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A note on Krein-Milman theorem

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ABSTRACT

In this note, we generalize a well-known theorem of Krein and Milman concerning the closed convex envelope of the extremal points of a compact convex set in a topological vector space to the case of an abstract convexity notion in a topological space with a convex structure defined on it. We also introduce some notions that will be useful to describe these generalizations.

RESUMEN

En esta nota generalizamos un teorema bien conocido de Krein y Milman respecto de la envolvente convexa cerrada de los puntos extremos de un conjunto compacto convexo en un espacio vectorial topológico al caso de una noción abstracta de convexidad en un espacio topológico con una estructura convexa definida en él. También introducimos algunas nociones que serán útiles para describir estas generalizaciones.

Keywords and Phrases: Convex sets, extremal points, Krein-Milman theorem.

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1 Introduction and results

The main pourpose of this paper is to prove a generalization of Krein and Milman Theorem (Theorem 1.14). We start with some definitions and properties which will be useful for our pourpose.

Definition 1.1. *Let X be a topological space. We say that a family $\mathcal{F} = \{\tau_{(x,y)} : [0,1] \to X : x, y \in X\}$ introduces a convex structure on X if:*

1. *The functions $\tau_{(x,y)}$ are continuous.*
2. *$\tau_{(x,y)}(0) = x$, $\tau_{(x,y)}(1) = y$.*

The space X with the a such family $\mathcal{F}$ we call a *space with convex structure.*

Definition 1.2. *Let X be a space with a convex structure. A subset A of X is said to be $\mathcal{F}-$convex if for all $x, y \in A$ we have that $\tau_{(x,y)}([0,1]) \subset A$.*

It is easy to see that the space X, the empty and the intersection of two $\mathcal{F}-$convex sets is an $\mathcal{F}-$convex set. Also the union of a monotone family of $\mathcal{F}-$convex sets is again an $\mathcal{F}-$convex set.

Example 1.3. *Let $X = S^2$ be a two dimensional sphere and let $\tau_{(x,y)} : [0,1] \to S^2$ be a parametrization of the geodesic line which joins the points x and y on this sphere. Then the family $\mathcal{F} = \{\tau_{(x,y)} : [0,1] \to X : x, y \in X\}$ introduces a convex structure on S^2. Particulary the spherical triangles are $\mathcal{F}-$convex sets.*

Example 1.4. *Let X be a topological vector space or an abstract topological cone. Then the family $\mathcal{F}_p = \{\tau_{(x,y)} : \tau_{(x,y)}(t) = ty + (1-t^p)^{\frac{1}{p}} x,\ t \in [0,1],\ x, y \in X\}$ introduces a convex structure on X for all $p > 0$.*

Definition 1.5. *Let X be space with a convex structure and A be a subset of X. We say that $B \subset A$ is an extremal subset of A if the condition $\tau_{(x,y)}(t) \in B$ for some $x, y \in A$ and $t \in (0,1)$ implies that $x, y \in B$.*

Lemma 1.6. *Let X be a topological space with a convex structure $\mathcal{F}$. Then every compact $\mathcal{F}-$convex subset A of X contains an $\mathcal{F}-$convex extreme B which is minimal with respect to inclusion.*

Proof. Let $\mathcal{M}$ be the family of all $\mathcal{F}-$convex closed extreme subsets of A ordered by inclusion. It is easy to observe that $\mathcal{M}$ is nonempty since $A \in \mathcal{M}$ and the intersection of any chain of elements of $\mathcal{M}$ also belongs to $\mathcal{M}$. Hence by the Kuratowski-Zorn Lemma there exists an minimal element of $\mathcal{M}$. □

Similarly we can prove the following:

Definition 1.7. *Let X be a topological space. We say that the family $\mathcal{G} = \{g_t : X \to \mathbb{R} : t \in T\}$ separates points of the space X, if for all $x, y \in X$, $x \neq y$ there exists $t \in T$ such that $g_t(x) \neq g_t(y)$.*

Definition 1.8. *We say that a family $\mathcal{G} = \{g_t : X \to \mathbb{R} : t \in T\}$ exposes faces of a compact $\mathcal{F}-$convex subset A if for any $g \in \mathcal{G}$ the set*

$$H_g^A = \left\{ x \in A : g(x) = \sup_{t \in A} g(t) \right\}$$

is an extremal subset of A.

Definition 1.9. *Let A be an $\mathcal{F}-$convex subset of X. We say that $x \in A$ is an extreme point of A if the set $\{x\}$ is an extreme subset of A. The set of all extreme points of the set A is denoted by Ext(A).*

Definition 1.10. *Let X be a topological space with a convex structure $\mathcal{F}$ and the family $\mathcal{G}$ of functions which separates points of X. We say that the family $\mathcal{G}$ is compatible with the family $\mathcal{F}$ on the class of $\mathcal{F}-$compact convex sets if for any compact $\mathcal{F}-$convex set A the set H_g^A is $\mathcal{F}-$convex for all $g \in \mathcal{G}$.*

Proposition 1.11. *Let X be a topological space with a convex structure $\mathcal{F}$. Assume that there exists a family $\mathcal{G}$ compatible with the family $\mathcal{F}$ on the class of $\mathcal{F}-$compact convex sets which separates points of X and exposes faces of compact $\mathcal{F}-$convex sets. Then every compact $\mathcal{F}-$convex subset A of X has extreme point.*

Proof. By Lemma 1.6 there exists a minimal extreme $\mathcal{F}-$convex subset B of the set A. Suppose that $x, y \in B$ then there exists $g \in \mathcal{G}$ such that $g(x) \neq g(y)$ but in this case the set H_g^B is also extreme $\mathcal{F}-$convex subset of A which is included in B and at least one of x, y does not belong to B. Contradiction. □

Definition 1.12. *Let X be a topological space with the convex structure $\mathcal{F}$. We say that the family $\mathcal{G}$ of real functions defined on X separates compact $\mathcal{F}-$convex sets from points if for any compact $\mathcal{F}-$convex set A and any $b \notin A$ there exists $g \in \mathcal{G}$ such that $g(x) < g(b)$ for all $x \in A$.*

Definition 1.13. *Let A be a subset of a topological space X with the convex structure $\mathcal{F}$. The $\mathcal{F}-$convex hull of the set A is defined as the intersection of all $\mathcal{F}-$convex subsets of X which contain the set A and we denote it by $conv_{\mathcal{F}}(A)$.*

Analogously the closed $\mathcal{F}-$convex hull of the set A is defined as the intersection of all $\mathcal{F}-$convex closed subsets of X which contains the set A and we denote it by $\overline{conv}_{\mathcal{F}}(A)$.

Clearly the $\mathcal{F}-$convex hull of any subset is $\mathcal{F}-$convex set and the closed $\mathcal{F}-$convex hull of any subset is closed and $\mathcal{F}-$convex set. Now we are prove theorem which generalizes the classical Krein-Milman theore.

Theorem 1.14. *Let X be a topological space with a convex structure $\mathcal{F}$ on X. Suppose that there exists a family $\mathcal{G}$ on X which is compatible with $\mathcal{F}$ on $\mathcal{F}-$compact convex subsets of X such that separates $\mathcal{F}-$convex compact subsets of X from points of X and exposes faces of compact $\mathcal{F}-$convex sets. Then every $\mathcal{F}-$convex compact subset A of X is equal to closed $\mathcal{F}-$convex hull of its extremal points. Symbolically*

$$A = \overline{conv}_{\mathcal{F}}(Ext\ (A)).$$

Proof. From Proposition 1.11 we have that the set $\mathrm{Ext}(A)$ is not empty. Obviously

$$K = \overline{\mathrm{conv}}_{\mathcal{F}}(\mathrm{Ext}(A)) \subset A.$$

Assume that $A \setminus K \neq \emptyset$ and let $x \in A \setminus K$. Since the set K is a closed $\mathcal{F}-$convex subset of the compact set A it is also compact. Now since the family $\mathcal{G}$ separates points from compact $\mathcal{F}-$convex sets then there exists $g \in \mathcal{G}$ such that $\sup_{t\in K} g(t) < g(x)$. Since the family exposes faces of compact $\mathcal{F}-$convex sets therefore the set H_g^A is extreme subset of A. From the compatibility of the family $\mathcal{G}$ with $\mathcal{F}$ on the compact $\mathcal{F}-$convex subsets we obtain that the set H_g^A is itself a compact $\mathcal{F}-$convex set and hence the set $\mathrm{Ext}(H_g^A)$ is not empty. Hence

$$\mathrm{Ext}(H_g^A) \subset \mathrm{Ext}(A) \subset K,$$

but this gives a contradiction since for $y \in \mathrm{Ext}(H_g^A)$ we have

$$g(y) = \sup_{t\in A} g(t) \geq g(x) > \sup_{t\in K} g(t) \geq g(y)$$

which ends the proof. □

Extremal points play an important role in mathematics and its applications. As was shown in [4] it plays an crucial role in proving continuity of convex functions. Hence the above theorem may be a possible tool for examining the continuty of some wider class of convex functions (*i.e.* convex functions defined by using abstract convex structure).

Remark 1.15. *If X is a locally convex topological vector space then the family*

$$\mathcal{F} = \{I_{xy} : [0,1] \to X : I_{xy}(t) = (1-t)x + ty,\ x, y \in X\}$$

defines a convex structure on X. It is clear that $\mathcal{F}-$convex sets are usual convex sets in this case. Denote by X' the topological dual of X i.e. the space of all real continous linear functionals *defined on X. From the geometric form of Hahn-Banach theorem [1] it follows that the family $\mathcal{G} = X'$ separates points from compact convex subsets of X. Moreover it is easy that the family $\mathcal{G}$ is compatible with $\mathcal{F}$ on convex subsets of X and the family $\mathcal{G}$ exposes faces of compact convex sets. Hence the assumptions of Theorem 1.14 are satisfied and from this theorem we obtain a classical version of Krein-Milman Theorem ([2, 3]) i.e.*

Theorem 1.16 (Krein-Milman theorem)**.** *Let X be a locally convex topological vector space and let A be a compact convex subset of X. Then A is equal to the closed convex envelope of the set of its extreme points. Symbolically,*

$$A = \overline{conv}(Ext(A)).$$

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