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Comparing the real genus and the symmetric crosscap number of a group

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ABSTRACT

Given a finite group G, there exist Klein surfaces, bordered X and unbordered non-orientable S, such that G acts as an automorphism group of X and of S. The minimum algebraic genus $\rho(G)$ of the surfaces X is called the real genus of G, and the minimal topological genus $\tilde{\sigma}(G)$ of the surfaces S is the symmetric crosscap number of G. In this work we study the relation between the real genus and the symmetric crosscap number of a group G and how both parameters can be compared. For instance, we see that there exist groups G such that the difference $\tilde{\sigma}(G) - \rho(G) = t$ for all even negative numbers t. In order to get it, we correct some inaccuracies in previous works, on these parameters for the groups $C_m \times D_n$ and $D_m \times D_n$. On the other hand, for some important families of groups, we prove that $\tilde{\sigma}(G) = \rho(G) + 1$. We use it to eliminate possible gaps in the symmetric crosscap spectrum, enforcing the conjecture that 3 is in fact the unique gap.

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RESUMEN

Dado un grupo finito G, existen superficies de Klein, con borde X y sin borde no-orientables S, tales que G actúa como un grupo de automorfismos de X y de S. El género algebraico mínimo $\rho(G)$ de las superficies X se llama el género real de G, y el género topológico mínimo $\tilde{\sigma}(G)$ de las superficies S es el "symmetric crosscap number" de G, que llamaremos género imaginario aunque no es una denominación estándar. En este trabajo, estudiamos la relación entre el género real y el imaginario de un grupo G y cómo se pueden comparar ambos parámetros. Por ejemplo, vemos que existen grupos G tales que la diferencia $\tilde{\sigma}(G) - \rho(G) = t$ para todos los números negativos pares t. Para ello, corregimos algunas inexactitudes en trabajos previos sobre estos parámetros para los grupos $C_m \times D_n$ y $D_m \times D_n$. Por otra parte, para algunas familias importantes de grupos, demostramos que $\tilde{\sigma}(G) = \rho(G) + 1$. Esto lo utilizamos para eliminar posibles huecos en el espectro simétrico imaginario, dando evidencia adicional a la conjetura de que 3 es, de hecho, el único hueco posible.

Keywords and Phrases: Real genus, symmetric crosscap number, Klein surfaces.

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1 Introduction and preliminaries

A Klein surface X is a compact surface endowed with a dianalytic structure. Klein surfaces may be seen as a generalization of Riemann surfaces, including bordered and non-orientable surfaces. An orientable unbordered Klein surface is a Riemann surface.

In the study of Klein surfaces and their automorphism groups, the non-Euclidean crystallographic groups (NEC groups, in short) play an essential role. An NEC group Γ is a discrete subgroup of the group of isometries of the hyperbolic plane $\mathcal{H}$ with compact quotient $\mathcal{H}/\Gamma$.

For the convenience of the reader we give a minimum of preliminaries about NEC groups and Klein surfaces (for details see [4]).

An NEC group Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathcal{H}$, including orientation reversing elements, with compact quotient $X = \mathcal{H}/\Gamma$. Every NEC group Γ has associated the following symbol called *signature*:

$$\sigma(\Gamma) = (g, \pm, [m_1, \ldots, m_r], \{(n_{i,1}, \ldots, n_{i,s_i}),\ i = 1, \ldots, k\}), \tag{1.1}$$

where the numbers g, r, k and s_i are non-negative integers, $m_i, n_{i,j}$ are integers such that $m_i, n_{i,j} \geq 2$. The number g is the topological genus of X, and the sign determines the orientability of X.

The numbers m_i are the *proper periods* corresponding to cone points in X. The brackets $(n_{i,1}, \ldots, n_{i,s_i})$ are the *period-cycles*. The number k of period-cycles is equal to the number of boundary components of X. Numbers $n_{i,j}$ are the periods of the period-cycle $(n_{i,1}, \ldots, n_{i,s_i})$ also called *link-periods*, corresponding to corner points in the boundary of X. The number $p = \alpha g + k - 1$, where $\alpha = 2$ or 1 according to the sign be "+" or "−", respectively, is called the *algebraic genus* of X.

An NEC group with the above signature is generated by x_i, $(i = 1, \ldots, r)$; e_i, $(i = 1, \ldots, k)$; $c_{i,j}$, $(i = 1, \ldots, k;\ j = 0, \ldots, s_i)$; and a_i, b_i $(i = 1, \ldots, g)$ if σ has sign "+" or d_i $(i = 1, \ldots, g)$ if σ has sign "−", and relations

$$\begin{aligned}
&x_i^{m_i} = 1; && i = 1, \ldots, r;\\
&c_{i,j-1}^2 = c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{i,j}} = 1; && i = 1, \ldots, k;\ j = 1, \ldots, s_i;\\
&e_i^{-1} c_{i,0} e_i c_{i,s_i} = 1; && i = 1, \ldots, k;\\
&\textstyle\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g (a_i b_i a_i^{-1} b_i^{-1}) = 1; && (\text{if } \sigma \text{ has sign "} + \text{"});\\
&\textstyle\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g d_i^2 = 1; && (\text{if } \sigma \text{ has sign "} - \text{"}).
\end{aligned}$$

The isometries x_i are elliptic, e_i, a_i, b_i are hyperbolic, c_i are reflections and d_i are glide reflections. They are called *canonical generators*.

Every NEC group Γ with signature (1.1) has associated a fundamental region whose area $\mu(\Gamma)$, called the *area of the group*, is

$$\mu(\Gamma) = 2\pi\left(\alpha g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{i,j}}\right)\right),$$

with $\alpha = 2$ or 1 according to the sign being "+" or "−". The group given by the presentation above can be represented as an NEC group with signature (1.1) if and only if its area is greater than 0. We denote by $|\Gamma|$ the expression $\mu(\Gamma)/2\pi$ and call it the *reduced area* of Γ.

If Γ is a subgroup of an NEC group Γ' of finite index N, then Γ is also an NEC group and the following Riemann-Hurwitz formula holds:

$$\mu(\Gamma) = N\mu(\Gamma').$$

If the group has neither proper periods nor link-periods, it is called a *surface group* and has the following signature

$$\sigma(\Gamma) = (g, \pm, [-], \{(-), \overset{k}{\ldots}, (-)\}),$$

For a Klein surface X with $p \geq 2$, there exists a NEC surface group Γ such that $X = \mathcal{H}/\Gamma$. A finite group G of order N is an automorphism group of $X = \mathcal{H}/\Gamma$ if and only if there exists an NEC group Λ such that Γ is a normal subgroup of Λ with index N and $G = \Lambda/\Gamma$. Since Γ is a surface group, it does not contain elements of finite order other than reflections. Therefore, there must be an epimorphism $\theta : \Lambda \to G$ with kernel Γ, such that the relations defining Λ are preserved by θ.

Given a finite group G there exist bordered Klein surfaces X such that G acts as an automorphism group of X, and also unbordered non-orientable surfaces S, such that G acts on S. The minimum algebraic genus of the surfaces X is called the *real genus* of G, $\rho(G)$, and the minimal topological genus of the surfaces S is the *symmetric crosscap number* of G, $\tilde{\sigma}(G)$. In order to obtain these parameters we need to study NEC groups Λ with minimal area such that $G = \Lambda/\Gamma$.

An extensive study has been made on both parameters $\rho(G)$ and $\tilde{\sigma}(G)$. The numbers which are $\rho(G)$ for some G form the *real genus spectrum*, whilst those which are $\tilde{\sigma}(G)$ form the *symmetric crosscap spectrum*. None of these spectra is still completely known, and the relationship between both parameters is a tool for that study. When an integer does not belong to either spectrum, it is called a gap of that spectrum.

Regarding the real genus, there is no group with real genus 2, 12 or 24 [14]. No other gap was currently known to exist, but in the very recent paper [6], it is proved that 72 is also a gap. Therefore, the first number for which it is not known whether it belongs to the spectrum is 84.

For the symmetric crosscap spectrum, the present knowledge is based on [1]. May proved that there does not exist any group G such that $\tilde{\sigma}(G) = 3$. For $N > 3$, if N is a gap of the symmetric crosscap spectrum then N lies in four congruence classes mod 120, namely 3, 51, 75 and 99, and it satisfies additional conditions. The present result will be given below in Theorem 3.4. However, many numbers satisfying those necessary conditions actually belong to the spectrum. In fact, no gap apart from 3 is currently known.

2 Results on real genus and symmetric crosscap number

The goal of the present work is to compare both parameters $\rho(G)$ and $\tilde{\sigma}(G)$. It is worth noting that very often

$$\tilde{\sigma}(G) = \rho(G) + 1. \tag{2.1}$$

This property holds for important classes of groups, but it is not true in general. When it holds for a group G, we say that G satisfies Property (2.1).

2.1 Groups of odd order

First, the authors proved in [1] that the Property (2.1) holds for all groups of odd order.

Theorem 2.1 ([1, Corollary 1]). *If G has odd order, then $\tilde{\sigma}(G) = \rho(G) + 1$.*

2.2 Abelian groups

Property (2.1) is also true for Abelian groups. In [18] J. Rodríguez mentions in Remark 6.2 that "the crosscap number of an Abelian group relates with its real genus straightforwardly: $\tilde{\sigma}(G) = \rho(G) + 1$". However, as far as we know this result has not appeared anywhere, and we are now providing its proof, taking into account that both parameters are already known in the case of Abelian groups, obtained by McCullough and Gromadzki in [16] and [11] respectively.

First, we quote the result on real genus.

Theorem 2.2 ([16]). *Let G be a non-cyclic Abelian group of order N, $G \neq C_2 \times C_2 \times C_2$, $C_2 \times C_{2k}$ $(k \geq 1)$. Write*

$$G = C_{e_1} \times \cdots \times C_{e_m} \times C_{d_1} \times \cdots \times C_{d_l} \times C_2^n,$$

e_i multiple of 4, d_j odd, $e_{i+1}|e_i$, $d_1|e_m$, $d_{j+1}|d_j$. Then $\rho(G)$ is

A) $1 + N\left(n + \sum_{i=1}^{m}\left(1 - \frac{1}{e_i}\right) + \sum_{j=1}^{l}\left(1 - \frac{1}{d_j}\right) - 1\right)$, $n < m$.

B) $1+N\left(m+t+\left(1-\frac{1}{2d_t}\right)+\sum_{j=t+1}^{l}\left(1-\frac{1}{d_j}\right)-2\right)$, *if* $m<n\leq m+2l-1$, $n-m=2t-1$.

C) $1+N\left(m+t+\sum_{j=t+1}^{l}\left(1-\frac{1}{d_j}\right)-1\right)$, *if* $m\leq n\leq m+2l$, $n-m=2t$.

D) $1+\frac{N(3m+2l+n-3)}{4}$, *if* $n\geq m+2l+1$.

On the other hand, for the symmetric crosscap number the result is the following

Theorem 2.3 ([11]). *Let G be a non-cyclic Abelian group of order N, $G\neq C_2\times C_2\times C_2$, $C_2\times C_{2k}$ $(k\geq 1)$. If G has non-cyclic 2-Sylow subgroup, write $G=C_{m_1}\times\cdots\times C_{m_k}\times C_2^s$, where $m_1,\ldots,m_l$ are odd, $m_{l+1}\ldots,m_k$ are even, $m_i|m_{i+1}$, and s is as large as possible. Then $\tilde{\sigma}(G)$ is*

i) $2+N\left(k-1-\sum_{i=1}^{k-s}\frac{1}{m_i}\right)$, *if* $s-(k-l)\leq 0$.

ii) $2+N(k-1)$, *if* $s-(k-l)=2l$.

iii) $2+N\left(k-1+\frac{s-k-l+1}{4}\right)$, *if* $s-(k-l)>2l$.

iv) $2+N\left(k-1-\sum_{i=1}^{(k+l-s)/2}\frac{1}{m_i}\right)$, *if* $0<s-(k-l)<2l$, $s-(k-l)$ *even.*

v) $2+N\left(k-1-\frac{1}{2m_{(k+l-s+1)/2}}-\sum_{i=1}^{(k+l-s-1)/2}\frac{1}{m_i}\right)$, *if* $0<s-(k-l)<2l$, $s-(k-l)$ *odd.*

And if N is odd, or G has cyclic 2-Sylow subgroup write $G=C_{m_1}\times\cdots\times C_{m_r}$, $m_i|m_{i+1}$ and then $\tilde{\sigma}(G)$ is

vi) $2+N\left(-1+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right)$.

Since in both Theorems the group G has been described in a different way, it is not too easy to compare $\rho(G)$ and $\tilde{\sigma}(G)$. We shall do it now, by proving

Theorem 2.4. *Let G be a non-cyclic Abelian group $G\neq C_2\times C_2\times C_2$, $C_2\times C_{2k}$ $(k\geq 1)$. Then $\tilde{\sigma}(G)=\rho(G)+1$.*

Proof. We start with each of the four possibilities for $\rho(G)$, namely A, B, C and D.

The translation of the parameters between both Theorems is as follows. In [16], m is the number of factors that are multiples of 4, l is the number of odd factors and n is the number of factors 2. Instead, in [11], $k-l$ is the number of factors multiple of 4, l is the number of odd factors and s is the number of factors 2.

We start with case A. Then $n<m$ in [16] is equivalent to $s<k-l$, what implies that $s-(k-l)<0$, and we are in case i) in [11]. Hence

$$\rho(G)=1+N\left(n+\sum_{i=1}^{l}\left(1-\frac{1}{d_i}\right)+\sum_{j=n+1}^{m}\left(1-\frac{1}{e_j}\right)-1\right)$$

translates to

$$\begin{aligned}\rho(G) &= 1 + N\left(s + l + (k - l - s) - 1 - \sum_{i=1}^{l} \frac{1}{d_i} - \sum_{j=l+s+1}^{m+l} \frac{1}{e_j}\right)\\ &= 1 + N\left(k - 1 - \sum_{i=1}^{l} \frac{1}{d_i} - \sum_{j=l+s+1}^{m+l} \frac{1}{e_j}\right) = 1 + N\left(k - 1 - \sum_{i=1}^{k-s} \frac{1}{m_i}\right) = \tilde{\sigma}(G) - 1.\end{aligned}$$

Now, we consider the case B. Then $m < n \le m + 2l - 1$, $n - m = 2t - 1$ odd. This implies $k - l < s \le k - l + 2l - 1 = k + l - 1$, $s - (k - l)$ odd, and so $0 < s - (k - l) \le 2l - 1$ with $s - (k - l)$ odd. We are in case $v)$ in [11]. Then

$$\rho(G) = 1 + N\left(m + t + \left(1 - \frac{1}{2d_t}\right) + \sum_{i=t+1}^{l} \left(1 - \frac{1}{d_i}\right) - 2\right)$$

translates to

$$\begin{aligned}\rho(G) &= 1 + N\left(k - l + \frac{s - k + l + 1}{2} + \left(1 - \frac{1}{2d_{(s-k+l+1)/2}}\right) + \sum_{i=(s-k+l+3)/2}^{l} \left(1 - \frac{1}{d_i}\right) - 2\right)\\ &= 1 + N\left(k - l + \frac{s - k + l + 1}{2} + 1 + l - \frac{s - k + l + 3}{2} + 1 - 2 - \frac{1}{2d_{(s-k+l+1)/2}} - \sum_{i=(s-k+l+3)/2}^{l} \frac{1}{d_i}\right)\\ &= 1 + N\left(k - l - \frac{1}{2m_{(l-s+k+1)/2}} - \sum_{i=1}^{(l-s+k-1)/2} \frac{1}{m_i}\right) = \tilde{\sigma}(G) - 1.\end{aligned}$$

We move to case D, where $n \ge m + 2l + 1$. This implies $s \ge (k - l) + 2l + 1$, and so $s - (k - l) \ge 2l + 1$. Hence $s - (k - l) > 2l$, and this corresponds to the case $iii)$. In this case

$$\rho(G) = \frac{1 + N(3m + 2l + n - 3)}{4}$$

corresponds to

$$\begin{aligned}\rho(G) &= 1 + N\frac{3k - 3l + 2l + s - 3}{4} = 1 + N\frac{3k - l + s - 3}{4}\\ &= 1 + N\left(k - 1 + \frac{s - k - l + 1}{4}\right) = \tilde{\sigma}(G) - 1.\end{aligned}$$

Finally, we must deal with the case C, where $m \le n \le m + 2l$, $n - m = 2t$ is even. This means that $k - l \le s \le k + l$, with $s - (k - l) = 2t$. This possibility splits into three subcases.

If $s - (k - l) = 0$, we are in case $i)$, and

$$\rho(G) = 1 + N\left(m + t + \sum_{i=t+1}^{l} \left(1 - \frac{1}{d_i}\right)\right)$$

means that

$$\rho(G) = 1 + N\left(k - l + l - \sum_{i=t+1}^{l} \frac{1}{d_i} - 1\right) = 1 + N\left(k - 1 - \sum_{i=1}^{k-s} \frac{1}{m_i}\right) = \tilde{\sigma}(G) - 1.$$

Now, if $s-(k-l) = 2l$, we are in case $ii)$ and $s-(k-l) = 2l$ implies $t = (n-m)/2 = (s-(k-l))/2 = l$, and so

$$\rho(G) = 1 + N\left(m + t + \sum_{i=t+1}^{l}\left(1 - \frac{1}{d_i}\right)\right) = 1 + N(k - l + l - 1) = 1 + N(k-1) = \tilde{\sigma}(G) - 1.$$

For the remaining values of $s - (k - l)$ we go to case $iv)$. Then

$$\begin{aligned}\rho(G) &= 1 + N\left(m + t - \sum_{i=t+1}^{l}\left(1 - \frac{1}{d_i}\right) - 1\right)\\ &= 1 + N\left(m + t + (l - t) - 1 - \sum_{i=t+1}^{l} \frac{1}{d_i}\right)\\ &= 1 + N\left(m + l - 1 - \sum_{i=t+1}^{l} \frac{1}{d_i}\right).\end{aligned}$$

Since $l - t = l - (s - (k - l))/2 = (k - s + l)/2$, we have

$$\rho(G) = 1 + N\left(k - 1 - \sum_{i=1}^{(k-s+l)/2} \frac{1}{m_i}\right) = \tilde{\sigma}(G) - 1. \qquad \square$$

Remark 2.5. *Theorem 2.4 enables a comparison of the results from both papers [2] and [15]. Call* $\mathcal{S}^c_{\text{ab}}$ *the set of numbers in the symmetric crosscap spectrum which are* $\tilde{\sigma}(A)$ *for some Abelian group* A*, and* $\mathcal{S}^r_{\text{ab}}$ *the set of numbers in the real genus spectrum which are* $\rho(A)$ *for some Abelian group* A*. The set* $\mathcal{S}^c_{\text{ab}}$ *was studied in [2], and the set* $\mathcal{S}^r_{\text{ab}}$ *in [15]. Since we have proved that* $\tilde{\sigma}(A) = \rho(A) + 1$ *for each Abelian group* A*, the results in both papers imply each other. For instance, if* n *is even, then* $n \in \mathcal{S}^c_{\text{ab}}$ *if and only if* $n \equiv 2 \pmod 4$ *(Theorem 2 of [2]), and if* n *is odd, then* $n \in \mathcal{S}^r_{\text{ab}}$ *if and only if* $n \equiv 1 \pmod 4$ *(Theorem 1 in [15]). In the same way, all partial results on the structure of each of both sets obtained in those two papers can be translated in terms of the other, by using the fundamental equality* $\tilde{\sigma}(A) = \rho(A) + 1$*.*

2.3 Groups $C_n \times DC_3$ and $C_n \times A_4$

Theorems 2.1 and 2.4 suggest that Property (2.1) holds often. Also other families of groups satisfy it. Consider the groups of order $12n$, $C_n \times DC_3$ and $C_n \times A_4$. The real genus and symmetric crosscap number of these groups were obtained in [5] and [9], respectively, and they are presented below

Table 1

n	$\rho(C_n \times DC_3)$	$\tilde{\sigma}(C_n \times DC_3)$
2	13	14
3	16	17
6	43	44
odd, $(n,6)=1$	$8n-2$	$8n-1$
odd, $3 \mid n,\ 9 \nmid n$	$8n-8$	$8n-7$
odd, $9 \mid n$	$8n-2$	$8n-1$
even, $4 \nmid n$	$9n-11$	$9n-10$
even, $4 \mid n$	$8n+1$	$8n+2$

Hence, for all n, $\tilde{\sigma}(C_n \times DC_3) = \rho(C_n \times DC_3) + 1$.

For the groups $C_n \times A_4$ with n divisible by 3 we have $\rho(C_n \times A_4) = 8n-11$ and $\tilde{\sigma}(C_n \times A_4) = 8n-10$. So there exist families of non-Abelian groups of even order satisfying Property (2.1).

2.4 Groups $C_m \times D_n$

Now, we consider the groups $C_m \times D_n$. Their real genus and symmetric crosscap number were obtained respectively in [10] and [7]. However, it is necessary to correct a mistake in [7]. In Proposition 2.3 of that paper, it was stated that $\tilde{\sigma}(C_m \times D_n) = 2 + n(m-2)$ if m is a multiple of 4 and n is odd. The proof included the claim that it is not possible to obtain a suitable epimorphism $\theta : \Lambda \to C_m \times D_n$ for a group Λ with signature $(0, +, [-], \{(\alpha), (-)\})$ for an $\alpha \geq 2$. As we will see this is wrong, and the genus of a surface on which $C_m \times D_n$ acts can be lowered for those values of m and n if $2n < m$, as follows.

Proposition 2.6. *Let m be a multiple of 4, n odd, and $2n < m$. Then $\tilde{\sigma}(C_m \times D_n) = 2 + m(n-1)$.*

Proof. Let X be a generator of C_m, A and B generators of D_n of order 2, and Λ be an NEC group with signature $(0, +, [-], \{(n), (-)\})$. We define a homomorphism θ from Λ to $C_m \times D_n$ by

$$\theta(e_1) = XAB, \quad \theta(e_2) = X^{-1}BA, \quad \theta(c_{1,0}) = A, \quad \theta(c_{1,1}) = BAB, \quad \theta(c_{2,0}) = X^{m/2}.$$

Then, $\theta(c_{1,1}c_{1,0}) = (BA)^2$, and so $\theta((c_{1,1}c_{1,0})^{(n+1)/2}) = BA$. Now, $\theta(e_1(c_{1,1}c_{1,0})^{(n+1)/2}) = X$; and so, $\theta(c_{2,0}(e_1(c_{1,1}c_{1,0})^{(n+1)/2})^{m/2}c_{1,0}) = X^{m/2}X^{m/2}A = A$. Finally, since BA and A are images of orientation-preserving elements of Λ, so is B.

The reduced area of Λ is $\frac{1}{2}(1-\frac{1}{n}) = \frac{n-1}{2n}$, and so $\tilde{\sigma}(C_m \times D_n) \leq 2 + \frac{n-1}{2n}2mn = 2 + m(n-1)$. We are going to see that this bound cannot be lowered. All possible signatures for the group Λ were already studied in the proof of Proposition 2.3 of [7], excepting those of the form $(0, +, [-], \{(\alpha), (-)\})$.

We complete the work now, considering these signatures. Therefore, suppose that there exists an epimorphism θ from an NEC group Λ with signature $(0,+,[-],\{(\alpha),(-)\})$ for an $\alpha \geq 2$ onto $C_m \times D_n$, and call ψ the composition of θ with the projection of $C_m \times D_n$ onto D_n. Since $c_{1,0}$ has order 2 and n is odd, necessarily $\psi(c_{1,0}) = (AB)^t A$ for a certain t. Then, $\psi(e_1)$ can have the form $(AB)^r$ or $(AB)^r A$. In any case those two images must generate D_n. If $\psi(e_1) = (AB)^r$, then $\psi(e_1 c_{1,0}) = (AB)^{r+t} A$ has order 2. So, in order to generate D_n, $(AB)^r$ must have order n. Besides, $\psi(c_{1,1}) = (BA)^r (AB)^t A (AB)^r = (AB)^{t-2r} A$, and so, $\psi(c_{1,0} c_{1,1}) = (AB)^t A (AB)^{t-2r} A = (AB)^{2r}$ has also order n. Thus, $\alpha = n$. On the other hand, if $\psi(e_1) = (AB)^r A$, then $\psi(e_1 c_{1,0}) = (AB)^{r-t}$, which must have order n. Since $\psi(c_{1,1}) = (AB)^r A (AB)^t A (AB)^r A = (AB)^{2t-t} A$, then $\psi(c_{1,0} c_{1,1}) = (AB)^t A (AB)^{2r-t} A = (AB)^{2t-2r}$. Now, both $\psi(e_1)$ and $\psi(c_{1,0})$ have order 2, and so $\psi(e_1 c_{1,0}) = (AB)^{r-t}$ must have order n. But then also $\psi(c_{1,0} c_{1,1})$ has order n, and again $\alpha = n$. We have finished, and the inequality $2 + m(n-1) < 2 + n(m-2)$ holds if and only if $2n < m$. □

By results in [10] and [7], and Proposition 2.6, we have the following Theorem where for an abuse of notation we write ρ and $\tilde{\sigma}$ for $\rho(C_m \times D_n)$ and $\tilde{\sigma}(C_m \times D_n)$.

Theorem 2.7. *The real genus and the symmetric crosscap number of the groups $C_m \times D_n$ are the following*

m odd, n even, $n < 2m$	$\rho = 1 + m(n-2)$	$\tilde{\sigma} = 2 + m(n-2)$
m odd, n even, $n \geq 2m$	$\rho = 1 + n(m-1)$	$\tilde{\sigma} = 2 + n(m-1)$
m, n odd, $m > n$	$\rho = 1 + m(n-1)$	$\tilde{\sigma} = 2 + mn - m - n$
m, n odd, $m < n$	$\rho = 1 + n(m-1)$	$\tilde{\sigma} = 2 + mn - m - n$
$m = n$ odd	$\rho = 1 + m(m-2)$	$\tilde{\sigma} = 2 + m(m-2)$
m, n even	$\rho = 1 + mn$	$\tilde{\sigma} = 2 + mn$
m a multiple of 4, n odd, $m < 2n$	$\rho = 1 + n(m-2)$	$\tilde{\sigma} = 2 + n(m-2)$
m a multiple of 4, n odd, $m > 2n$	$\rho = 1 + m(n-1)$	$\tilde{\sigma} = 2 + m(n-1)$

Corollary 2.8. *Observe that $\tilde{\sigma}(C_m \times D_n) = \rho(C_m \times D_n) + 1$, except when m and n are different odd numbers. In such a case, for $m > n$, $\tilde{\sigma}(C_m \times D_n) = \rho(C_m \times D_n) + 1 - n$; and if $n > m$, $\tilde{\sigma}(C_m \times D_n) = \rho(C_m \times D_n) + 1 - m$. Both results provide all even negative numbers for the difference $\tilde{\sigma}(G) - \rho(G)$.*

2.5 Groups $D_m \times D_n$

Now we shall consider the groups $D_m \times D_n$. Their symmetric crosscap number was obtained in [7], and the real genus in [5]. Observe that the real genus for m and n odd was calculated in Proposition 2(a) of [5], and included with a misprint in Theorem 3 there. The result should be read as follows: If m and n are odd, $n < m$, then $\rho(D_m \times D_n) = 1 + m(n-1)$.

In turn, the mistake stated above on $\tilde{\sigma}(C_m \times D_n)$ produced a couple of wrong results on $\tilde{\sigma}(D_m \times D_n)$ which we must correct here. For m odd and n even, Proposition 8 in [7] states that $\tilde{\sigma}(D_m \times D_n) = m(n-2)+2$. This is correct for $n \leq 2m$, but if $2m < n$, then the symmetric crosscap number of $D_m \times D_n$ is in fact smaller, as given by the forthcoming two results.

Proposition 2.9. *Let m be an odd number, n an even number with $n/2$ odd and $2m < n$. Then, $\tilde{\sigma}(D_m \times D_n) = 2 + (m-1)n$.*

Proof. Let A and B be generators of D_m of order 2, and C and D generators of D_n of order 2. Take Λ to be an NEC group with signature $(0, +, [-], \{(2m, 2, 2, 2)\})$, and define a homomorphism θ from Λ to $D_m \times D_n$ by

$$\theta(c_{1,0}) = A, \quad \theta(c_{1,1}) = BD, \quad \theta(c_{1,2}) = B(CD)^{n/2}, \quad \theta(c_{1,3}) = (CD)^{n/2}C, \quad \theta(c_{1,4}) = A.$$

Then, $\theta(c_{1,0}c_{1,1}) = ABD$, and so, $\theta((c_{1,0}c_{1,1})^m) = D$, $\theta((c_{1,0}c_{1,1})^{m+1}) = AB$. Now, $\theta(c_{1,1}c_{1,3}) = B(DC)^{n/2+1}$. Since $(DC)^{n/2+1}$ has order $n/2$ which is odd, $\theta((c_{1,1}c_{1,3})^{n/2}) = B$. And so,

$$\theta((c_{1,0}c_{1,1})^{m+1}(c_{1,1}c_{1,3})^{n/2}) = A.$$

Finally, $\theta(c_{1,2}c_{1,3}) = BC$, and so, $\theta((c_{1,1}c_{1,3})^{n/2}c_{1,2}c_{1,3}) = C$. So $D_m \times D_n$ is generated by the images of orientation-preserving elements of Λ.

The reduced area of Λ is $\frac{1}{4} - \frac{1}{4m}$, and so $\tilde{\sigma}(D_m \times D_n) \leq 2 + 4mn\left(\frac{1}{4} - \frac{1}{4m}\right) = 2 + (m-1)n$.

We now prove that this is in fact $\tilde{\sigma}(D_m \times D_n)$ by comparing with $\tilde{\sigma}(C_m \times D_n)$ as obtained in [7]. By Proposition 2.2.i) of that paper, for m odd, n even, with $2m < n$, $\tilde{\sigma}(C_m \times D_n) = 2 + n(m-1)$. Since $\tilde{\sigma}(D_m \times D_n) \geq \tilde{\sigma}(C_m \times D_n)$, we have finished. □

Proposition 2.10. *Let m be an odd number, n a multiple of 4, and $2m < n$. Then, $\tilde{\sigma}(D_m \times D_n) = 2 + (m-1)n$.*

Proof. Let A and B generators of D_m, and C and D generators of D_n, all of them of order 2. Take Λ an NEC group of signature $(0, +, [-], \{(2m, 2, 2, 2)\})$, and define a homomorphism θ from Λ to $D_m \times D_n$ by

$$\theta(c_{1,0}) = A, \quad \theta(c_{1,1}) = BD, \quad \theta(c_{1,2}) = (CD)^{n/2}, \quad \theta(c_{1,3}) = C, \quad \theta(c_{1,4}) = A.$$

Then, $\theta(c_{1,0}c_{1,1}) = ABD$. Since m is odd, $\theta((c_{1,0}c_{1,1})^m) = D$, and $\theta((c_{1,0}c_{1,1})^{m+1}) = AB$. Now, $\theta(c_{1,3}(c_{1,0}c_{1,1})^m) = CD$, and so, $\theta((c_{1,3}(c_{1,0}c_{1,1})^m)^{n/2}) = (CD)^{n/2}$. So,

$$\theta(c_{1,0}c_{1,2}(c_{1,3}(c_{1,0}c_{1,1})^m)^{n/2}) = A \quad \text{and} \quad \theta(c_{1,3}c_{1,2}(c_{1,3}(c_{1,0}c_{1,1})^m)^{n/2}) = C.$$

Finally, $\theta(c_{1,0}c_{1,2}(c_{1,3}(c_{1,0}c_{1,1})^m)^{n/2}(c_{1,0}c_{1,1})^{m+1}) = B$. So, $D_m \times D_n$ is generated by the images of orientation-preserving elements of Λ.

The reduced area of Λ is $\frac{1}{4} - \frac{1}{4m}$, and so $\tilde{\sigma}(D_m \times D_n) \leq 2 + 4mn\left(\frac{1}{4} - \frac{1}{4m}\right) = 2 + (m-1)n = \tilde{\sigma}(C_m \times D_n)$. The proof is finished. □

Hence, from [10] and [8] along with Propositions 2.8 and 2.9, we have the following Theorem.

Theorem 2.11. *The real genus and symmetric crosscap number of the groups $D_m \times D_n$ are the following*

m *odd,* n *even,* $n < 2m$	$\rho = 1 + m(n-2)$	$\tilde{\sigma} = 2 + m(n-2)$
m *odd,* n *even,* $n \geq 2m$	$\rho = 1 + n(m-1)$	$\tilde{\sigma} = 2 + n(m-1)$
m*,* n *odd,* $m > n$	$\rho = 1 + m(n-1)$	$\tilde{\sigma} = 1 + (m-1)(n-1)$
$m = n$ *odd*	$\rho = 1 + m(m-2)$	$\tilde{\sigma} = 2 + m(m-2)$
m*,* n *even*	$\rho = 1 + mn$	$\tilde{\sigma} = 2 + mn$

Remark 2.12. *Thus, the groups $D_m \times D_n$ satisfy Property* (2.1)*, except when m and n are different odd numbers. In that case, $\tilde{\sigma}(D_m \times D_n) - \rho(D_m \times D_n) = 1 - n$, and so this difference provides again, as in Corollary 2.8, all even negative numbers.*

3 Gaps in the symmetric crosscap spectrum

Our next results are inspired by [14, Theorem 6]. In that result, C. L. May studied the groups $C_n \times G_{pq}$.

Let $p < q$ be two odd primes such that $p \mid q-1$. Then there exists a non-Abelian group of order pq, denoted by G_{pq}. This group admits a presentation given by generators S and T, and relations $S^q = T^p = 1$, $T^{-1}ST = S^r$, where $r^p \equiv 1 (\text{mod } q)$, $r \not\equiv 1 (\text{mod } q)$. Then ST has order p, and so $X = T$, $Y = ST$, are two generators of G_{pq} of order p. It follows that $\rho(G_{pq}) = q(p-2)+1$, [13, Theorem 4], and, applying Theorem 2.1, we have:

Theorem 3.1. *Let $p < q$ be two odd primes such that $p \mid q-1$. Then $\tilde{\sigma}(G_{pq}) = q(p-2)+2$.*

Now consider the groups $G = C_n \times G_{pq}$. We are going to study the real genus and the symmetric crosscap number of G. In the case when n is coprime with pq, the real genus of G is given by the following theorem of May:

Theorem 3.2 ([14], Theorem 6)**.** *Let $p < q$ be two odd primes such that $p \mid q-1$, and n an integer coprime with pq. Then $\rho(C_n \times G_{pq}) = 1 + q(pn - n - 1)$.*

Now we turn to the symmetric crosscap number of these groups.

Theorem 3.3. *Let $p < q$ be two odd primes such that $p \mid q-1$, and n an integer coprime with pq. Then $\tilde{\sigma}(C_n \times G_{pq}) = 2 + q(pn - n - 1)$.*

Proof. If n is odd, then $C_n \times G_{pq}$ has odd order, and we apply Theorem 2.1 and Theorem 3.2.

Now, we show that these groups satisfy Property (2.1) also in the case when n is even. Let us take X and Y to be the generators of G_{pq} of order p as above, and denote by A the generator of C_n. Consider an NEC group Γ with signature $(0, +, [p, np], \{(-)\})$, and define the epimorphism θ from Γ onto $C_n \times G_{pq}$ by $\theta(x_1) = X$, $\theta(x_2) = AY$, $\theta(e_1) = (AXY)^{-1}$, $\theta(c_{1,0}) = A^{n/2}$. Since n and p are coprime, there exist integers α, β, such that $\alpha n + \beta p = 1$. Then, $\theta(x_2^{\alpha n}) = (AY)^{\alpha n} = Y^{\alpha n} = Y^{1-\beta p} = Y$, $\theta(x_2^{\beta p}) = (AY)^{\beta p} = A^{\beta p} = A^{1-\alpha n} = A$. Besides, $\theta\left(x_2^{\beta pn/2} c_{1,0}\right) = 1$, and so the kernel contains an orientation reversing element. So, $\tilde{\sigma}(C_n \times G_{pq}) \leq 2 + q(pn - n - 1)$.

Now we need to see that the area of Γ is minimal. The only possibility to reduce the area is to substitute n with one of its factors, say k, and take signature

$$(0, +, [p, kp], \{(-)\}) \quad \text{or} \quad (1, -, [p, kp], \{-\}).$$

Then the image of x_2 must be $A^{n/k}Y$, and either the image of $c_{1,0}$ is $A^{n/2}$ or the image of d_1 is $A^{(n-n/k)/2}(XY)^{(p-1)/2}$.

In the first case it is not possible to generate A as an image of an orientation preserving element, because the image of any word with an even number of copies of $c_{1,0}$ will have, as projection onto C_n, a power of $A^{n/k}$. In the second case, the exponent n/k must be even, in order to get that the image of $d_1^2 x_1 x_2$ be 1. But then also the orientation preserving elements contain an even number of copies of d_1, and so only powers of A with even exponent can be obtained. Therefore, also in this case the element A is not the image of an orientation preserving element.

Thus the area of Γ is minimal, and we have that $\tilde{\sigma}(C_n \times G_{pq}) = 2 + q(pn - n - 1)$, and these groups satisfy Property (2.1). □

We are now going to use the above results to eliminate many possible gaps in the symmetric crosscap spectrum. This problem was studied in [1], and the main result was the following:

Theorem 3.4 ([1], Theorem 2)**.** *Let $N > 3$ be a gap of the symmetric crosscap spectrum. Then $N \equiv 3$, 51, 75 or 99 (mod 120), $N \not\equiv 651$ (mod 660), $N - 2$ is not a square, and $N - 2$ has some prime factor $p \equiv 5$ (mod 6).*

These conditions, necessary for a number to be a gap, are not sufficient. For $N < 10000$, they left sixty-seven numbers which were possible gaps. Three of them are in fact the symmetric crosscap number of a group, thanks to Theorems 2.3 and 3.1. We show them in the Table 2, where we indicate N, its class (mod 120), the prime factors of $N - 2$, and the group G such that $\tilde{\sigma}(G) = N$.

Table 2

N	$N \equiv \pmod{120}$	$N-2$	$G,\ \tilde{\sigma}(G)=N$
1443	3	$1441 = 11 \cdot 131$	$G_{13\cdot 131}$
4875	15	$4873 = 11 \cdot 443$	$G_{13\cdot 443}$
6051	51	$6049 = 23 \cdot 263$	$C_{23} \times C_{276}$

This leaves sixty-four numbers which are candidates for being a gap, but forty of them are actually $\tilde{\sigma}(C_n \times G_{pq})$ for some n, p, q as obtained in Theorem 3.3. We display the respective data in Table 3.

Table 3

N	$N \equiv \pmod{120}$	$N-2$	$G,\ \tilde{\sigma}(G)=N$
915	75	$913 = 11 \cdot 83$	$C_{21} \times G_{5\cdot 11}$
1179	99	$1177 = 11 \cdot 107$	$C_{27} \times G_{5\cdot 11}$
1539	99	$1537 = 29 \cdot 53$	$C_{9} \times G_{7\cdot 29}$
1635	75	$1633 = 23 \cdot 71$	$C_{6} \times G_{5\cdot 71}$
1923	3	$1921 = 17 \cdot 113$	$C_{3} \times G_{7\cdot 113}$
2235	75	$2233 = 7 \cdot 11 \cdot 29$	$C_{13} \times G_{7\cdot 29}$
2499	99	$2497 = 11 \cdot 227$	$C_{57} \times G_{5\cdot 11}$
2739	99	$2737 = 7 \cdot 17 \cdot 23$	$C_{12} \times G_{11\cdot 23}$
2763	3	$2761 = 11 \cdot 251$	$C_{3} \times G_{5\cdot 251}$
3339	99	$3337 = 47 \cdot 71$	$C_{8} \times G_{7\cdot 71}$
3555	75	$3553 = 11 \cdot 17 \cdot 19$	$C_{81} \times G_{5\cdot 11}$
3819	99	$3817 = 11 \cdot 347$	$C_{87} \times G_{5\cdot 11}$
4083	3	$4081 = 7 \cdot 11 \cdot 53$	$C_{93} \times G_{5\cdot 11}$
4323	3	$4321 = 29 \cdot 149$	$C_{25} \times G_{7\cdot 29}$
4395	75	$4393 = 23 \cdot 191$	$C_{6} \times G_{5\cdot 191}$
4899	99	$4897 = 59 \cdot 83$	$C_{3} \times G_{29\cdot 59}$
5139	99	$5137 = 11 \cdot 467$	$C_{117} \times G_{5\cdot 11}$
5403	3	$5401 = 11 \cdot 491$	$C_{3} \times G_{5\cdot 491}$
5499	99	$5497 = 23 \cdot 239$	$C_{4} \times G_{7\cdot 239}$
5595	75	$5593 = 7 \cdot 17 \cdot 47$	$C_{400} \times G_{3\cdot 7}$

N	$N \equiv$ (mod 120)	$N-2$	$G,\ \tilde{\sigma}(G)=N$
5715	75	$5713 = 29 \cdot 197$	$C_5 \times G_{7\cdot 197}$
6195	75	$6193 = 11 \cdot 563$	$C_{141} \times G_{5\cdot 11}$
6411	51	$6409 = 13 \cdot 17 \cdot 29$	$C_{37} \times G_{7\cdot 29}$
6459	99	$6457 = 11 \cdot 587$	$C_{147} \times G_{5\cdot 11}$
6723	3	$6721 = 11 \cdot 13 \cdot 47$	$C_{259} \times G_{3\cdot 13}$
7155	75	$7153 = 23 \cdot 311$	$C_6 \times G_{5\cdot 311}$
7515	75	$7513 = 11 \cdot 683$	$C_{171} \times G_{5\cdot 11}$
7635	75	$7633 = 17 \cdot 449$	$C_3 \times G_{7\cdot 449}$
7731	51	$7729 = 59 \cdot 131$	$C_5 \times G_{13\cdot 131}$
7779	99	$7777 = 7 \cdot 11 \cdot 101$	$C_{177} \times G_{5\cdot 11}$
7803	3	$7801 = 29 \cdot 269$	$C_{45} \times G_{7\cdot 29}$
8043	3	$8041 = 11 \cdot 17 \cdot 43$	$C_{94} \times G_{3\cdot 43}$
8259	99	$8257 = 23 \cdot 359$	$C_{36} \times G_{11\cdot 23}$
8451	51	$8449 = 7 \cdot 17 \cdot 71$	$C_{20} \times G_{7\cdot 71}$
8835	75	$8833 = 11^2 \cdot 73$	$C_{61} \times G_{3\cdot 73}$
8979	99	$8977 = 47 \cdot 191$	$C_{12} \times G_{5\cdot 191}$
9099	99	$9097 = 11 \cdot 827$	$C_{207} \times G_{5\cdot 11}$
9195	75	$9193 = 29 \cdot 317$	$C_{53} \times G_{7\cdot 29}$
9363	3	$9361 = 11 \cdot 23 \cdot 37$	$C_{127} \times G_{3\cdot 37}$
9915	75	$9913 = 23 \cdot 431$	$C_6 \times G_{5\cdot 431}$

According to above results only twenty-four numbers N remain as potential gaps in the symmetric crosscap spectrum, with $3 < N < 10000$. They are shown in Table 4.

These results reinforce the conjecture that there is no other gap besides 3 in the spectrum of the symmetric crosscap number.

Now, we are going to study the particular case $N = 699$, the smallest number for which it is unknown whether it represents a gap in the spectrum. This will demonstrate how to use the relationship between the real genus and symmetric crosscap number, and how Property (2.1) is useful when it holds. Unfortunately, this is not the case for this value of N and the group G already known to satisfy $\rho(G) = N - 1$.

Table 4: Table 4

N	$N \equiv \pmod{120}$	$N-2$
699	99	$697 = 17 \cdot 41$
1083	3	$1081 = 23 \cdot 47$
1515	75	$1513 = 17 \cdot 89$
2331	51	$2329 = 17 \cdot 137$
3651	51	$3649 = 41 \cdot 89$
3843	3	$3841 = 23 \cdot 167$
3963	3	$3961 = 17 \cdot 233$
4371	51	$4369 = 17 \cdot 257$
4635	75	$4633 = 41 \cdot 113$
5019	99	$5017 = 29 \cdot 173$
5355	75	$5353 = 53 \cdot 101$
5619	99	$5617 = 41 \cdot 137$
6003	3	$6001 = 17 \cdot 353$
6315	75	$6313 = 59 \cdot 107$
6819	99	$6817 = 17 \cdot 401$
7851	51	$7849 = 47 \cdot 167$
7899	99	$7897 = 53 \cdot 149$
8499	99	$8497 = 29 \cdot 293$
8811	51	$8809 = 23 \cdot 383$
8859	99	$8857 = 17 \cdot 521$
8883	3	$8881 = 83 \cdot 107$
9171	51	$9169 = 53 \cdot 173$
9555	75	$9553 = 41 \cdot 233$
9675	75	$9673 = 17 \cdot 569$

Since $41 \equiv 1 \pmod 4$, there exists a semidirect product $C_4 \rtimes C_{41}$, with presentation

$$\langle X, Y \mid Y^4 = X^{41} = 1, XY = YX^9 \rangle.$$

Now call $G = C_9 \times (C_4 \rtimes C_{41})$, and Z a generator of C_9. This group G has real genus 698, see Corollary 6 of [14]. So, if it satisfies Property (2.1), we have a group with symmetric crosscap number 699. Let us study G. Its elements of order 2 lie in $C_4 \rtimes C_{41}$, and they have the form X^kY^2. For, $(X^kY^2)^2 = X^kY^2X^kY^2 = YX^{9k}YX^kY^2 = Y^2X^{81k}X^kY^2 = Y^2X^{82k}Y^2 = 1$, and it is clear that no other element has order 2.

Now, consider an NEC group Λ with signature $(0, +, [2, 36], \{(-)\})$ and an epimorphism $\theta : \Lambda \to G$ defined by

$$\theta(x_1) = XY^2, \quad \theta(x_2) = YZ, \quad \theta(e_1) = YX^{-1}Z^{-1}, \quad \theta(c_{1,0}) = X^{10}Y^2.$$

The kernel of this epimorphism is a non-orientable unbordered surface group, because $o(XY^2) = 2$, $o(YZ) = 36$, and

$$\begin{aligned}\theta(x_1x_2e_1) &= XY^2YZYX^{-1}Z^{-1} = XY^4X^{-1} = 1,\\ \theta(e_1^{-1}c_{1,0}e_1c_{1,0}) &= XY^3ZX^{10}Y^2YX^{-1}Z^{-1}X^{10}Y^2 = XY^3X^{10}Y^3X^9Y^2\\ &= Y^3X^{729}X^{10}Y^3X^9Y^2 = Y^3X^{739}Y^3X^9Y^2 = Y^6X^{739\cdot 729+9}Y^2 = Y^8 = 1.\end{aligned}$$

Besides, $\theta(\Lambda^+) = G$, because

$$\theta(x_2^9) = (YZ)^9 = Y \quad \theta(x_2^{28}) = (YZ)^{28} = Z \quad \theta(x_1x_2^{18}) = (XY^2)Y^2 = X$$

The genus of the corresponding surface is

$$(9\cdot 4\cdot 41)\left(1-\frac{1}{2}+1-\frac{1}{36}-1\right)+2 = 9\cdot 4\cdot 41\cdot\frac{17}{36}+2 = 17\cdot 41+2 = 699.$$

It only remains to prove that this is the minimum genus of a non-orientable unbordered surface on which G acts. But this is not the case. Consider an NEC group Γ with signature $(0, +, [36], \{(41)\})$ and an epimorphism $\theta : \Gamma \to G$ defined by

$$\theta(x_1) = YZ, \quad \theta(e_1) = Y^{-1}Z^{-1}, \quad \theta(c_{1,0}) = XY^2, \quad \theta(c_{1,1}) = X^{32}Y^2.$$

Then,

$$\theta(e_1^{-1}c_{1,0}e_1c_{1,1}) = YZXY^2Y^{-1}Z^{-1}X^{32}Y^2 = YXYX^{32}Y^2 = YYX^9X^{32}Y^2 = 1.$$

Besides, $\theta(\Gamma^+) = G$, because

$$\begin{aligned}\theta(x_1^{28}) &= Z,\\ \theta(x_1^9) &= Y,\\ \theta(c_{1,0}c_{1,1}) &= XY^2X^{32}Y^2 = Y^2X^{81}X^{32}Y^2 = Y^2X^{31}Y^2 = Y^4X^{31\cdot 81} = X^{10}.\end{aligned}$$

So that, $\theta((c_{1,0}c_{1,1})^{37}) = X^{370} = X$. Now, we compute the genus, and it is

$$(9\cdot 4\cdot 41)\left(\left(1-\frac{1}{36}\right)+\frac{1}{2}\left(1-\frac{1}{41}\right)-1\right)+2 = 9\cdot 4\cdot 41\cdot\left(\frac{20}{41}-\frac{1}{36}\right)+2 = 20\cdot 9\cdot 4-41+2 = 681.$$

Hence $\tilde{\sigma}(G) \leq 681$, in fact it equals 681, and so the group G does not satisfy Property (2.1), and no group with symmetric crosscap number 699 is known yet.

4 Gaps in the real genus spectrum

All odd numbers belong to the real genus spectrum, since C. L. May proved in [12] that the dicyclic group DC_n of order $4n$ has real genus $2n+1$. So the problem of determining the spectrum of the real genus restricts to even numbers. It is known that 2, 12, 24 and 72 are not the real genus of any group. In his paper [14], C. L. May obtained families of groups whose real genera cover most of the even numbers. For instance, for $N < 10000$, his results leave 328 numbers for which it is unknown whether they belong to the real genus spectrum. M. Pires has calculated explicitly those numbers in [17]. Most of them are multiple of 12, but there are also numbers $N \equiv 2, 6, 8 \pmod{12}$.

Unfortunately, the groups G for which we know that $\tilde{\sigma}(G) \equiv 1, 7, 9 \pmod{12}$ do not satisfy Property (2.1) and cannot be used to eliminate gaps in the real genus spectrum. The situation is very different for $N \equiv 2 \pmod{12}$. According to [17], the numbers $N \equiv 2 \pmod{12}$ with $N < 10000$, which are not yet known to belong to the real genus spectrum are 1082, 3842, 6266, 7850, 8810 and 8882. Let us pay attention to $6266 \equiv 26 \pmod{60}$. In [1] it was proved that for each $k \geq 0$, a semidirect product $G_k = C_5 \rtimes C_{8+16k}$ satisfies $\tilde{\sigma}(G_k) = 60k + 27$. We are going to show that these groups satisfy Property (2.1), and so $\rho(G_k) = 60k + 26$.

Proposition 4.1. *Let $k \geq 0$, and $G_k = C_5 \rtimes C_{8+16k}$, with presentation $\langle A, B \mid B^5 = A^{8+16k} = 1, BA = AB^2 \rangle$. Then, $\rho(G_k) = 60k + 26$.*

Proof. One can see in [1] or [17] that the element BA^{2+4k} has order 4, and A^{4+8k} is the unique element of G_k of order 2. Since BA^{2+4k} and A generate G_k, take an NEC group Λ with signature $(0, +, [4, 8+16k], \{(-)\})$, and define $\theta : \Lambda \to G_k$ by

$$\theta(x_1) = BA^{2+4k}, \quad \theta(x_2) = A, \quad \theta(e_1) = A^{5+12k}B^{-1}, \quad \theta(c_{1,0}) = 1.$$

Then, θ is an epimorphism, the reduced area of Λ is $|\Lambda| = \frac{5+12k}{8+16k}$, and $\rho(G_k) \leq 1 + o(G_k)|\Lambda| = 1 + (40+80k)\frac{5+12k}{8+16k} = 60k + 26$. In order to see that this is in fact $\rho(G_k)$, recall that the signature of the suitable group Λ must have a period-cycle with two consecutive link-periods equal to 2, or an empty period-cycle, see [3]. Since G_k has a unique element of order 2, the first possibility does not hold. So, Λ must have an empty period-cycle, and for getting a smaller reduced area, its signature must have the form $(0, +, [m_1, m_2], \{(-)\})$. Then, by using the same arguments as in Proposition 7 of [1], it follows that the minimal area is indeed attained for the signature $(0, +, [4, 8+16k], \{(-)\})$. Thus, $\rho(G_k) = 60k + 26$. Observe that in particular $\rho(G_{104}) = 6266$. □

On the contrary, for the five other values of N, namely 1082, 3842, 7850, 8810 and 8882, it is not known whether $N+1$ belongs to the symmetric crosscap spectrum, see Table 4. Hence, these pairs $(N, N+1)$ seem to be a convenient target for identifying possible gaps in both spectra.

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

All necessary data are stated and quoted in the paper.

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