$, $\alpha=3/4$, $a=1$, $\tau=2/3$, $h(x)=\frac{1}{10}\sin(x)+\frac{9}{10}x$, which yields $H(x)=\frac{1}{10}(x-\sin(x))$.

Also,

$$g(t,x)=\frac{1}{50}\cos(t)\arctan(x), \quad f(t,x,z)=\frac{1}{100}(\sin(x)+\cos(z))e^{-t},$$

and

$$\psi(t)=0.02\cos(\pi t).$$

We define the set $\mathbb{M}=\{x\in\mathbb{C}:\|x\|\le R\}$ with $R=0.5$.

Now on the domain $\mathbb{M}_R=[-0.5,0.5]$, $h(x)$ is strictly increasing since

$$h'(x)=\frac{1}{10}\cos(x)+\frac{9}{10}\ge\frac{1}{10}\cos(0.5)+0.9\approx 0.987>0.$$

It can be verified that conditions $(H3)$-$(H4)$ also hold, making $H(x)$ a large contraction.

The Lipschitz constant for $H(x)$ is

$$k=\sup_{x\in U_R}|H'(x)|=\sup_{x\in U_R}\left|\frac{1}{10}(1-\cos(x))\right|\le\frac{1}{10}(1-\cos(0.5))\approx 0.001224.$$

Also,

$$|g(t,x)-g(t,y)|\le\frac{1}{50}|x-y|, \quad |f(t,x,z)-f(t,y,w)|\le\frac{1}{100}(|x-y|+|z-w|)$$

Thus, $L_g=0.02$ and $L_f=1/100=0.01$.

It must also be noted that $\sigma_g=\sup|g(t,0)|=0$ and $\sigma_f=\sup|f(t,0,0)|=\frac{1}{100}e^{-1}\approx 0.00368$, $\psi(1)=-0.02\Longrightarrow H(\psi(1))\approx 0$ and $g(1,\psi(2/3))\approx -0.000108$.

To verify the main conditions, we must select an endpoint T_1 for the time scale. Let us choose $T_1=9/4$. A rigorous numerical calculation using the definitions of the q-Gamma function and q-power function yields the q-integral bound

$$K_A=\frac{(T_1-1)_q^{\alpha}}{\Gamma_q(\alpha+1)}=\frac{(9/4-1)_{2/3}^{3/4}}{\Gamma_{2/3}(7/4)}\approx 1.4331.$$

It must also be noted that $|H(0.5)| \approx 0.00206$ and with $J = 5$

$$\frac{J-1}{J}R = \frac{4}{5}(0.5) = 0.4.$$

Hence, showing that Lemma 3.2 holds. Moreover, to verify condition (3.3), we have

$$\begin{aligned}
&|\psi(a)| + |H(\psi(a))| + |g(a, \psi(\tau a))| + RL_g + \sigma_g + (2RL_f + \sigma_f)K_A \\
&\qquad = 0.02 + 0 + 0.000108 + (0.5)(0.02) + 0 + (2(0.5)(0.01) + 0.003679)(1.4331) = 0.0497 \\
&\qquad \le 0.1.
\end{aligned}$$

Thus, condition (3.3) hold. It therefore follows from Theorem 3.3 that Equation (5.1) has at least one solution in $\mathbb{M}$.

To verify the stability of solutions we verify condition (4.1). Thus,

$$k + L_g + 2L_f K_A = 0.01224 + 0.02 + 2(0.01)(1.4331) = 0.03224 + 0.02866 \le 1.$$

Thus, by Theorem 4.1 the solutions of Equation (5.1) are stable.

6 Conclusion

This paper has established sufficient conditions for the existence and stability of solutions to a class of totally nonlinear neutral Caputo q-fractional difference equations. The Krasnoselskii-Burton fixed point theorem was a key tool in proving existence, by decomposing the solution operator into a compact part and a large contraction. The stability analysis provides criteria based on the Lipschitz constants of the involved functions and the bound on the q-integral operator. The presented theoretical framework generalizes existing results by considering a more comprehensive nonlinear and neutral structure. The illustrative example demonstrates the method of verifying the derived conditions. Future work could explore specific applications of these equations or investigate uniqueness conditions and other qualitative properties.

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