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Some inequalities associated with a partial differential operator

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ABSTRACT

We study uncertainty principles for a generalized Fourier transform $\mathcal{F}_\alpha$, associated with the pair of partial differential operators (D, D_α) originally introduced by Flensted-Jensen and later extended by Trimèche. This transform, is defined via the Jacobi kernel and an appropriate weighted measure. We establish an $\mathrm{L}^p - \mathrm{L}^q$ version of Miyachi's theorem, from which we deduce Cowling-Price-type results. Additionally, we establish a local uncertainty principle in the sense of Faris and provide related numerical estimates.

RESUMEN

Estudiamos principios de incertidumbre para una transformada de Fourier generalizada $\mathcal{F}_\alpha$, asociada al par de operadores diferenciales parciales (D, D_α) originalmente introducidos por Flensted-Jensen y luego extendidos por Trimèche. Esta transformada está definida a través del núcleo de Jacobi y una medida pesada apropiada. Establecemos una versión $\mathrm{L}^p - \mathrm{L}^q$ del teorema de Miyachi, a partir del cual deducimos resultados de tipo Cowling-Price. Adicionalmente, establecemos un principio de incertidumbre local en el sentido de Faris y entregamos estimaciones numéricas relacionadas.

Keywords and Phrases: Partial differential operators, generalized Fourier transform, Jacobi kernel, Miyachi theorem, Cowling-Price theorem, uncertainty principle.

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1 Introduction

In the context of harmonic analysis on symmetric spaces, Flensted-Jensen [7] introduced a pair of partial differential operators fundamental to the study of spherical functions on simply connected semisimple Lie groups:

$$D = \frac{\partial}{\partial\theta} \quad \text{and} \quad D_n = \frac{\partial^2}{\partial y^2} + [(2n-1)\coth y + \tanh y]\,\frac{\partial}{\partial y} - \frac{1}{\cosh^2 y}\frac{\partial^2}{\partial\theta^2},$$

where n is a positive integer. Trimèche [16] extended these operators by generalizing the integer parameter $n-1$ to a positive real parameter $\alpha > 0$, thereby developing an associated harmonic analysis framework centered around a generalized Fourier transform $\mathcal{F}_\alpha$. For suitable functions, this transform is given by

$$\mathcal{F}_\alpha f(\lambda,\mu) = \iint_{\mathbb{R}_+\times\mathbb{R}} f(y,\theta)\varphi_{-\lambda,\mu}(y,\theta)\,dm_\alpha(y,\theta),$$

where $\varphi_{-\lambda,\mu}$ is constructed from the classical Jacobi kernel $\varphi_\mu^{\alpha,\lambda}$ via the formula:

$$\varphi_{\lambda,\mu}(y,\theta) = e^{i\lambda\theta}(\cosh y)^\lambda \varphi_\mu^{\alpha,\lambda}(y)$$

and the measure

$$dm_\alpha(y,\theta) = 2^{2(\alpha+1)}(\sinh y)^{2\alpha+1}\cosh y\,dy\,d\theta$$

reflects the intrinsic non-Euclidean geometry of the underlying space. Unlike classical Jacobi transforms, where λ is fixed, $\mathcal{F}_\alpha$ treats λ as a spectral variable. This key innovation makes $\mathcal{F}_\alpha$ a natural and powerful tool for analyzing radial functions on the universal covering group of $\mathbf{U}(n,1)$. Although significant work has been done to explore various aspects of this transform [7, 9, 12, 16], its potential within the framework of uncertainty principles remains largely unexplored. This paper aims to address this gap by establishing several uncertainty principles for $\mathcal{F}_\alpha(f)$. We begin by recalling that classical examples of such principles include decay-based results like Hardy's theorem [8], which states that if

$$|f(x)| \le ce^{-ax^2} \quad \text{and} \quad |\widehat{f}(y)| \le ce^{-by^2},$$

then $f = 0$ when $ab > \frac{1}{4}$, and f is Gaussian otherwise. Cowling-Price [2] extended this to $\mathrm{L}^p - \mathrm{L}^q$ integrability conditions, while Miyachi [13] introduced logarithmic integrability conditions, requiring

$$e^{ax^2} f \in \mathrm{L}^1(\mathbb{R}) + \mathrm{L}^\infty(\mathbb{R}), \qquad \int_{\mathbb{R}} \log^+\left(\frac{|\widehat{f}(y)|e^{by^2}}{\beta}\right) dy < \infty,$$

where $\log^+ x = \max(\log x, 0)$. Miyachi extended Hardy's theorem by replacing pointwise decay with logarithmic integrability conditions, thereby enlarging the class of admissible functions.

This work establishes an analogue of Miyachi's theorem for the generalized Fourier transform $\mathcal{F}_\alpha$, associated with the operator pair (D, D_α). Our approach, which leverages sharp estimates of the generalized Jacobi kernel, distinguishes itself from previous techniques. These include methods reliant on Bessel operators and the Dunkl setting [1,10], Laguerre polynomials for Riemann-Liouville operators [10], or Abel transforms and heat kernels in Jacobi analysis [3]. This builds upon several related studies on uncertainty principles found in [4,9,12,14].

Alongside these decay-based principles, a distinct, support-based perspective was developed by Faris and Price. This approach quantifies uncertainty not through rates of decay, but through the spatial concentration of a function and the frequency dispersion of its transform. The Faris-Price [5,15] expresses this idea via measurable sets: for $f \in \mathrm{L}^2(\mathbb{R}^n)$ and a measurable set $E \subset \mathbb{R}^n$, one has

$$\int_E |\widehat{f}(\xi)|^2 d\xi \leq K_\alpha |E|^{\frac{2\alpha}{n}} \||x|^\alpha f\|_2^2, \quad 0 < \alpha < \frac{n}{2}.$$

Such support-based principles provide explicit constants that govern the trade-off between spatial localization and spectral dispersion.

A second main contribution is the establishment of a local uncertainty principle of Faris-type for $\mathcal{F}_\alpha$. The theoretical result guarantees the existence of an optimal constant $K_{\alpha,a,q}(\gamma_\alpha(F))$ but does not provide its explicit form. To bridge this gap, we employ numerical optimization techniques to compute this constant, quantifying the precise trade-off between spatial and spectral localization.

The paper is organized as follows. Section 2 develops the harmonic analysis framework for $\mathcal{F}_\alpha$ and provides the necessary kernel bounds. Section 3 proves Miyachi- and Cowling-Price-type theorems. Section 4 establishes the Faris-type principle and conducts a numerical investigation to compute the associated optimal constants.

2 Mathematical framework

2.1 Generalized Jacobi Kernel

Let α be a positive real number and let $\mathbb{K} = [0, +\infty[\times\mathbb{R}$. Following [16], we consider the differential operators:

$$\begin{cases} D = \dfrac{\partial}{\partial\theta}, \\ D_\alpha = \dfrac{\partial^2}{\partial y^2} + [(2\alpha+1)\coth y + \tanh y]\dfrac{\partial}{\partial y} - \dfrac{1}{\cosh^2 y}\dfrac{\partial^2}{\partial\theta^2} + (\alpha+1)^2. \end{cases} \tag{2.1}$$

For complex parameters $\lambda, \mu \in \mathbb{C}$, the system

$$\begin{cases} Du = i\lambda u, \\ D_\alpha u = -\mu^2 u, \\ u(0,0) = 1, \quad \frac{\partial u}{\partial y}(0,\theta) = 0 \text{ for } \theta \in \mathbb{R} \end{cases} \tag{2.2}$$

has a unique solution given by the generalized Jacobi kernel:

$$\varphi_{\lambda,\mu}(y,\theta) = e^{i\lambda\theta}(\cosh y)^\lambda \varphi_\mu^{\alpha,\lambda}(y), \tag{2.3}$$

where $\varphi_\mu^{\alpha,\lambda}$ is the Jacobi kernel [6]:

$$\varphi_\mu^{\alpha,\lambda}(y) = {}_2F_1\left(\frac{\alpha+\lambda+1+i\mu}{2}, \frac{\alpha+\lambda+1-i\mu}{2}; \alpha+1; -\sinh^2 y\right), \tag{2.4}$$

expressed in terms of the Gaussian hypergeometric function ${}_2F_1$.

For $y > 0$ and $\theta \in \mathbb{R}$, the kernel admits the integral representation [16]:

$$\varphi_{\lambda,\mu}(y,\theta) = \frac{2^\alpha \alpha}{\pi}(\sinh y)^{-2\alpha} \int_0^y \int_{-\omega}^{\omega} (\cosh y \cos\psi - \cosh s)^{\alpha-1} \cos(\mu s) e^{i\lambda(\theta+\psi)} d\psi\, ds, \tag{2.5}$$

where $\omega = \omega(s,y) = \arccos(\cosh s/\cosh y)$. When $y = 0$, the kernel simplifies to $\varphi_{\lambda,\mu}(0,\theta) = e^{i\lambda\theta}$.

The spectral space $\widehat{\mathbb{K}} = \mathbb{L} \cup \Omega$ consists of:

$$\mathbb{L} = \mathbb{R} \times [0, +\infty[, \quad \Omega = \bigcup_{m\in\mathbb{N}} (D_m^+ \cup D_m^-),$$

where:

$$D_m^+ = \{(\alpha+2m+1+\eta, i\eta) \mid \eta > 0\} \quad \text{and} \quad D_m^- = \{(-\alpha-2m-1-\eta, i\eta) \mid \eta > 0\}.$$

The kernel satisfies the uniform bound [16]:

$$\forall (\lambda,\mu) \in \widehat{\mathbb{K}}, \quad \sup_{(y,\theta)\in\mathbb{K}} |\varphi_{\lambda,\mu}(y,\theta)| = 1. \tag{2.6}$$

The kernel relates to the generalized Riemann-Liouville transform $\mathfrak{X}_\alpha$ through:

$$\varphi_{\lambda,\mu}(y,\theta) = \mathfrak{X}_\alpha\left(\cos(\mu\cdot)e^{i\lambda\cdot}\right)(y,\theta),$$

where

$$\mathfrak{X}_\alpha f(y,\theta) = \int_{\mathbb{K}} f(x,t) \mathrm{K}(x,t,y,\theta) dx\, dt$$

with kernel

$$\mathrm{K}(x,t,y,\theta) = \frac{2^\alpha \alpha}{\pi} \chi_{[0,y]}(x) \chi_{[-\omega,\omega]}(t-\theta)(\cosh y \cos(t-\theta) - \cosh x)^{\alpha-1}(\sinh y)^{-2\alpha}.$$

For the constant function $\mathbf{1}$, we have the bound:

$$\mathfrak{X}_\alpha(\mathbf{1})(y,\theta) = \int_{\mathbb{K}} \mathrm{K}(x,t,y,\theta)dx\,dt \leq 1. \tag{2.7}$$

2.2 Generalized Fourier transform

For $p \in [1,+\infty]$, we define the weighted Lebesgue spaces as follows:

- For $1 \leq p < \infty$, the space $\mathrm{L}^p_\alpha(\mathbb{K})$ consists of measurable functions $f : \mathbb{K} \to \mathbb{C}$ satisfying

$$\|f\|_{p,m_\alpha} = \left(\int_{\mathbb{K}} |f(y,\theta)|^p dm_\alpha(y,\theta)\right)^{1/p} < \infty,$$

  where the measure is given by

$$dm_\alpha(y,\theta) = 2^{2(\alpha+1)}(\sinh y)^{2\alpha+1}\cosh y\,dy\,d\theta. \tag{2.8}$$

- For $p = \infty$, the space $\mathrm{L}^\infty_\alpha(\mathbb{K})$ consists of measurable functions with finite essential supremum norm

$$\|f\|_{\infty,m_\alpha} = \operatorname*{ess\,sup}_{(y,\theta)\in\mathbb{K}} |f(y,\theta)|.$$

The generalized Fourier transform $\mathcal{F}_\alpha$ on $\mathrm{L}^1_\alpha(\mathbb{K})$ is defined by:

$$\mathcal{F}_\alpha f(\lambda,\mu) = \int_{\mathbb{K}} f(y,\theta)\varphi_{-\lambda,\mu}(y,\theta)dm_\alpha(y,\theta),$$

satisfying the following inequality:

$$\forall(\lambda,\mu) \in \widehat{\mathbb{K}}, \quad |\mathcal{F}_\alpha f(\lambda,\mu)| \leq \|f\|_{1,m_\alpha}. \tag{2.9}$$

The Plancherel measure $d\gamma_\alpha$ combines continuous and discrete parts:

$$\begin{aligned}\int_{\widehat{\mathbb{K}}} g(\lambda,\mu)d\gamma_\alpha(\lambda,\mu) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}\times[0,+\infty[} g(\lambda,\mu)\frac{d\lambda\,d\mu}{|C_1(\lambda,\mu)|^2} \\ &+ \frac{1}{(2\pi)^2}\sum_{m=0}^{\infty}\left\{\int_0^\infty g(\kappa+\eta,i\eta)C_2(\kappa+\eta,i\eta)d\eta + \int_0^\infty g(-\kappa-\eta,i\eta)C_2(-\kappa-\eta,i\eta)d\eta\right\},\end{aligned}$$

where $\kappa = \alpha + 2m + 1$ and:

$$C_1(\lambda,\mu) = \frac{2^{\alpha+1-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma\left(\frac{\alpha+\lambda+1+i\mu}{2}\right)\Gamma\left(\frac{\alpha-\lambda+1+i\mu}{2}\right)}, \quad C_2(\lambda,\mu) = -2i\pi\operatorname*{Res}_{z=\mu}[C_1(\lambda,z)C_1(\lambda,-z)]^{-1}.$$

The weight functions satisfy [16]:

$$K_1|\mu|^2 \le |C_1(\lambda,\mu)|^{-2} \le K_2(1+|\lambda|^2+|\mu|^2)^{2[\alpha+\frac{1}{2}]+1}. \tag{2.10}$$

$$|C_2(\lambda,\mu)| \le K_3(1+|\lambda|^2+|\mu|^2)^{2[\alpha+\frac{1}{2}]+1}. \tag{2.11}$$

The transform $\mathcal{F}_\alpha$ satisfies the Plancherel identity

$$\|\mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha} = \|f\|_{2,m_\alpha}.$$

For $1 \le p \le 2$, the Hausdorff-Young inequality holds:

$$\|\mathcal{F}_\alpha(f)\|_{q,\gamma_\alpha} \le \|f\|_{p,m_\alpha}, \tag{2.12}$$

where q is the conjugate of p. The inversion formula is given by:

$$f(y,\theta) = \int_{\widehat{\mathbb{K}}} \mathcal{F}_\alpha(f)(\lambda,\mu)\varphi_{\lambda,\mu}(y,\theta)d\gamma_\alpha(\lambda,\mu) \tag{2.13}$$

The heat kernel relates to Gaussians via:

$$\mathrm{E}_a^\alpha(y,\theta) = \int_{\widehat{\mathbb{K}}} e^{-a(\lambda^2+\mu^2+(\alpha+1)^2)}\varphi_{\lambda,\mu}(y,\theta)d\gamma_\alpha(\lambda,\mu), \tag{2.14}$$

with more general heat functions:

$$W_{k,j}^\alpha(a,(y,\theta)) = i^k \int_{\widehat{\mathbb{K}}} \lambda^k(-\mu)^{2j} e^{-a(\lambda^2+\mu^2+(\alpha+1)^2)}\varphi_{\lambda,\mu}(y,\theta)d\gamma_\alpha(\lambda,\mu). \tag{2.15}$$

3 Miyachi-type theorem for the generalized Fourier transform

To establish our main result, we first derive kernel estimates on $\mathbb{C}^2$.

Proposition 3.1. *For all $(\lambda,\mu) \in \mathbb{C}^2$ and $(y,\theta) \in \mathbb{K}$,*

$$|\varphi_{\lambda,\mu}(y,\theta)| \le C(1+y)e^{(|\Im\mu|-(\alpha+1))y}e^{|\Im\lambda|(|\theta|+\pi)}, \tag{3.1}$$

where $C > 0$. Moreover, since $y \ge 0$,

$$|\varphi_{\lambda,\mu}(y,\theta)| \le Ce^{|\Im\mu|y+|\Im\lambda|(|\theta|+\pi)}. \tag{3.2}$$

Proof. By [11, Lemma 2.3], for $\lambda = \mu = 0$,

$$\varphi_{0,0}(y,\theta) \le C(1+y)e^{-(\alpha+1)y}.$$

Using $|\cos(\mu s)| \le e^{|\Im\mu|s}$ and the integral representation (2.5),

$$|\varphi_{\lambda,\mu}(y,\theta)| \le Ce^{|\Im\mu|y+|\Im\lambda|(|\theta|+\pi)}\varphi_{0,0}(y,\theta),$$

since $\omega \in [-\pi,\pi]$. This proves (3.1). Inequality (3.2) follows by analyzing the decay of $f(y) = (1+y)e^{-(\alpha+1)y}$ on $[0,+\infty[$. □

We now state a Phragmén-Lindelöf-type lemma sufficient for our needs:

Lemma 3.2 ([10])**.** *Let h be entire on $\mathbb{C}^2$. Suppose there exist constants $C, B > 0$ such that*

$$|h(z_1,z_2)| \le Ce^{B((\Re z_1)^2+(\Re z_2)^2)} \quad \textit{and} \quad \int_{\mathbb{R}^2} \log^+|h(x,y)|dx\,dy < \infty.$$

Then h is constant.

Lemma 3.3. *Let $p,q \in [1,+\infty]$ and f be measurable on $\mathbb{K}$ satisfying*

$$e^{a(y^2+(|\theta|+\pi)^2)+2(\alpha+1)y} f \in \mathrm{L}^p_\alpha(\mathbb{K}) + \mathrm{L}^q_\alpha(\mathbb{K}), \quad a > 0. \tag{3.3}$$

Then $\mathcal{F}_\alpha(f)$ is well-defined and entire on $\mathbb{C}^2$. Moreover, for all $(\lambda,\mu) \in \mathbb{C}^2$,

$$|\mathcal{F}_\alpha(f)(\lambda,\mu)| \le Ce^{\frac{|\Im\lambda|^2+|\Im\mu|^2}{4a}}. \tag{3.4}$$

Proof. The function $(\lambda,\mu) \mapsto \varphi_{-\lambda,\mu}(y,\theta)$ is entire by (2.3) and (2.4). Using Proposition 3.1,

$$|f(y,\theta)\varphi_{-\lambda,\mu}(y,\theta)m_\alpha(y,\theta)| \le Ce^{|\Im\lambda|(|\theta|+\pi)+|\Im\mu|y}|f(y,\theta)|m_\alpha(y,\theta).$$

By (3.3), there exist $f_1 \in \mathrm{L}^p_\alpha(\mathbb{K})$ and $f_2 \in \mathrm{L}^q_\alpha(\mathbb{K})$ such that

$$|f\varphi_{-\lambda,\mu}m_\alpha| \le \sum_{k=1}^{2} g_k(\lambda,\mu,y,\theta),$$

where

$$g_k(\lambda,\mu,y,\theta) = Ce^{|\Im\lambda|(|\theta|+\pi)+|\Im\mu|y}e^{-a(y^2+(|\theta|+\pi)^2)-2(\alpha+1)y}|f_k(y,\theta)|m_\alpha(y,\theta).$$

Observe that

$$|\Im\lambda|(|\theta|+\pi)+|\Im\mu|y - a(y^2+(|\theta|+\pi)^2) = -\Delta_{\lambda,\mu}(y,\theta) + \frac{|\Im\lambda|^2+|\Im\mu|^2}{4a},$$

where

$$\Delta_{\lambda,\mu}(y,\theta) = \left(\sqrt{a}y - \frac{|\Im\mu|}{2\sqrt{a}}\right)^2 + \left(\sqrt{a}(|\theta|+\pi) - \frac{|\Im\lambda|}{2\sqrt{a}}\right)^2 \geq 0.$$

Thus,

$$g_k(\lambda,\mu,y,\theta) \leq Ce^{\frac{|\Im\lambda|^2+|\Im\mu|^2}{4a}} e^{-\Delta_{\lambda,\mu}(y,\theta)} |f_k(y,\theta)| e^{-2(\alpha+1)y} m_\alpha(y,\theta).$$

For a compact $K \subset \mathbb{C}^2$, there exists $(\lambda_0,\mu_0) \in K$ such that

$$\min_{(\lambda,\mu)\in K} \Delta_{\lambda,\mu}(y,\theta) = \Delta_{\lambda_0,\mu_0}(y,\theta).$$

Since $e^{\frac{|\Im\lambda|^2+|\Im\mu|^2}{4a}}$ is bounded on K,

$$g_k(\lambda,\mu,y,\theta) \leq G_k(y,\theta) = Ce^{-\Delta_{\lambda_0,\mu_0}(y,\theta)} |f_k(y,\theta)| e^{-2(\alpha+1)y} m_\alpha(y,\theta).$$

To show $\mathcal{F}_\alpha f$ is entire, it suffices to prove $G_k \in \mathrm{L}^1_\alpha(\mathbb{K})$. By Hölder's inequality,

$$\int_{\mathbb{K}} |G_1(y,\theta)| dy\, d\theta \leq C \left\| f_1 e^{-\frac{2(\alpha+1)y}{p}} \right\|_{p,m_\alpha} \left(\int_{\mathbb{K}} e^{-\Delta_{\lambda_0,\mu_0}(y,\theta)p'} e^{-2(\alpha+1)y} m_\alpha(y,\theta) dy\, d\theta \right)^{\frac{1}{p'}}.$$

Using (2.8), $e^{-2(\alpha+1)y} m_\alpha(y,\theta) \leq C$, so

$$\int_{\mathbb{K}} |G_1(y,\theta)| dy\, d\theta \leq C \|f_1\|_{p,m_\alpha} \left(\int_{\mathbb{K}} e^{-\Delta_{\lambda_0,\mu_0}(y,\theta)p'} dy\, d\theta \right)^{\frac{1}{p'}} < \infty.$$

Similarly, for q' conjugate to q,

$$\int_{\mathbb{K}} |G_2(y,\theta)| dy\, d\theta < \infty.$$

Thus $\mathcal{F}_\alpha f$ is entire.

To prove (3.4), apply Hölder's inequality to g_1 and g_2:

$$|\mathcal{F}_\alpha f(\lambda,\mu)| \leq Ce^{\frac{|\Im\lambda|^2+|\Im\mu|^2}{4a}} \left(\|f_1\|_{p,m_\alpha} \left(\int_{\mathbb{K}} e^{-\Delta_{\lambda,\mu}(y,\theta)p'} dy\, d\theta \right)^{\frac{1}{p'}} \right.$$
$$\left. + \|f_2\|_{q,m_\alpha} \left(\int_{\mathbb{K}} e^{-\Delta_{\lambda,\mu}(y,\theta)q'} dy\, d\theta \right)^{\frac{1}{q'}} \right) \leq Ce^{\frac{|\Im\lambda|^2+|\Im\mu|^2}{4a}} \left(\|f_1\|_{p,m_\alpha} + \|f_2\|_{q,m_\alpha} \right). \quad \square$$

Remark 3.4. *Condition* (3.3) *implies* $f \in \mathrm{L}^1_\alpha(\mathbb{K})$. *Indeed, by* (2.8) *and Hölder's inequality,*

$$\begin{aligned} \int_{\mathbb{K}} |f(y,\theta)| dm_\alpha(y,\theta) &\leq \left\| f_1 e^{-\frac{2(\alpha+1)y}{p}} \right\|_{p,m_\alpha} \left(\int_{\mathbb{K}} e^{-ap'(y^2+(|\theta|+\pi)^2)} e^{-2(\alpha+1)y} dm_\alpha \right)^{\frac{1}{p'}} \\ &\quad + \left\| f_2 e^{-\frac{2(\alpha+1)y}{q}} \right\|_{q,m_\alpha} \left(\int_{\mathbb{K}} e^{-aq'(y^2+(|\theta|+\pi)^2)} e^{-2(\alpha+1)y} dm_\alpha \right)^{\frac{1}{q'}} \\ &\lesssim \|f_1\|_{p,m_\alpha} + \|f_2\|_{q,m_\alpha} < \infty. \end{aligned}$$

Theorem 3.5. *Let $a, b, \beta > 0$, $p, q \in [1, \infty]$, and f be measurable on $\mathbb{R}^2$, even in the first variable, satisfying*

$$e^{a(y^2+(|\theta|+\pi)^2)+2(\alpha+1)y} f \in \mathrm{L}^p_\alpha(\mathbb{K}) + \mathrm{L}^q_\alpha(\mathbb{K})$$

and

$$\int_{\mathbb{R}^2} \log^+ \frac{|\mathcal{F}_\alpha f(\lambda,\mu)| e^{b(\mu^2+\lambda^2)}}{\beta} d\lambda\, d\mu < \infty. \tag{3.5}$$

Then:

- *If $ab > \frac{1}{4}$, then $f = 0$ a.e.*
- *If $ab = \frac{1}{4}$, then $f = C\mathrm{E}^\alpha_{\frac{1}{4a}}$ with $|C| \leq \beta$, where $\mathrm{E}^\alpha_{\frac{1}{4a}}$ is the heat kernel* (2.14).

Proof. Define $h(\lambda,\mu) = e^{\frac{\lambda^2+\mu^2}{4a}} \mathcal{F}_\alpha f(\lambda,\mu)$. By Lemma 3.3, h is entire and satisfies

$$|h(\lambda,\mu)| \leq C e^{\frac{(\Re\lambda)^2+(\Re\mu)^2}{4a}}.$$

Now consider

$$\int_{\mathbb{R}^2} \log^+ |h(\lambda,\mu)| d\lambda\, d\mu = \int_{\mathbb{R}^2} \log^+ \left(|\mathcal{F}_\alpha f(\lambda,\mu)| e^{b(\mu^2+\lambda^2)} e^{(\frac{1}{4a}-b)(\lambda^2+\mu^2)} \right) d\lambda\, d\mu.$$

- **Case** $ab > \frac{1}{4}$**:** Since $e^{(\frac{1}{4a}-b)(\lambda^2+\mu^2)} \leq 1$ and $\int_{\mathbb{R}^2} e^{(\frac{1}{4a}-b)(\lambda^2+\mu^2)} d\lambda\, d\mu < \infty$,

$$\int_{\mathbb{R}^2} \log^+ |h(\lambda,\mu)| d\lambda\, d\mu < \infty.$$

  Lemma 3.2 implies h is constant, so $\mathcal{F}_\alpha f = C e^{-\frac{\lambda^2+\mu^2}{4a}}$. Condition (3.5) forces $C = 0$ when $ab > \frac{1}{4}$, so $f = 0$ by injectivity of $\mathcal{F}_\alpha$.

- **Case** $ab = \frac{1}{4}$**:** Then

$$\int_{\mathbb{R}^2} \log^+ |h(\lambda,\mu)| d\lambda\, d\mu \leq \int_{\mathbb{R}^2} \log^+ \frac{|\mathcal{F}_\alpha f(\lambda,\mu)| e^{b(\mu^2+\lambda^2)}}{\beta} d\lambda\, d\mu < \infty.$$

  Lemma 3.2 gives $\mathcal{F}_\alpha f = C e^{-\frac{\lambda^2+\mu^2}{4a}}$, and (3.5) implies $|C| \leq \beta$. Inverting $\mathcal{F}_\alpha$ yields $f = C\mathrm{E}^\alpha_{\frac{1}{4a}}$. □

Corollary 3.6. *Let $a, b > 0$, $p, q \in [1, \infty]$, $1 \leq r < \infty$, and f measurable on $\mathbb{R}^2$, even in the first variable, satisfying*

$$e^{a(y^2+(|\theta|+\pi)^2)+2(\alpha+1)y} f \in \mathrm{L}^p_\alpha(\mathbb{K}) + \mathrm{L}^q_\alpha(\mathbb{K})$$

and

$$\int_{\mathbb{R}^2} e^{br(\mu^2+\lambda^2)} |\mathcal{F}_\alpha f(\lambda,\mu)|^r d\lambda\, d\mu < \infty. \tag{3.6}$$

If $ab \geq \frac{1}{4}$, then $f = 0$ a.e.

Proof. Since $\log^+ x \leq x$ for $x > 0$,

$$\log^+ \frac{|\mathcal{F}_\alpha f(\lambda,\mu)|e^{b(\mu^2+\lambda^2)}}{\beta} \leq \left(\frac{|\mathcal{F}_\alpha f(\lambda,\mu)|e^{b(\mu^2+\lambda^2)}}{\beta}\right)^r.$$

Choosing $\beta = 1$, (3.6) implies

$$\int_{\mathbb{R}^2} \log^+ |\mathcal{F}_\alpha f(\lambda,\mu)|e^{b(\mu^2+\lambda^2)} d\lambda\, d\mu < \infty.$$

By Theorem 3.5, $f = 0$ if $ab > \frac{1}{4}$. If $ab = \frac{1}{4}$, $f = C\mathrm{E}^\alpha_{\frac{1}{4a}}$ with $|C| \leq 1$, but (3.6) holds only if $C = 0$. ☐

Theorem 3.7 (Cowling-Price Type)**.** *Let f be measurable on $\mathbb{R}^2$, even in the first variable, with $a, b > 0$, $1 \leq p, q < \infty$, satisfying*

$$e^{a(y^2+(|\theta|+\pi)^2)+2(\alpha+1)y} f \in \mathrm{L}^p_\alpha(\mathbb{K})$$

and

$$e^{b(\mu^2+\lambda^2)}|\mathcal{F}_\alpha f(\lambda,\mu)| \in \mathrm{L}^q_\alpha(\widehat{\mathbb{K}}). \tag{3.7}$$

If $ab \geq \frac{1}{4}$, then $f = 0$ a.e.

Proof. Since $\mathrm{L}^p(\mathbb{K}) \subset \mathrm{L}^p(\mathbb{K}) + \mathrm{L}^q(\mathbb{K})$, (3.3) holds. From (3.7) and (2.10),

$$\int_{\mathbb{L}} e^{bq(\mu^2+\lambda^2)}|\mathcal{F}_\alpha f(\lambda,\mu)|^q |C_1(\lambda,\mu)|^{-2} d\lambda\, d\mu < \infty$$

implies

$$\int_{\mathbb{R}^2} e^{bq(\mu^2+\lambda^2)}|\mathcal{F}_\alpha f(\lambda,\mu)|^q d\lambda\, d\mu < \infty$$

by the evenness of $\mathcal{F}_\alpha f$ in μ. Corollary 3.6 with $r = q$ completes the proof. ☐

Remark 3.8. *This work establishes a Cowling-Price-type uncertainty principle (Theorem 3.7) within the Miyachi framework. It is instructive to compare this result with those derived from the Beurling-Hörmander framework, such as the one found in [12]. The two approaches are distinct in their hypotheses and their conclusions, particularly at the critical exponent $ab = 1/4$.*

(1) ***Comparison of hypotheses:***

- *In the Miyachi framework requires strict exponential decay without polynomial weights:*

$$e^{a(y^2+(|\theta|+\pi)^2)+2(\alpha+1)y} f \in \mathrm{L}^p_\alpha(\mathbb{K}), \quad e^{b(\lambda^2+\mu^2)}|\mathcal{F}_\alpha f| \in \mathrm{L}^q_\alpha(\widehat{\mathbb{K}}).$$

- *In the Beurling-Hörmander framework [12] permits a tempered decay, allowing polynomial weights:*

$$\int_{\mathbb{K}} \frac{|f| e^{a|y+\pi,\theta|^2}}{(1+|y,\theta|)^N}\, dm_\alpha < \infty, \quad \int_{\widehat{\mathbb{K}}} \frac{|\mathcal{F}_\alpha f| e^{b|\lambda,\mu|^2}}{(1+|\lambda,\mu|)^N}\, d\gamma_\alpha < \infty.$$

(2) ***Comparison of conclusions at*** $ab = 1/4$***:***

- *Under the Miyachi hypotheses, the conclusion is a sharp uniqueness result:* $f = 0$ *is the only function that satisfies the conditions.*
- *Under the Beurling-Hörmander hypotheses, the conclusion is a characterization result: the function* f *must be a finite linear combination of heat kernel modes:*

$$f(y,\theta) = \sum_{k+j<N-1} a_{k,j} \mathcal{W}^\alpha_{k,j}(y,\theta),$$

where $\mathcal{W}^\alpha_{k,j}$ *are defined by relation* (2.15)*.*

4 Local uncertainty principle and numerical study

In this section, we provide a local uncertainty principle of Faris-type for the generalized Fourier transform $\mathcal{F}_\alpha$. This result quantifies the impossibility of a function f and its transform $\mathcal{F}_\alpha(f)$ being simultaneously concentrated on sets of finite measure. We derive an inequality bounding the concentration of $\mathcal{F}_\alpha(f)$ on a set F by the spatial dispersion of f. We then compute the optimal constant numerically, quantifying the precise trade-off between spatial and spectral localization.

4.1 Faris-type local uncertainty principle

Faris local uncertainty theorem for the generalized Fourier $\mathcal{F}_\alpha$ states

Theorem 4.1. *If* $1 < p \leq 2$, $q = \frac{p}{p-1}$ *and* $0 < a < \frac{2}{q}$ *then for all* $f \in \mathrm{L}^p_\alpha(\mathbb{K})$ *and all measurable subset* $F \subset \widehat{\mathbb{K}}$ *satisfying* $0 < \gamma_\alpha(F) < +\infty$,

$$\left(\int_F |\mathcal{F}_\alpha f(\lambda,\mu)|^q \, d\gamma_\alpha(\lambda,\mu)\right)^{\frac{1}{q}} \leq K_{\alpha,a,q}\left(\gamma_\alpha(F)\right) \left(\int_{\mathbb{K}} |(y,\theta)|^p \, |f(y,\theta)|^p \, dm_\alpha(y,\theta)\right)^{\frac{1}{p}}, \tag{4.1}$$

where $K_{\alpha,a,q}$ *is a constant which depend on the measure of the subset* F, $\gamma_\alpha(F)$*.*

Proof. Let F be a measurable subset of $\widehat{\mathbb{K}}$. Let us denote B the Euclidean ball of radius $r > 0$.

$$B = \left\{(y,\theta) \in \mathbb{K}, \quad |(y,\theta)| = \sqrt{y^2+\theta^2} < r\right\}.$$

We get

$$\|\mathcal{F}_\alpha(f)\ \chi_F\|_{q,\gamma_\alpha} \leq \ \|\mathcal{F}_\alpha(f\chi_B)\ \chi_F\|_{q,\gamma_\alpha} + \|\mathcal{F}_\alpha(f\chi_{B^c})\ \chi_F\|_{q,\gamma_\alpha}.$$

On the other hand

$$\begin{aligned}\|\mathcal{F}_\alpha(f\chi_B)\ \chi_F\|^q_{q,\gamma_\alpha} &= \int_{\widehat{\mathbb{K}}} |\mathcal{F}_\alpha(f\chi_B)(\lambda,\mu)\ \chi_F(\lambda,\mu)|^q\ \ d\gamma_\alpha(\lambda,\mu)\\ &\leq \|\mathcal{F}_\alpha(f\chi_B)\|^q_{\infty,\gamma_\alpha} \int_{\widehat{\mathbb{K}}} \chi_F(\lambda,\mu)\ d\gamma_\alpha(\lambda,\mu).\end{aligned}$$

Then

$$\|\mathcal{F}_\alpha(f\chi_B)\ \chi_F\|_{q,\gamma_\alpha} \leq (\gamma_\alpha(F))^{\frac{1}{q}}\ \ \|\mathcal{F}_\alpha(f\chi_B)\|_{\infty,\gamma_\alpha}. \tag{4.2}$$

Moreover

$$\|\mathcal{F}_\alpha(f\chi_{B^c})\ \chi_F\|_{q,\gamma_\alpha} \leq \|\mathcal{F}_\alpha(f\chi_{B^c})\|_{q,\gamma_\alpha}. \tag{4.3}$$

According to relations (4.2) and (4.3), we obtain

$$\|\mathcal{F}_\alpha(f)\ \chi_F\|_{q,\gamma_\alpha} \leq (\gamma_\alpha(F))^{\frac{1}{q}}\ \|\mathcal{F}_\alpha(f\chi_B)\|_{\infty,\gamma_\alpha} + \|\mathcal{F}_\alpha(f\chi_{B^c})\|_{q,\gamma_\alpha}.$$

Therefore (2.9) and (2.12) yield to

$$\|\mathcal{F}_\alpha f\chi_F\|_{q,\gamma_\alpha} \leq (\gamma_\alpha(F))^{\frac{1}{q}}\ \|f\chi_B\|_{1,m_\alpha} + \|f\chi_{B^c}\|_{p,m_\alpha}. \tag{4.4}$$

Using Hölder inequality, we get

$$\|f\chi_B\|_{1,m_\alpha} \leq \left(\int_{\mathbb{K}} |f(y,\theta)|^p\ \ |(y,\theta)|^{ap}\, dm_\alpha(y,\theta)\right)^{\frac{1}{p}} \left(\int_{\mathbb{K}} |(y,\theta)|^{-aq}\, \chi_B(y,\theta) dm_\alpha(y,\theta)\right)^{\frac{1}{q}}.$$

Applying polar coordinates we get

$$\int_{\mathbb{K}} \frac{\chi_B(y,\theta)}{\|(y,\theta)\|^{aq}} dy\, d\theta = \frac{\pi}{2-qa} r^{2-qa}.$$

Since

$$\int_{\mathbb{K}} |(y,\theta)|^{-aq}\, \chi_B(y,\theta) dm_\alpha(y,\theta) \leq 2^{2(\alpha+1)} e^{2(\alpha+1)r} \int_{\mathbb{K}} \frac{\chi_B(y,\theta)}{|(y,\theta)|^{aq}} dy\, d\theta$$

then we deduce that

$$\|f\chi_B\|_{1,m_\alpha} \leq C_{\alpha,a,q} e^{\frac{2}{q}(\alpha+1)r}\ \ r^{\frac{2}{q}-a}\ \|\ \ |(y,\theta)|^a\ f\|_{p,m_\alpha}, \tag{4.5}$$

where

$$C_{\alpha,a,q} = \left(\frac{\pi\, 2^{2(\alpha+1)}}{2-qa}\right)^{\frac{1}{q}}. \tag{4.6}$$

According to relations (4.5) and (4.4) and the fact that

$$\|f\chi_{B^c}\|_{p,m_\alpha}^p \leq \| \, |(y,\theta)|^a \, f\|_{p,m_\alpha}^p \| \, |(y,\theta)|^{-ap} \, \chi_B^c \|_{\infty,m_\alpha} \leq r^{-ap} \| \, |(y,\theta)|^a \, f\|_{p,m_\alpha}^p$$

we conclude that

$$\|\mathcal{F}_\alpha f \chi_F\|_{q,\gamma_\alpha} \leq g(r) \| \, |(y,\theta)|^a \, f\|_{p,m_\alpha}, \tag{4.7}$$

where g is a function from $]0,+\infty[$ into $\mathbb{R}$, given by

$$g(r) = Ae^{br} r^c + r^{-a}, \tag{4.8}$$

where

$$A = C_{\alpha,a,q}\big(\gamma_\alpha(F)\big)^{\frac{1}{q}} > 0, \quad b = \frac{2}{q}(\alpha+1) > 0, \quad c = \frac{2}{q} - a > 0. \tag{4.9}$$

The function g is continuous and coercive on $]0,+\infty[$ since

$$\lim_{r\to 0^+} g(r) = +\infty \quad \text{and} \quad \lim_{r\to +\infty} g(r) = +\infty.$$

Thus, g attains a minimum. Differentiating, we get

$$g'(r) = Ae^{br} r^{c-1}(br + c) - ar^{-a-1}. \tag{4.10}$$

Setting $g'(r) = 0$ is equivalent to solving

$$h(r) := Ae^{br} r^{c+a}(br + c) - a.$$

Since $c + a = \frac{2}{q} > 0$, the function h is continuous and strictly increasing on $]0,+\infty[$, with

$$\lim_{r\to 0^+} h(r) = -a < 0, \quad \lim_{r\to +\infty} h(r) = +\infty.$$

Therefore, there exists a unique $r^* > 0$ such that $h(r^*) = a$, so $g'(r^*) = 0$. Since g is coercive, this critical point is the unique global minimum of g. Let us denote this unique minimum of g by

$$K_{\alpha,a,q}\big(\gamma_\alpha(F)\big) := \min_{r>0} g(r). \tag{4.11}$$

Finally, relation (4.7) yields (4.1), completing the proof. $\square$

4.2 Numerical study of the optimal constant

This section presents a comprehensive numerical investigation of the function $g(r)$ defined in Equation (4.7), which determines the optimal constant $K_{\alpha,a,q}(\gamma_\alpha(F))$ in Theorem 4.1. We recall that

$$g(r) = Ae^{br}r^c + r^{-a},$$

where the parameters are defined in relation (4.9).

To find the global minimizer $r^* > 0$ of $g(r)$, we implement the Newton-Raphson method to solve the equation $g'(r) = 0$. The first and second derivatives of $g(r)$ are:

$$\begin{aligned} g'(r) &= Ae^{br}r^{c-1}(br + c) - ar^{-a-1}, \\ g''(r) &= Ae^{br}r^{c-2}\left[(br + c)^2 + (c-1)(br + c) - c\right] + a(a+1)r^{-a-2}. \end{aligned}$$

The Newton-Raphson iteration scheme is given by:

$$r_{n+1} = r_n - \frac{g'(r_n)}{g''(r_n)}.$$

We initialize the algorithm with $r_0 = 0.1$ and use a convergence criterion of

$$|r_{n+1} - r_n| < 10^{-6}.$$

- **Numerical computation.** We choose specific parameter values:

$$\begin{cases} p = 1.5 & \to \text{ so } q = 3, \\ \alpha = 0.5, & \\ a = 0.5 & \to \text{ satisfies } a < \frac{2}{q}, \\ \gamma_\alpha(F) = 1 & \to \text{ for simplicity.} \end{cases}$$

  Now compute the constants:

$$\begin{cases} A = C_{\alpha,a,q} \cdot \left(\gamma_\alpha(F)\right)^{1/3} = \left(\frac{\pi \cdot 2^{2(0.5+1)}}{2-3\cdot 0.5}\right)^{1/3} = \left(\frac{\pi \cdot 2^3}{0.5}\right)^{1/3} \approx (50.265)^{1/3} \approx 3.691, \\ b = \frac{2}{3}(0.5+1) = 1, \\ c = \frac{2}{3} - 0.5 \approx 0.1667. \end{cases}$$

  Thus, the function simplifies to

$$g(r) \approx 3.691 \cdot e^r \cdot r^{0.1667} + r^{-0.5}.$$

The Newton-Raphson method converges rapidly to the solution, as demonstrated in Table 1.

Table 1: Newton-Raphson iterations.

Iteration (n)	r_n	$g'(r_n)$
0	0.100000	-12.456
1	0.157832	-2.891
2	0.180214	-0.327
3	0.183105	-0.006
4	0.183127	-0.000012
5	0.183127	≈ 0

The algorithm converges in 5 iterations to $r^* \approx 0.1831$, yielding the minimum value $g(r^*) \approx 5.677$. The following Figure 1 illustrates the behavior of $g(r)$, confirming the existence of a unique minimum where the term r^{-a} dominates as $r \to 0^+$ and the term $Ae^{br}r^c$ dominates as $r \to +\infty$.

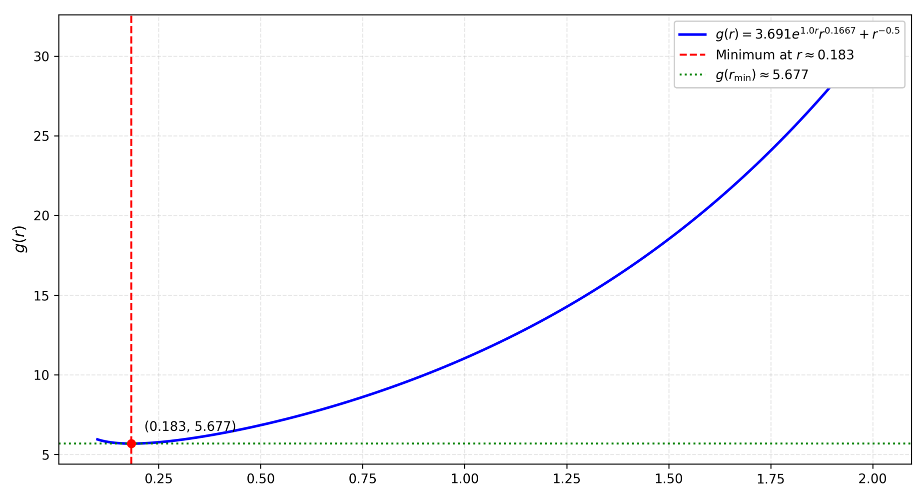


Figure 1: Behavior of $g(r)$ for $p = 1.5$, $\alpha = 0.5$, $a = 0.5$.

4.3 Asymptotic behavior of $K_{\alpha,a,q}(\gamma_\alpha(F))$

In the previous numerical study, the measure of the frequency set was fixed at $\gamma_\alpha(F) = 1$ to compute a specific value for the optimal constant. We now analyze the behavior of $K_{\alpha,a,q}(\gamma_\alpha(F))$ over the full range of its domain, particularly in the asymptotic regimes where $\gamma_\alpha(F) \to 0^+$ or $\gamma_\alpha(F) \to +\infty$. This analysis reveals the intrinsic scaling properties of the uncertainty principle and provides practical insight into the trade-off between spatial and frequency localization governed by the parameters α, a, p.

- **Behavior as** $\gamma_\alpha(F) \to 0^+$

  When $\gamma_\alpha(F) \to 0^+$, by relation (4.9) we have $A \to 0^+$. From (4.8), the dominant term in $g(r)$ becomes r^{-a}, so we expect the minimizing r^* to grow. We have

$$g'(r) = 0 \Longleftrightarrow Ae^{br}r^{c-1}(br+c) - ar^{-a-1} = 0 \Longleftrightarrow Ae^{br}r^{c-1}(br+c) = ar^{-a-1}.$$

  Applying logarithms, we get

$$\ln A + br + (c+a)\ln r + \ln(br+c) = \ln a.$$

  For small A, the term br dominates, so we approximate:

$$br^* \approx \ln\left(\frac{a}{A}\right) \implies r^* \approx \frac{1}{b}\ln\left(\frac{a}{A}\right).$$

  By substituting into $g(r^*)$, we obtain

$$g(r^*) \approx Ae^{br^*}(r^*)^c + (r^*)^{-a} \approx a(r^*)^{-a} \approx a\left(\frac{b}{\ln(a/A)}\right)^a.$$

  Since A is proportional to $\left(\gamma_\alpha(F)\right)^{\frac{1}{q}}$, we derive

$$K_{\alpha,a,q}(\gamma_\alpha(F)) \sim a\left(\frac{b}{\ln\left(\frac{a}{C_{\alpha,a,q}\gamma_\alpha(F)^{1/q}}\right)}\right)^a \quad \text{as} \quad \gamma_\alpha(F) \to 0^+,$$

  where $C_{\alpha,a,q}$ is given by (4.6).

- **Behavior as** $\gamma_\alpha(F) \to +\infty$

  Since $\gamma_\alpha(F) \to +\infty$, then $A \to +\infty$. On the other hand, the dominant term in $g(r)$ is $Ae^{br}r^c$, so we expect the minimizing r^* to shrink. The equation $g'(r) = 0$ gives us

$$Ae^{br}r^{c-1}(br+c) = ar^{-a-1}.$$

  For large A, the left hand side dominates, so we balance terms by taking $r^* \to 0^+$. Assume r^* is small and expand $e^{br} \approx 1 + br$. Then

$$A(1+br^*)(r^*)^{c-1}(br^*+c) \approx a(r^*)^{-a-1}.$$

  Yields to

$$Ac(r^*)^{c-1} \approx a(r^*)^{-a-1} \implies (r^*)^{c+a} \approx \frac{a}{Ac}.$$

Thus:

$$r^* \approx \left(\frac{a}{Ac}\right)^{\frac{1}{c+a}} = \left(\frac{a}{C_{\alpha,a,q}c\gamma_\alpha(F)^{1/q}}\right)^{\frac{1}{c+a}}.$$

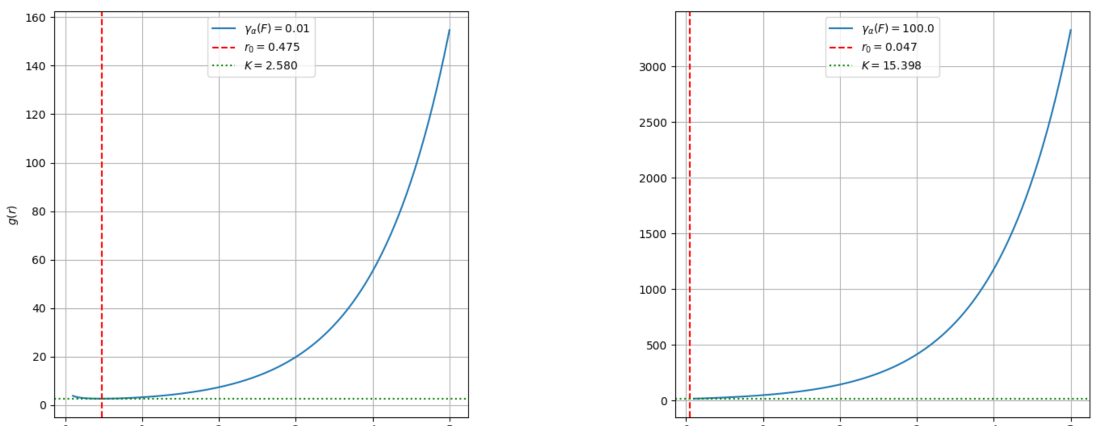
Substituting into $g(r^*)$:

$$g(r^*) \approx Ae^{br^*}(r^*)^c + (r^*)^{-a} \approx A(r^*)^c + (r^*)^{-a}.$$

Since $r^* \to 0^+$, the second term dominates:

$$K_{\alpha,a,q}(\gamma_\alpha(F)) \approx (r^*)^{-a} \approx \left(\frac{C_{\alpha,a,q}c\gamma_\alpha(F)^{1/q}}{a}\right)^{\frac{a}{c+a}}.$$

This contrasting behavior is illustrated in Figure 2, which shows the function $g(r)$ for extreme values of $\gamma_\alpha(F)$. The left panel shows the slow logarithmic decay for $\gamma_\alpha(F) \to 0^+$, while the right panel demonstrates the power-law growth for $\gamma_\alpha(F) \to +\infty$. The vertical dashed lines indicate the minimizing radius r^* in each case.



(a) Behavior of $g(r)$ for small $\gamma_\alpha(F)$.

(b) Behavior of $g(r)$ for large $\gamma_\alpha(F)$.

Figure 2

- **Numerical computation** The following table presents numerical values of the minimizing radius r_0 and the optimal constant $K_{\alpha,a,q}(\gamma_\alpha(F))$ for different values of $\gamma_\alpha(F)$, using the parameters:

$$p = 1.5, \quad \alpha = 0.5, \quad a = 0.5.$$

Table 2: Numerical values of the optimal radius r^* and constant $K_{\alpha,a,q}$.

$\gamma_\alpha(F)$	A	r^*	$K_{\alpha,a,q}$
10^{-6}	0.037	13.12	0.276
10^{-5}	0.079	11.72	0.295
10^{-4}	0.171	10.32	0.316
10^{-3}	0.369	8.92	0.341
10^{-2}	0.795	7.52	0.372
10^{-1}	1.713	6.12	0.404
1	3.691	0.183	5.677
10	7.937	0.089	12.309
10^{2}	17.088	0.042	24.891
10^{3}	36.913	0.020	48.712
10^{4}	79.370	0.009	94.868

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